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On certain applications of the two-point Padé approximants by using extended epsilon algorithm

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Abstract. The epsilon algorithm is closely related to the table of Padé approximants. In this paper, we extend this algorithm to compute the two-point Padé approximants recursively and generate the table of these approximants. The connection between this extended algorithm and two-point Padé approximants is established. Some examples in numerical analysis are treated.

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1. Introduction

The so-called two-point Padé approximants studied by several authors [5, 7, 10] arise as a natural generalization of the classical (one-point) Padé approximants. These approximants are rational functions whose expansions in ascending and descending powers coincide with Taylor and Laurent series as far as possible. The epsilon algorithm discovered by Wynn [11] has been studied extensively by himself and many other authors as Brezinski [2, 3, 4]. The epsilon algorithm was also studied in the vector case in [9]. This algorithm can be used to compute the half table of the classical Padé approximants. To generate the table of two-point Padé approximants, an algorithm as epsilon algorithm in two-point case, is needed. This paper is organized as follows. In section 2, two-point Padé approximants have been defined by means of orthogonal polynomials. Two-point Padé approximants have been written as a report of ratio of determinant. In section 3, the extended epsilon algorithm has been defined and has been established the link between two-point Padé approximants and this algorithm. In section 4, this algorithm has been applied to some examples, treated in [1, 6, 8], in order to illustrate the results.

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2. Two-point Padé approximants

Let $f(z)$ be a function which admits the following Taylor and Laurent expansions

$$f_0(z) = \sum_{i=0}^{+\infty} c_i z^i, \tag{1}$$

$$f_\infty(z) = -\sum_{i=1}^{+\infty} c_{-i} z^{-i}. \tag{2}$$

Let $l \in \mathbb{Z}$, the linear functional $c^{(l)}$ is defined as follows, with $c^{(0)}$ will be denoted by c

$$c^{(l)}(t^i) = c_{l+i}, \; i \in \mathbb{Z}. \tag{3}$$

$f(z)$ can formally be rewritten as

$$f(z) = c\left(\frac{1}{1-tz}\right). \tag{4}$$

Let m be an integer and $0 \le k \le m$, and let $V_{k,m}$ be a polynomial of the form

$$V_{k,m}(z) = \sum_{i=0}^{m} b_{m-i}^{(k,m)} z^i. \tag{5}$$

The associated polynomial of degree $m-1$ is defined by

$$W_{k,m}(z) = c^{(k-m)}\Big(\frac{z^{m-k}V_{k,m}(t) - t^{m-k}V_{k,m}(z)}{t-z}\Big). \tag{6}$$

It is defined as,

$$\widetilde{V}_{k,m}(z) = z^m V_{k,m}(z^{-1}) = \sum_{i=0}^{m} b_i^{(k,m)} z^i, \tag{7}$$

and

$$\widetilde{W}_{k,m}(z) = z^{m-1} W_{k,m}(z^{-1}). \tag{8}$$

Theorem 2.1.

$$f(z)\widetilde{V}_{k,m}(z) - \widetilde{W}_{k,m}(z) = z^k c^{(k-m)}\left(\frac{V_{k,m}(t)}{1-tz}\right). \tag{9}$$

Proof. For $\widetilde{V}_{k,m}(z)$ and $\widetilde{W}_{k,m}(z)$ defined as above, we have

$$\begin{aligned}
f(z)\widetilde{V}_{k,m}(z) &= c^{(k-m)}\left(\frac{t^{m-k}}{1-tz}\right)\widetilde{V}_{k,m}(z) \\
&= z^m c^{(k-m)}\left(\frac{t^{m-k}V_{k,m}(z^{-1})}{1-tz}\right) \\
&= z^{m-1} c^{(k-m)}\left(\frac{t^{m-k}V_{k,m}(z^{-1}) - z^{-(m-k)}V_{k,m}(t)}{z^{-1}-t}\right) + z^k c^{(k-m)}\left(\frac{V_{k,m}(t)}{1-tz}\right) \\
&= \widetilde{W}_{k,m}(z) + z^k c^{(k-m)}\left(\frac{V_{k,m}(t)}{1-tz}\right).
\end{aligned}$$

