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Multiple solutions for a (p, q)-Laplacian Steklov problem

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Abstract. In this paper we study the existence of at least three nontrivial solutions for the nonlinear (p, q)-Laplacian problem, with nonlinear boundary conditions. We establish that there exist at least three non-zero solutions, under assumptions on the asymptotic behavior of the quotients $f(x, s)/|s|^{p-2}s$ and $pF(x, s)/|s|^p$ which extends the classical results with Dirichlet boundary conditions that for a.e. $x \in \partial\Omega$.

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1. Introduction

Consider the following (p, q)-Laplacian Steklov eigenvalue problem:

$$(S_{p,q}) \begin{cases} \Delta_p u + \mu\Delta_q u = |u|^{p-2}u + \mu|u|^{q-2}u & \text{in } \Omega, \\ \langle |\nabla u|^{p-2}\nabla u + \mu|\nabla u|^{q-2}\nabla u, \nu\rangle = f(x, u) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\langle ., .\rangle$ is the scalar product of $\mathbb{R}^N$, $1 < q < p < \infty$ and Δ_r, $r > 1$, denotes the r-Laplacian, namely $\Delta_r := \operatorname{div}(|\nabla u|^{r-2}\nabla u)\ \forall u \in W^{1,r}(\Omega)$, while the reaction term $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition.

Elliptic equations involving differential operators of the form

$$Au := \operatorname{div}(D(u)\nabla u) = \Delta_p u + \Delta_q u,$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$, usually called (p, q)-Laplacian, occur in many important concrete situations. For instance, this happens when one seeks stationary solutions to the reaction-diffusion system

$$u_t = Au + c(x, u), \tag{1}$$

This system has a wide range of applications in physics and related sciences like chemical reaction design [7], biophysics [15] and plasma physics [24]. In such applications, the function u describes a concentration, the first term on the right-hand side of (1) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x; u)$ has a polynomial form with respect to the concentration. In the last few years, the (p, q)-Laplace attracts a lot of attention

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and has been studied by many authors (see [21, 11, 27, 30]). However, there are few results one the eigenvalue problems for the (p, q)-Laplacian, we cite [8, 10, 18, 25].

In the case where $\mu = 0$, our problem $(S_{p,q})$ becomes the problem driven by p-Laplacian operator

$$(S_p)\begin{cases} \Delta_p u &= |u|^{p-2}u \quad \text{in } \Omega, \\ \langle |\nabla u|^{p-2}\nabla u, \nu\rangle &= f(x,u) \quad \text{on } \partial\Omega \end{cases}$$

In [3], the authors was studied the solvability of the Steklov problem (S_p), the existence of nontrivial solutions for this problem was also proved in [13], and the non resonance of solutions under and between the two first eigenvalues for this problem was studied in [4]. Under the zero Dirichlet boundary condition in Ω, the authors established in [22] that the the existence of at least three nontrivial solutions of problem $(S_{p,q})$, one greatest negative, another smallest positive, and the third nodal. In [23] it was studied the existence of multiple solutions via variational methods, truncation-comparison techniques, and Morse theory. We have studied in [28], the existence and non-existence results of a positive solution for our problem $(S_{p,q})$ in the case that f has the form $f(x,u) = \lambda[m_p(x)|u|^{p-2}u + \mu m_q(x)|u|^{q-2}u]$, at non resonance cases and in [29] at resonance cases. In [9], we have constructed a continuous curve in plane, which becomes a threshold between the existence and non-existence of positive solutions, in the case $f(x,u) = \alpha|u|^{p-2}u + \beta|u|^{q-2}u$. Our purpose of this work is to extend some of the known results with Dirichlet boundary conditions on bounded domain, (see, [22, 23]). This paper is organized as follows. In Section 2, we give some basic assumptions and preliminary results, that will be useful to prove the principal results of this article. In Section 3, we state and prove our main results.