□

The rational approximant $\dfrac{\widetilde{W}_{k,m}(z)}{\widetilde{V}_{k,m}(z)}$, called two-point Padé-type approximant and denoted by $(k/m)_f$. When $k = m$, then the standard Padé-type approximants [3] has been dealt with.
If we do not choose $V_{k,m}$ randomly, but impose the conditions

$$c^{(k-2m)}\left(t^i V_{k,m}(t)\right) = 0, \;\; 0 \le i \le m-1. \tag{10}$$

then $V_{k,m}(t)$ is called the orthogonal polynomial of degree m with respect to the functional $c^{(k-2m)}$.
For $V_{k,m}(t)$ satisfying (10) the two-point Padé approximation condition

$$f_0(z) - \frac{\widetilde{W}_{k,m}(z)}{\widetilde{V}_{k,m}(z)} = \frac{z^k}{\widetilde{V}_{k,m}(z)} c^{(k-2m)}\left(t^m \frac{V_{k,m}(t)}{1-tz}\right),$$

and

$$\begin{aligned} f_\infty(z) - \frac{\widetilde{W}_{k,m}(z)}{\widetilde{V}_{k,m}(z)} &= \frac{z^k}{\widetilde{V}_{k,m}(z)} c^{(k-m)}\left(\frac{V_{k,m}(t)}{1-tz}\right) \\ &= \frac{z^{k-m}}{\widetilde{V}_{k,m}(z)} c^{(k-2m)}\left(\frac{V_{k,m}(t)}{1-tz}\right), \end{aligned}$$

hold. The rational approximant called two-point Padé approximants $\dfrac{\widetilde{W}_{k,m}(z)}{\widetilde{V}_{k,m}(z)}$ called two-point Padé approximant and denoted $[k/m]_f$. The case $k = 2m$ correspond to Padé approximants [3]. These two-point Padé approximants can be ordered in a table as

$$\begin{array}{cccc} & & & \iddots \\ & & [0/2]_f & \cdots \\ & [0/1]_f & [1/2]_f & \cdots \\ [0/0]_f = 0 & [1/1]_f & [2/2]_f & \cdots \\ & [2/1]_f & [3/2]_f & \cdots \\ & & [4/2]_f & \cdots \\ & & & \ddots \end{array} \tag{11}$$

As it is well known there is a close relationship between orthogonal polynomials and Hankel determinants. The Hankel determinants of order $n \in \mathbb{N}$ of the sequence $(c_i)_{i\in\mathbb{Z}}$ are

$$H_n^{(l)} = \begin{vmatrix} c_l & c_{l+1} & \cdots & c_{l+n-1} \\ c_{l+1} & c_{l+2} & \cdots & c_{l+n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{l+n-1} & c_{l+n} & \cdots & c_{l+2n-2} \end{vmatrix}, \text{ with } H_0^{(l)} = 1. \tag{12}$$

We also call the functional $c^{(l)}$ definite if

$$H_n^{(l)} \neq 0, \ \ n \geq 0.$$

In the sequel of the text we shall assume that $c^{(l)}$ is definite $\forall l \in \mathbb{Z}$.
Also we shall assume that $V_{k,m}(z)$ satisfies (10), $\widetilde{V}_{k,m}(z)$ is given by

$$\widetilde{V}_{k,m}(z) = \begin{vmatrix} c_{k-2m} & c_{k-2m+1} & \cdots & c_{k-m} \\ c_{k-2m+1} & c_{k-2m+2} & \cdots & c_{k-m+1} \\ \vdots & & & \vdots \\ c_{k-m-1} & c_{k-m} & \cdots & c_{k-1} \\ z^m & z^{m-1} & \cdots & 1 \end{vmatrix}. \tag{13}$$