2. Basic assumptions and preliminary results

Let $(X, \|.\|)$ be real Banach space. Given a set $V \subseteq X$, write $\overline{V}$ for the closure of V, ∂V for the boundary of V, and $\text{int}(V)$ for the interior of V. If $x \in X$ and $\delta > 0$ then $B_\delta(x) := \{z \in X : \|z - x\| < \delta\}$.
The symbol $(X^*, \|.\|_{X^*})$ denotes the dual space of X, $\langle ., .\rangle$ indicates the duality pairing between X and X^*, while $x_n \to x$ (respectively, $x_n \rightharpoonup x$) in X means the sequence $\{x_n\}$ converges strongly(respectively, weakly) in X. A function $\Phi : X \to \mathbb{R}$ fulfilling

$$\lim_{\|x\|\to+\infty} \Phi(x) = +\infty$$

is called coercive. Let $\Phi \in C^1(X)$. We say that Φ satisfies the Palais-Smale condition when
$(PS)_\Phi$ *Every sequence* $\{x_n\} \subseteq X$ *such that* $\{\Phi(x_n)\}$ *is bounded and*

$$\lim_{n\to\infty} \|\Phi'(x)\|_{X^*} = 0$$

possesses a convergent subsequence.
If $c \in \mathbb{R}$ then, as usual, $\Phi^c = \{x \in X : \Phi(x) \leq c\}$, while $K_c(\Phi) := K(\Phi) \cap \Phi^{-1}(c)$, with $K(\Phi)$ being the critical set of Φ, i.e. $K(\Phi) := \{x \in X : \Phi'(x) = 0\}$. An operator $A : X \to X^*$ is called of type $(S)_+$ provided

$$x_n \rightharpoonup x \text{ in } X, \ \limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle \leq 0 \ \Rightarrow \ x_n \to x \text{ in } X.$$

The next simple result is more or less known([21], Proposition 2.3).

Proposition 2.1. *Let X be reflexive and let $\Phi \in C^1(X)$ be coercive. Assume that: $\Phi' = A + B$, where $A : X \to X^*$ is of type $(S)_+$ while $B : X \to X^*$ is compact. Then Φ satisfies the Palais-Smale condition (PS).*

In the analysis of problem $(S_{p,q})$ we will use the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$ as well as the order cone

$$C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) \geq 0 \text{ for every } x \in \Omega\}.$$

This cone has a non-empty interior described as follows:

$$\text{int}C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \overline{\Omega}\}.$$

Let $u,\, v : \Omega \to \mathbb{R}$ be measurable functions and let $t \in \mathbb{R}$. The symbol $u \leq v$ means $u(x) \leq v(x)$ for almost every $x \in \overline{\Omega}$, $t^{\pm} := \max\{\pm t, 0\}, u^{\pm}(.) := u(.)^{\pm}$. If $p \in [1, +\infty)$ then $p' := p/(p-1)$ is the conjugate exponent of p and p^* indicates the Sobolev conjugate in dimension N, namely

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{when } p < N, \\ \text{any } q > 1 & \text{for } p = N, \\ +\infty & \text{otherwise.} \end{cases}$$

We introduce, provided $r \in [1, +\infty[$, the usual norm of $W^{1,r}(\Omega)$

$$\|u\|_{1,r} := \left(\int_\Omega (|\nabla u|^r + |u|^r) dx \right)^{1/r} \quad u \in W^{1,r}(\Omega),$$

$W^{1,r}(\Omega)^*$ denotes the dual space of $W^{1,r}(\Omega)$ while $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ is the nonlinear operator defined by:

$$\langle A_r(u), v\rangle := \int_\Omega (|\nabla u|^{r-2} \nabla u \cdot \nabla v + |u|^{r-2} uv) dx \;\forall u, v \in W^{1,r}(\Omega). \tag{2}$$

Denote by $\lambda_{1,r}$(respectively, $\lambda_{2,r}$) the first (respectively, second) eigenvalue of the operator Δ_r in $W^{1,r}(\Omega)$. The following properties of $\lambda_{1,r}$, $\lambda_{2,r}$ and A_r can be found in [2, 3, 5], see also [17],([16], Section 6.2).
(p_1) $0 < \lambda_{1,r} < \lambda_{2,r}$
(p_2) $\|u\|^r_{L^r(\partial\Omega)} \leq \frac{1}{\lambda_{1,r}} \|u\|^r_{1,r}$
(p_3)There exists an eigenfunction $\phi_{1,r}$ corresponding to $\lambda_{1,r}$ such that $\phi_{1,r} \in \text{int}(C^1(\overline{\Omega})_+)$ as well as $\|\phi_{1,r}\|_{L^r(\partial\Omega)} = 1$.
(p_4) The operator A_r is maximal monotone and of type $(S)_+$.

Let $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,0) = 0$ for all $x \in \partial\Omega$ and let

$$F(x,t) := \int_0^t f(x,s)ds, \; (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3}$$

The hypotheses below will be posited in the sequel. By convention, $p = q$ whenever $\mu = 0$.
(f_1) There exist $c_1 > 0, r \in [p, p*)$ satisfying

$$|f(x,t)| \leq c_1(1 + |t|^{r-1}) \text{ for all}(x,t) \in \partial\Omega \times \mathbb{R}.$$

(f_2) There exists $\theta \in L^\infty(\partial\Omega)\backslash\{\lambda_{1,p}\}$, where $0 \leq \theta < \lambda_{1,p}$ satisfying

$$\limsup_{|t|\to+\infty} \frac{pF(x,t)}{|t|^p} \leq \theta(x) \text{ uniformly with respect to a.e. } \ x \in \partial\Omega.$$

(f_3) for suitable $c_3 \geq c_2 > c(\mu)$, with

$$c(\mu) = \begin{cases} \lambda_{2,q} & \text{if } \mu = 0, \\ \mu\lambda_{2,q} & \text{otherwise,} \end{cases}$$

one has

$$c_2 \leq \liminf_{t\to 0} \frac{f(x,t)}{|t|^{q-2}t} \leq \limsup_{t\to 0} \frac{f(x,t)}{|t|^{q-2}t} \leq c_3 \text{ uniformly with respect to a.e. } \ x \in \partial\Omega.$$

(f_4) f is bounded on bounded sets.

Lemma 2.2. *If $\theta \in L^\infty(\partial\Omega)$, $\theta \leq \lambda_{1,r}$ a.e. in $\overline{\Omega}$ and $\theta \neq \lambda_{1,r}$ then there exists a constant $c(\theta) > 0$ such that*

$$\|u\|_{1,r}^r - \int_{\partial\Omega} \theta|u|^r d\sigma \geq c(\theta)\|u\|_{1,r}^r \ \forall u \in W^{1,r}(\Omega).$$

Proof. Let $\psi(u) = \|u\|_{1,r}^r - \int_{\partial\Omega} \theta|u|^r d\sigma$. It is clear that $\psi \geq 0$. Suppose that the lemma is not true. Since ψ is r-homogeneous, we can find a sequence $\{u_n\} \subset W^{1,r}(\Omega)$ such that $\|u_n\|_{1,r} = 1$ and $\psi(u_n) \downarrow 0$ as $n \to \infty$. By passing to a suitable subsequence if necessary, we may assume that

$$u_n \rightharpoonup u \text{ in } W^{1,r}(\Omega), \ u_n \to u \text{ in } L^r(\partial\Omega) \tag{4}$$

(recall that $W^{1,r}(\Omega)$ is embedded compactly in $L^r(\partial\Omega)$) and $|u_n(x)| \leq k(x)$ a.e. on $\overline{\Omega}$, for all $n \geq 1$, with $k \in L^r(\partial\Omega)$.
We have that $\|u\|_{1,r}^r \leq \liminf_{n\to\infty} \|u_n\|_{1,r}^r$ while from the dominated convergence theorem, it follows that $\int_{\partial\Omega} \theta|u_n|^r d\sigma \to \int_{\partial\Omega} \theta|u|^r d\sigma$. So, $\psi(u) \leq \liminf_{n\to\infty} \psi(u_n) = 0$. Consequently,

$$\|u\|_{1,r}^r \leq \int_{\partial\Omega} \theta|u|^r d\sigma \leq \lambda_{1,r}\|u\|_{L^r(\partial\Omega)}^r. \tag{5}$$

If $u = 0$, then from (5) applied to u_n, and (4), we see that $u_n \to 0$ in $W^{1,r}(\Omega)$ a contradiction to the fact that $\|u_n\|_{1,r} = 1$ for all $n \geq 1$. Hence, $u \neq 0$. But from (5) and by (6)

$$\lambda_{1,r} := \inf\left\{ \frac{\|u\|_{1,r}^r}{\|u\|_{L^r(\partial\Omega)}^r} : \|u\|_{L^r(\partial\Omega)}^r = 1 \right\}, \tag{6}$$

we have $\|u\|_{1,r}^r = \lambda_{1,r}\|u\|_{L^r(\partial\Omega)}^r$, and so $u = \pm t\hat{u}_1$ for some $t > 0$. Recalling that $\hat{u}_1(x) > 0$ for all $x \in \overline{\Omega}$. From (5) and hypothesis on θ we have $\|u\|_{1,r}^r < \lambda_{1,r}\|u\|_{L^r(\partial\Omega)}^r$, again a contradiction. The lemma is thus proved. □