It denotes

$$S_l(z) = \begin{cases} \displaystyle\sum_{i=0}^{l-1} c_i z^i & \text{if } l \geq 1 \\ 0 & \text{if } l = 0 \\ \displaystyle -\sum_{i=1}^{-l} c_{-i} z^{-i} & \text{if } l \leq -1. \end{cases} \tag{14}$$

The polynomial $\widetilde{W}_{k,m}(z)$ can be expressed explicitly by

$$\widetilde{W}_{k,m}(z) = \begin{vmatrix} c_{k-2m} & \cdots & c_{k-m-1} & c_{k-m} \\ c_{k-2m+1} & \cdots & c_{k-m} & c_{k-m+1} \\ \vdots & & & \vdots \\ c_{k-m-1} & \cdots & c_{k-2} & c_{k-1} \\ & & & \\ z^m S_{k-2m}(z) & \cdots & z S_{k-m-1}(z) & S_{k-m}(z) \end{vmatrix}. \tag{15}$$

So, for l with $-2n \leq l \leq 0$ and $n \in \mathbb{N}$, the two-point Padé approximant $[l+2n/n]_f(z)$ can be written as

$$[l+2n/n]_f(z) = \frac{\begin{vmatrix} c_l & \cdots & c_{l+n-1} & c_{l+n} \\ c_{l+1} & \cdots & c_{l+n} & c_{l+n+1} \\ \vdots & & & \vdots \\ c_{l+n-1} & \cdots & c_{l+2n-2} & c_{l+2n-1} \\ & & & \\ z^n S_l(z) & \cdots & z S_{l+n-1}(z) & S_{l+n}(z) \end{vmatrix}}{\begin{vmatrix} c_l & \cdots & c_{l+n-1} & c_{l+n} \\ c_{l+1} & \cdots & c_{l+n} & c_{l+n+1} \\ \vdots & & & \vdots \\ c_{l+n-1} & \cdots & c_{l+2n-2} & c_{l+2n-1} \\ z^n & \cdots & z & 1 \end{vmatrix}}. \tag{16}$$

3. Extended epsilon algorithm

The Hankel determinants of order $n \in \mathbb{N}$ of the sequence $(S_l(z))_{l \in \mathbb{Z}}$ are

$$H_n(S_l(z)) = \begin{vmatrix} S_l(z) & S_{l+1}(z) & \dots & S_{l+n-1}(z) \\ S_{l+1}(z) & S_{l+2}(z) & \dots & S_{l+n}(z) \\ \vdots & \vdots & \vdots & \vdots \\ S_{l+n-1}(z) & S_{l+n}(z) & \dots & S_{l+2n-2}(z) \end{vmatrix}, \text{ with } \ H_0(S_l(z)) = 1. \quad (17)$$

The Shanks's transformation which denoted by $e_n(S_l(z))$ is given by

$$e_n(S_l(z)) = \frac{H_{n+1}(S_l(z))}{H_n(\Delta^2 S_l(z))}. \quad (18)$$

Where Δ is the usual forward difference operator whose powers are defined by

$$\Delta^{i+1} S_l(z) = \Delta^i S_{l+1}(z) - \Delta^i S_l(z). \quad (19)$$

The rule of extended epsilon algorithm is as follows

$$\varepsilon_{j+1}^{(i)} = \varepsilon_{j-1}^{(i+1)} + \frac{1}{\varepsilon_j^{(i+1)} - \varepsilon_j^{(i)}}, \qquad i \in \mathbb{Z},\ j \in \mathbb{N}, \quad (20)$$

with

$$\varepsilon_{-1}^{(i)} = 0 \qquad i = 0, \pm 1, \pm 2, \dots \quad (21a)$$

$$\varepsilon_0^{(i)} = S_i(z) \qquad i = 0, \pm 1, \pm 2, \dots \quad (21b)$$

The extended epsilon algorithm and Shanks's transformation are related by

$$\varepsilon_{2n}^{(l)} = e_n(S_l(z)) \quad \text{and} \quad \varepsilon_{2n+1}^{(l)} = \frac{1}{e_n(\Delta S_l(z))}, \quad l \in \mathbb{Z},\ n \in \mathbb{N}. \quad (22)$$

Thus

$$\varepsilon_{2n}^{(l)} = \frac{H_{n+1}(S_l(z))}{H_n(\Delta^2 S_l(z))} \quad \text{and} \quad \varepsilon_{2n+1}^{(l)} = \frac{H_n(\Delta^3 S_l(z))}{H_{n+1}(\Delta S_l(z))}, \quad l \in \mathbb{Z},\ n \in \mathbb{N}. \quad (23)$$

The $\varepsilon_j^{(i)}$ can be ordered in a table as

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & \vdots & \varepsilon_1^{(-2)} & \vdots & \cdots \\
\vdots & \varepsilon_0^{(-1)} & & \varepsilon_1^{(-2)} & \\
0 & & \varepsilon_1^{(-1)} & & \cdots \\
 & \varepsilon_0^{(0)} & & \varepsilon_2^{(-1)} & \\
0 & & \varepsilon_1^{(0)} & & \cdots \\
\vdots & \varepsilon_0^{(1)} & \vdots & \varepsilon_2^{(0)} & \\
 & \vdots & & \vdots &
\end{array}$$

so the index j refers to a column while i refers to a diagonal.

Theorem 3.1. *For l such that $-2n \leq l \leq 0$ and $n \in \mathbb{N}$, let us apply the extended epsilon algorithm to the sequence $(S_l(z))_{l\in\mathbb{Z}}$ defined in (14), it holds that*

$$\varepsilon_{2n}^{(l)} = [l+2n/n]_f(z). \tag{24}$$

Proof.

$$\varepsilon_{2n}^{(l)} = \frac{\begin{vmatrix} S_l(z) & S_{l+1}(z) & \cdots & S_{l+n}(z) \\ \Delta S_l(z) & \Delta S_{l+1}(z) & \cdots & \Delta S_{l+n}(z) \\ \vdots & & & \vdots \\ \Delta S_{l+n-1}(z) & \Delta S_{l+n}(z) & \cdots & \Delta S_{l+2n-1}(z) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_l(z) & \Delta S_{l+1}(z) & \cdots & \Delta S_{l+n}(z) \\ \vdots & & & \vdots \\ \Delta S_{l+n-1}(z) & \Delta S_{l+n}(z) & \cdots & \Delta S_{l+2n-1}(z) \end{vmatrix}},$$

where

$$\Delta S_l(z) = S_{l+1}(z) - S_l(z) = c_l\, z^l.$$

Then

$$\varepsilon_{2n}^{(l)} = \frac{\begin{vmatrix} S_l(z) & S_{l+1}(z) & \cdots & S_{l+n}(z) \\ c_l z^l & c_{l+1} z^{l+1} & \cdots & c_{l+n} z^{l+n} \\ \vdots & & & \vdots \\ c_{l+n-1} z^{l+n-1} & c_{l+n} z^{l+n} & \cdots & c_{l+2n-1} z^{l+2n-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_l z^l & c_{l+1} z^{l+1} & \cdots & c_{l+n} z^{l+n} \\ \vdots & & & \vdots \\ c_{l+n-1} z^{l+n-1} & c_{l+n} z^{l+n} & \cdots & c_{l+2n-1} z^{l+2n-1} \end{vmatrix}}.$$

Multiply the first column of the numerator and denominator by z^n, the second by z^{n-1}, etc. and the last ones by 1, and divide the second lines of the numerator and the denominator by z^{l+n}, the third by z^{l+n+1}, and so on, and the last by z^{l+2n-1}. This proves the theorem. □

4. Numerical results

Let us apply the extended epsilon algorithm rules for the approximation of certain functions. It should be noted that the extended epsilon algorithm cannot be used in the case where the sequence $(c_i)_{i\in\mathbb{Z}}$ is not definite. So, some technicals to approximate such functions are shown.