Lemma 2.3. *If $\beta > 0$ and $\alpha > \mu\lambda_{1,q}$ then the problem*

$$(P_{\alpha,\beta}) = \begin{cases} \Delta_p u + \mu\Delta_q u = \mu|u|^{q-2}u + |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\nabla u\frac{\partial u}{\partial\nu} + |\nabla u|^{q-2}\nabla u\frac{\partial u}{\partial\nu} = \alpha|u|^{q-2}u - \beta|u|^{p-2}u & \text{on } \partial\Omega \end{cases}$$

possesses a unique nontrivial positive solution $\hat{u} \in int(C^1(\overline{\Omega})_+)$. Further, $-\hat{u}$ is the unique nontrivial negative solution of $(P_{\alpha,\beta})$.

Proof. Define, for every $u \in W^{1,p}(\Omega)$,

$$\psi_+(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{q}\|u\|_{1,q}^q - \frac{\alpha}{q}\|u^+\|_{L^q(\partial\Omega)}^q + \frac{\beta}{p}\|u^+\|_{L^p(\partial\Omega)}^p$$

Evidently, the functional ψ_+ belongs to $C^1(\overline{\Omega})$, is coercive, because $p > q$, and weakly sequentially lower semi-continuous. So, there exists $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\psi_+(\hat{u}) = \min_{u\in W^{1,p}(\Omega)} \psi_+(u). \tag{7}$$

Through (p_3), besides the conditions $\alpha > \mu\lambda_{1,q}$ and $p > q$, one has

$$\psi_+(t\phi_{1,q}) \leq \frac{t^p}{p}\left(\|\phi_{1,q}\|_{1,p}^p + \beta\|\phi_{1,q}\|_{L^p(\partial\Omega)}^p\right) + \frac{t^q}{q}(\mu\lambda_{1,q} - \alpha) < 0 \tag{8}$$

for any $t > 0$ small enough. Hence $\psi_+(\hat{u}) < 0$, which implies $\hat{u} \neq 0$. Now, from (7) it follows

$$\langle A_p(\hat{u}) + \mu A_q(\hat{u}), v\rangle = \int_{\partial\Omega}(\alpha|\hat{u}^+|^{q-2}\hat{u}^+ - \beta|\hat{u}^+|^{p-2}\hat{u}^+)vd\sigma \;\forall v \in W^{1,p}(\Omega). \tag{9}$$

Setting $v := -\hat{u}^-$ in (9) we obtain $\hat{u}^- = 0$. Thus, $\hat{u} \geq 0$ and, a fortiori, the function $\hat{u}$ solves $(P_{\alpha,\beta})$. By the regularity proven in [1], $\hat{u} \in C^{1,\alpha}(\overline{\Omega})$. From the first equation of $(P_{\alpha,\beta})$ we conclude

$$\begin{aligned}\Delta_p\hat{u} + \mu\Delta_q\hat{u} &= |\hat{u}|^{p-2}\hat{u} + \mu|\hat{u}|^{q-2}\hat{u}\\ &\leq \hat{u}^{p-1} + \mu\hat{u}^{q-1} \;\leq\; \mu'\hat{u}^{q-1}\end{aligned}$$