Example 4.1. Consider the following function

$$f(z) = \frac{1}{z} \ln\left(1 + \frac{z}{1+z}\right).$$

The expansions in the neighborhood of zero and infinity respectively are given by

$$f_0(z) = \sum_{i=0}^{+\infty} \frac{(-1)^{i+1}(1-2^{i+1})}{i+1} z^i$$

$$f_\infty(z) = \frac{\ln(2)}{z} + \sum_{i=0}^{+\infty} \frac{(-1)^{i+1}(1-2^{-i-1})}{i+1} \frac{1}{z^{i+2}}.$$

The use of (20) and (21) yields the following results.
For $z = 0.01$, the exact value is 0.98522964430116

			$\varepsilon_6^{(-6)} = .98519594183238$
		$\varepsilon_4^{(-4)} = .98408544821482$	$\varepsilon_6^{(-5)} = .98523013566874$
	$\varepsilon_2^{(-2)} = .94776718479701$	$\varepsilon_4^{(-3)} = .98524634478430$	$\varepsilon_6^{(-4)} = .98522963714019$
$\varepsilon_0^{(0)} = 0$	$\varepsilon_2^{(-1)} = .98577822641337$	$\varepsilon_4^{(-2)} = .98522940083594$	$\varepsilon_6^{(-3)} = .98522964440548$
	$\varepsilon_2^{(0)} = .98522167487684$	$\varepsilon_4^{(-1)} = .98522964784664$	$\varepsilon_6^{(-2)} = .98522964429964$
		$\varepsilon_4^{(0)} = .98522964424959$	$\varepsilon_6^{(-1)} = .98522964430118$
			$\varepsilon_6^{(0)} = .98522964430116$

For $z = 100$, the exact value is 0.00688184391217

			$\varepsilon_6^{(-6)} = .00688184391217$
		$\varepsilon_4^{(-4)} = .00688184391215$	$\varepsilon_6^{(-5)} = .00688184391217$
	$\varepsilon_2^{(-2)} = .00688182989628$	$\varepsilon_4^{(-3)} = .00688184391533$	$\varepsilon_6^{(-4)} = .00688184391217$
$\varepsilon_0^{(0)} = 0$	$\varepsilon_2^{(-1)} = .00688375723639$	$\varepsilon_4^{(-2)} = .00688184347947$	$\varepsilon_6^{(-3)} = .00688184391227$
	$\varepsilon_2^{(0)} = .00662251655629$	$\varepsilon_4^{(-1)} = .00688190317753$	$\varepsilon_6^{(-2)} = .00688184389926$
		$\varepsilon_4^{(0)} = .00687373867653$	$\varepsilon_6^{(-1)} = .00688184568247$
			$\varepsilon_6^{(0)} = .00688160128456$

Example 4.2. The Dawson function (see [6]) is defined as follows

$$F(z) = e^{-z^2} \int_0^z e^{t^2}\, dt,$$

which admits expansions in the neighborhood of zero and infinity

$$F_0(z) = \sum_{i=0}^{\infty} \frac{(-1)^i 2^i}{1 \cdot 3 \cdots (2i+1)} z^{2i+1},$$

$$F_\infty(z) = \sum_{i=1}^{\infty} \frac{1 \cdot 3 \cdots (2i-3)}{2^i} \frac{1}{z^{2i-1}}.$$

Hence $F(z) = zf(z^2)$, with

$$f_0(z) = \sum_{i=0}^{\infty} \frac{(-1)^i 2^i}{1 \cdot 3 \cdots (2i+1)} z^i,$$

$$f_\infty(z) = \frac{1}{z} + \sum_{i=2}^{\infty} \frac{1 \cdot 3 \cdots (2i-3)}{2^i} \frac{1}{z^i}.$$

Thus, we apply the extended epsilon algorithm on function f in order to generate the approximants of F denoted by $\tilde{\varepsilon}_{2j}^{(i)}$. For $z = 0.01$, the exact value of function F is .00999933335999.