Setting $\beta(s) = \mu' s^{q-1}$ for $s > 0$ allows us to apply Vázquez's strong maximum principle [26] shows that $\hat{u} > 0$ for a.e. $x \in \Omega$. if there exists $x_0 \in \partial\Omega$ such that $\hat{u}(x_0) = 0$, we obtain by applying again Vázquez's strong maximum principle that $\frac{\partial\hat{u}}{\partial\nu}(x_0) < 0$, but the boundary condition impose $\frac{\partial\hat{u}}{\partial\nu}(x_0) = 0$ a condition. Hence, $\hat{u}(x) > 0$ in $\overline{\Omega}$ and therefore, we get $\hat{u} \in \mathrm{int}(C^1(\overline{\Omega})_+)$.
Let us verify that $\hat{u}$ is unique. The functional $\eta(u) : L^1(\Omega) \to [0, +\infty]$ given by

$$\eta(u) := \begin{cases} \frac{1}{p}\|u^{1/q}\|_{1,p}^p + \frac{\mu}{q}\|u^{1/q}\|_{1,q}^q & \text{if } u \geq 0,\; u^{1/q} \in W^{1,p}(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

is convex. In fact, pick $u_1, u_2 \in dom(\eta)$, $\theta \in [0, 1]$ and define $w := (\theta u_1+(1-\theta)u_2)^{1/q}$. The proof of [[11],Lemma 1] ensures that

$$|\nabla w|^q + |w|^q \leq (\theta|\nabla u_1^{1/q}|^q + |u_1^{1/q}|^q) + (1-\theta)(|\nabla u_2^{1/q}|^q + |u_2^{1/q}|^q). \tag{10}$$

$$\begin{aligned}\eta(\theta u_1 + (1-\theta)u_2) &= \frac{1}{p}\|w\|_{1,p}^p + \frac{\mu}{q}\|w\|_{1,q}^q\\ &= \frac{1}{p}\int_\Omega(|\nabla w|^p + |w|^p)dx + \frac{\mu}{q}\int_\Omega(|\nabla w|^q + |w|^q)dx\\ &\leq \theta(\frac{1}{p}\|u_1^{1/q}\|_{1,p}^p + \frac{\mu}{q}\|u_1^{1/q}\|_{1,q}^q) + (1-\theta)(\frac{1}{p}\|u_2^{1/q}\|_{1,p}^p + \frac{\mu}{q}\|u_2^{1/q}\|_{1,q}^q)\\ &= \theta\eta(u_1) + (1-\theta)\eta(u_2)\end{aligned}$$

Through Fatou's lemma we see that η is also lower semi-continuous. Now, suppose u_1, u_2 are solutions of $(P_{\alpha,\beta})$ lying in $\mathrm{int}C^1(\overline{\Omega})_+$ while $v \in C^1(\overline{\Omega})$. Obviously, $u_1^q +$

$tv, u_2^q + tv \in C^1(\overline{\Omega})_+$
for any small t. Thus, by the chain rule,

$$\begin{aligned}\langle \eta'(u_1^q), v\rangle &= \int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 \nabla v + |u_1|^{p-2}u_1 v)(\frac{u_1^{1-q}}{q})dx \\ &+ \mu \int_\Omega (|\nabla u_1|^{q-2}\nabla u_1 \nabla v + |u_1|^{q-2}u_1 v)(\frac{u_1^{1-q}}{q})dx \\ &= \int_{\partial\Omega} (|\nabla u_1|^{p-2} + |\nabla u_1|^{q-2})\nabla u_1 \frac{\partial u_1}{\partial \nu}\frac{u_1^{1-q}}{q} v d\sigma \\ &= \int_{\partial\Omega} (\alpha |u_1|^{q-2}u_1 - \beta |u_1|^{p-2}u_1)\frac{u_1^{1-q}}{q} v d\sigma\end{aligned}$$

$$\begin{aligned}\langle \eta'(u_2^q), v\rangle &= \int_\Omega (|\nabla u_2|^{p-2}\nabla u_2 \nabla v + |u_2|^{p-2}u_2 v)(\frac{u_2^{1-q}}{q})dx \\ &+ \mu \int_\Omega (|\nabla u_2|^{q-2}\nabla u_2 \nabla v + |u_2|^{q-2}u_2 v)(\frac{u_2^{1-q}}{q})dx \\ &= \int_{\partial\Omega} (\alpha |u_2|^{q-2}u_2 - \beta |u_2|^{p-2}u_2)\frac{u_2^{1-q}}{q} v d\sigma\end{aligned}$$

exploiting the monotonicity of η', this entails

$$\begin{aligned}0 &\leq \langle \eta'(u_1^q) - \eta'(u_2^q), u_1 - u_2\rangle \\ &= \int_{\partial\Omega} (\alpha |u_1|^{q-2}u_1 - \beta |u_1|^{p-2}u_1)\frac{u_1^{1-q}}{q}(u_1 - u_2) d\sigma \\ &- \int_{\partial\Omega} (\alpha |u_2|^{q-2}u_2 - \beta |u_2|^{p-2}u_2)\frac{u_2^{1-q}}{q}(u_1 - u_2) d\sigma \\ &= \int_{\partial\Omega} (u_2^{p-q} - u_1^{p-q})(u_1 - u_2) d\sigma \leq 0.\end{aligned}$$

Since $t \mapsto t^{p-q}$, $t \geq 0$, is strictly increasing, $u_1 = u_2$, and the uniqueness of $\hat{u}$ follows. □

To simplify notation, define $X = W^{1,p}(\Omega)$. Let F be as in(3) and let

$$\varphi(u) := \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{p}\|u\|_{1,q}^q - \int_{\partial\Omega} F(x, x(u))d\sigma,\ x \in X. \tag{11}$$

Obviously, $\varphi \in C^1(X)$. Moreover, critical points of φ are weak solutions to $(S_{p,q})$, and vice-versa.

Lemma 2.4. *Suppose (f_1)-(f_2) hold true. Then φ turns out to be weakly sequentially lower semi-continuous and coercive.*

Proof. Since X compactly embeds in $L^p(\partial\Omega)$ while $W^{1,q}(\Omega)^* \subseteq W^{1,p}(\Omega)^*$ the functional φ is weakly sequentially lower semi-continuous. Pick $\varepsilon \in (0, c_\theta \lambda_{1,p})$, with c_θ coming from Lemma 2.2. By (f_1)-(f_2), there exists $c_4 > 0$ such that

$$F(x, z) \leq \frac{\theta(x) + \varepsilon}{p}|z|^p + c_4\ \forall (x, z) \in \partial\Omega \times \mathbb{R}.$$

Hence, on account of (p_2),

$$\varphi(u) \geq \frac{1}{p}\left(\|u\|_{1,p}^p - \int_{\partial\Omega} \theta(x)|u(x)|^p d\sigma - \frac{\varepsilon}{\lambda_{1,p}}\|u\|_{1,p}^p\right) - c_4 m(\partial\Omega) \text{ in} X.$$

Due to Lemma 2.2 this implies

$$\varphi(u) \geq \frac{1}{p}\left(c_\theta - \frac{\varepsilon}{\lambda_{1,p}}\right)\|u\|_{1,p}^p - c_4 m(\partial\Omega).\ u \in X.$$

and the conclusion follows. □

Gathering Proposition 2.1, (p_4), and Lemma 2.4 together we easily infer the next result.

Lemma 2.5. *Under assumptions (f_1)-(f_2), the functional φ satisfies the Palais-Smale condition (PS).*

In fact, one has

$$\langle \varphi'(u), v\rangle = \langle A_p(u) + \mu A_q(u), v\rangle + \langle B(u), v\rangle \ \forall u, v \in X,$$

where $\langle B(u), v\rangle = -\int_{\partial\Omega} f(x, u(x))v(x)d\sigma$.
By (p_5) the operator $A_p + \mu A_q$ turns out to be of type $(S)_+$, while $B : X \to X^*$ is compact. Indeed, let $(u_n)_n$ be a bounded sequence in X: Up to a subsequence denoted also by $(u_n)_n$, we have

$$u_n \rightharpoonup u \text{ in } X,$$

by the compact embedding X into $L^p(\partial\Omega)$, we have

$$u_n \to u \text{ a.e in } \partial\Omega.$$

Since f is Carathéodory function which also verifies the condition (f_1),

$$f(x, u_n)u_n \to f(x, u)u \text{ a.e in } \partial\Omega.$$

By using Hölder's inequality and Sobolev's embedding and according to Dominated convergence theorem, we obtain

$$B(u_n) \to B(u).$$

3. Existence of multiple solutions

In this section, we can formulate our main results about the existence of three non-trivial solutions of our problem $(S_{p,q})$ in the following theorem.