$$
\begin{array}{c|c|c|c}
 & & & \tilde{\varepsilon}_6^{(-6)} = -.02201133928313 \\
 & & \tilde{\varepsilon}_4^{(-4)} = -.01667266884524 & \tilde{\varepsilon}_6^{(-5)} = .00998867479374 \\
 & \tilde{\varepsilon}_2^{(-2)} = -.01000200040008 & \tilde{\varepsilon}_4^{(-3)} = .00999400279864 & \tilde{\varepsilon}_6^{(-4)} = .00999933293347 \\
\tilde{\varepsilon}_0^{(0)} = 0 & \tilde{\varepsilon}_2^{(-1)} = .00999800039992 & \tilde{\varepsilon}_4^{(-2)} = .00999933328890 & \tilde{\varepsilon}_6^{(-3)} = .00999933335999 \\
 & \tilde{\varepsilon}_2^{(0)} = .00999933337777 & \tilde{\varepsilon}_4^{(-1)} = .00999933335999 & \tilde{\varepsilon}_6^{(-2)} = .00999933335999 \\
 & & \tilde{\varepsilon}_4^{(0)} = .00999933335999 & \tilde{\varepsilon}_6^{(-1)} = .00999933335999 \\
 & & & \tilde{\varepsilon}_6^{(0)} = .00999933335999
\end{array}
$$

For $z = 100$, the exact value of function F is .00500025003751.

$$
\begin{array}{c|c|c|c}
 & & & \tilde{\varepsilon}_6^{(-6)} = .00500025003751 \\
 & & \tilde{\varepsilon}_4^{(-4)} = .00500025003751 & \tilde{\varepsilon}_6^{(-5)} = .00500025003751 \\
 & \tilde{\varepsilon}_2^{(-2)} = .00500025001251 & \tilde{\varepsilon}_4^{(-3)} = .00500025003751 & \tilde{\varepsilon}_6^{(-4)} = .00500025003751 \\
\tilde{\varepsilon}_0^{(0)} = 0 & \tilde{\varepsilon}_2^{(-1)} = .00499975001250 & \tilde{\varepsilon}_4^{(-2)} = .00500024993751 & \tilde{\varepsilon}_6^{(-3)} = .00500025003748 \\
 & \tilde{\varepsilon}_2^{(0)} = .01499775033745 & \tilde{\varepsilon}_4^{(-1)} = .00500224913767 & \tilde{\varepsilon}_6^{(-2)} = .00500025063712 \\
 & & \tilde{\varepsilon}_4^{(0)} = -.00832042239410 & \tilde{\varepsilon}_6^{(-1)} = .00499625543421 \\
 & & & \tilde{\varepsilon}_6^{(0)} = .02096748289651
\end{array}
$$

Example 4.3. Consider the error function (see [1])

$$erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^{+\infty} e^{-t^2} dt.$$

The following function

$$f(z) = \sqrt{\pi} e^{z^2} erfc(z),$$

admits expansions in the neighborhood of zero and infinity

$$f_0(z) = \sqrt{\pi} - 2z + \sqrt{\pi} z^2 + \ldots$$
$$f_\infty(z) = \frac{1}{z} - \frac{1}{2z^3} + \frac{3}{4z^5} + \ldots$$

Take

$$g(z) = f(z) \frac{z+1}{z+2}.$$

Since the coefficients of even power equal zeroes in the expansion in the neighborhood of infinity of f, the extended epsilon algorithm is applied on the function g in order to generate the approximations of f. The table below gives the errors between approximations and values accurate for each $k = 3, 4, 5$ with $m = 4$ and for different value of z.

z	$Error(k=3, m=4)$	$Error(k=4, m=4)$	$Error(k=5, m=4)$
0.001	$.162212 \times 10^{-9}$	$.824462 \times 10^{-14}$	$.566445 \times 10^{-18}$
0.01	$.151036 \times 10^{-6}$	$.788370 \times 10^{-10}$	$.527860 \times 10^{-13}$
0.1	$.780162 \times 10^{-4}$	$.508519 \times 10^{-6}$	$.256365 \times 10^{-8}$
1	$.858429 \times 10^{-3}$	$.132008 \times 10^{-3}$	$.747088 \times 10^{-5}$
10	$.322968 \times 10^{-6}$	$.109271 \times 10^{-5}$	$.257499 \times 10^{-5}$
100	$.657059 \times 10^{-12}$	$.258565 \times 10^{-10}$	$.730461 \times 10^{-9}$
1000	$.704689 \times 10^{-18}$	$.281755 \times 10^{-15}$	$.810789 \times 10^{-13}$

Example 4.4. The transformation of Laplace (see [6, 8]) is defined as below

$$F(p) = \int_0^{+\infty} e^{-px} f(x)dx.$$

Numerical inversion of Laplace transform by using continued fractions is studied by Grundy in [8]. Here, by using extended epsilon algorithm, the computation of numerical inversion is very easy. One of the examples given by Grundy is considered

$$F(p) = \frac{1}{\sqrt{p}(\sqrt{p}+a)}.$$

Take $a = 1$, the expansions in the neighborhood of zero and infinity

$$F(p) = \sum_{i=0}^{\infty} (-1)^i p^{\frac{i-1}{2}} \quad for\ \sqrt{p} < 1$$
$$F(p) = \sum_{i=0}^{\infty} (-1)^i p^{-\frac{i+2}{2}} \quad for\ \sqrt{p} > 1,$$

which gives, see [8]

$$f(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(1+\frac{i}{2})} x^{\frac{i}{2}} \quad for\ x\ small,$$
$$f(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(\frac{1-i}{2})} x^{-\frac{i+1}{2}} \quad for\ x\ large.$$

It is noted that $1/\Gamma(\frac{1-i}{2}) = 0$ if i is odd. For $z = \sqrt{x}$ and $g(z) = f(z^2)$, the expansion in the neighborhood of zero

$$g_0(z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(1+\frac{i}{2})} z^i,$$

and the expansion in the neighborhood of infinity

$$g_\infty(z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(\frac{1-i}{2})} z^{-i-1}.$$

We take

$$h(z) = g(z) + \frac{1}{z+2}.$$

Because there are coefficients equal zeroes in the expansions in the neighborhood of infinity of g, the extended epsilon algorithm is used on the function h in the purpose of generating the approximations of g and subsequently, it is resulted in those of the function f. In this example, the error between the exact value and approximate value is known because the inverse transform is

$$f(x) = \exp(x) erfc(\sqrt{x}).$$

The following table presents the absolute errors for each $k = 3, 4, 5$ with $m = 4$ and for different value of z.

z	$Error(k=3,m=4)$	$Error(k=4,m=4)$	$Error(k=5,m=4)$
0.001	$.125338 \times 10^{-6}$	$.390758 \times 10^{-8}$	$.628374 \times 10^{-10}$
0.01	$.300709 \times 10^{-5}$	$.307151 \times 10^{-6}$	$.141634 \times 10^{-7}$
0.1	$.413014 \times 10^{-4}$	$.150008 \times 10^{-4}$	$.164935 \times 10^{-5}$
1	$.131799 \times 10^{-3}$	$.233521 \times 10^{-3}$	$.376753 \times 10^{-4}$
10	$.273863 \times 10^{-4}$	$.968935 \times 10^{-3}$	$.505376 \times 10^{-4}$
100	$.338378 \times 10^{-6}$	$.379039 \times 10^{-5}$	$.440588 \times 10^{-5}$
1000	$.864093 \times 10^{-9}$	$.190817 \times 10^{-7}$	$.938192 \times 10^{-7}$

5. Conclusion

This work has shown how the two-point Padé approximants can been computed recursively in easier way. Some examples have been studied to illustrate our approach.

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