Theorem 3.1. *Let (f_1)-(f_4) be fulfilled. Then, there exist three functions $u_1 \in$ int(C_+), $u_2 \in -$int(C_+), and $u_3 \in C^1(\overline{\Omega})$ that solves Problem $(S_{p,q})$.*

Proof. First, we prove the existence of u_1 and u_2 which are local minimizers of φ. We define, for every $x \in \partial\Omega, t, \in \mathbb{R}$,

$$f_+(x,t) = \begin{cases} 0 & \text{if } t \leq 0 \\ f(x,t) & \text{if } t > 0, \end{cases} \quad \text{and } f_-(x,t) = \begin{cases} 0 & \text{if } t \geq 0 \\ f(x,t) & \text{if } t < 0, \end{cases}$$

which are a Carathéodory functions. Setting

$$F_\mp(x,t) := \int_0^t f_\mp(x,\tau)d\tau.$$

as well as

$$\varphi_{\mp}(u) := \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{q}\|u\|_{1,q}^q - \int_{\partial\Omega} F_{\mp}(x,u(x))d\sigma \ \forall u \in X. \tag{12}$$

Remark 3.1. Obviously, both φ_+ and φ_- fulfill the properties stated in Lemmas 2.4 and 2.5 concerning φ.

By Remark 3.1 the functional φ_+ turns out to be coercive. A simple argument, based on the compact embedding $X \subset L^p(\partial\Omega)$, shows that it is also weakly sequentially lower semi-continuous. So, there exists $u_1 \in X$ satisfying

$$\varphi_+(u_1) = \inf_{u\in X} \varphi_+(u). \tag{13}$$

We claim that $u_1 \neq 0$. In fact, because of (f_3) one has

$$\frac{c_5}{q}|z|^q \leq F(x,z) \ (x,z) \in \partial\Omega \times [-\delta,\delta], \tag{14}$$

for suitable $c_5 \in (c(\mu), c_2)$ and $\delta > 0$. If $t > 0$ is so small that

$$0 \leq t\phi_{1,q}(x) \leq \delta \ \forall x \in \overline{\Omega}.$$

where $\phi_{1,q}$ comes from (p_3), then

$$\varphi_+(t\phi_{1,q}) \leq \frac{t^p}{p}\|\phi_{1,q}\|_{1,p}^p + \mu\frac{t^q}{q}\lambda_{1,q} - \frac{t^q}{q}c_5 = \frac{t^q}{q}\left(t^{p-q}\|\phi_{1,q}\|_{1,p}^p + \mu\lambda_{1,q} - c_5\right). \tag{15}$$

Since $p > q$ while $c_5 > \mu\lambda_{1,q}$, by decreasing t when necessary, (15) furnishes $\varphi_+(t\phi_{1,q}) < 0$. Hence,

$$\varphi_+(u_1) = \inf_{u\in X} \varphi_+(u) = \varphi_+(0), \tag{16}$$

which clearly means $u_1 \neq 0$, as desired. Now, from (13) it follows, $\varphi'_+(u_1) = 0$, namely

$$\langle A_p(u_1) + \mu A_q(u_1), v\rangle = \int_{\partial\Omega} f_+(x,u_1(x))v(x)d\sigma \ \forall v \in X. \tag{17}$$

Through (17) written for $v := -u_1^-$ we obtain $\|u_1^-\|_{1,p}^p + \mu\|u_1^-\|_{1,q}^q = 0$. Arguing exactly as in the proof of lemma 2.3 yields $u_1 \in C_+ \setminus \{0\}$. Let $\rho := \|u_1\|_{L^\infty(\partial\Omega)}$. The conditions (f_3), (f_4) imply the existence of constant $c_f > 0$ such that

$$|f(x,s)| \leq c_f s^{p-1} \text{ for a.e. } x \in \partial\Omega \text{ and all } \ -\rho \leq s \leq \rho. \tag{18}$$

In order to prove that u_1 is strictly positive in the closure of Ω, we suppose there exists $x_0 \in \partial\Omega$ such that $u_0(x_0) = 0$. By applying the maximum principle (see [26], Theorem 5), we obtain $\frac{\partial u_1}{\partial\nu}(x_0) < 0$. But taking into account $f(x_0, u_1(x_0)) = f(x_0, 0) = 0$ along with the boundary condition in $(S_{p,q})$ yields $\frac{\partial u_1}{\partial\nu}(x_0) = 0$, which is a contradiction. Thus, $u_1 > 0$ in $\overline{\Omega}$ which proves $u_1 \in \text{int}(C^1(\overline{\Omega})_+)$.
The same reasoning, with φ_- instead of φ_+, gives a solution $u_2 \in -\text{int}(C^1(\overline{\Omega})_+)$ to (2). So, the proof is completed once we show that both u_1 and u_2 are local minimizers for φ. If $u_n \to u_1$ in X then $u_n^+ \to u_1$ and $u_n^- \to 0$ in X because $u_1 \in \text{int}(C^1(\overline{\Omega})_+)$. Let

$$\partial\Omega_n := \{x \in \partial\Omega : u_n(x) < 0\}, \ n \in \mathbb{N}.$$

For every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \partial\Omega$ such that $m(\partial\Omega \backslash K_\varepsilon) < \varepsilon$. Observe that

$$\begin{aligned}\|u_n - u_1\|^p_{L^p(\partial\Omega)} &\geq \int_{\partial\Omega_n \cap K_\varepsilon} |u_n(x) - u_1(x)|^p d\sigma \\ &\geq \int_{\partial\Omega_n \cap K_\varepsilon} u_1(x)^p d\sigma \geq \left(\min_{x\in K_\varepsilon} u_1(x)\right)^p m(\partial\Omega_n \cap K_\varepsilon) \ \forall\, n \in \mathbb{N},\end{aligned}$$

which evidently forces

$$\lim_{n\to+\infty} m(\partial\Omega_n \cap K_\varepsilon) = 0.$$

Consequently.

$$\limsup_{n\to+\infty} m(\partial\Omega_n) = \limsup_{n\to+\infty}(m(\partial\Omega_n \cap K_\varepsilon) + m(\partial\Omega_n \backslash K_\varepsilon)) \leq m(\partial\Omega\backslash K_\varepsilon) < \varepsilon.$$

As ε was arbitrary, we actually have

$$\lim_{n\to+\infty} m(\partial\Omega_n) = 0. \tag{19}$$

Assumptions (f_2)-(f_3) provide a constant $c_6 > 0$ satisfying

$$F(x, s) \leq c_6(|s|^p + |s|^q) \ (x, s) \in \partial\Omega \times \mathbb{R}. \tag{20}$$

Let us verify that

$$\frac{1}{p}\|u_n^-\|^p_{1,p} > c_6\|u_n^-\|^p_{L^p(\partial\Omega)} \tag{21}$$

for any sufficiently large n. If this is not true then, passing to a subsequence when necessary, $\|w_n\|^p_{L^p(\partial\Omega)} \leq c_8 p \ \forall\, n \in \mathbb{N}$, where $w_n := \frac{1}{\|u_n^-\|^p_{L^p(\partial\Omega)}} u_n^-$.
Hence, we may assume that $w_n \to w$ in $L^p(\partial\Omega)$ for some $w \in X$. Since $w \geq 0$ and $\|w\|_{L^p(\partial\Omega)} = 1$, there exists $\delta > 0$ fulfilling

$$m(\partial\Omega_\delta) > 0, \tag{22}$$

with $m(\partial\Omega_\delta) := \{x \in \partial\Omega : w(x) \geq \delta\}$. On the other hand,

$$\|w_n - w\|^p_{L^p(\partial\Omega)} \geq \int_{\partial\Omega_\beta\backslash\partial\Omega_n} w(x)^p d\sigma \geq \delta^p(m(\partial\Omega_\beta) - m(\partial\Omega_\beta \cap \partial\Omega_n))$$

for all $n \in \mathbb{N}$. On account of (19) this entails, as $n \to +\infty$. $\delta^p(m(\partial\Omega_\beta)) = 0$, which contradicts (22). Therefore, (21) holds true. A similar reasoning ensures that

$$\frac{\mu}{q}\|u_n^-\|^q_{1,q} > c_6\|u_n^-\|^q_{L^q(\partial\Omega)} \tag{23}$$

provided n is big enough. Gathering (20), (21) and (23) together yields

$$\begin{aligned}\varphi(u_n) &\geq \varphi(u_n^+) + \frac{1}{p}\|u_n^-\|^p_{1,p} + \frac{\mu}{q}\|u_n^-\|^q_{1,q} - c_6\|u_n^-\|^q_{L^q(\partial\Omega)} - c_6\|u_n^-\|^p_{L^p(\partial\Omega)} \\ &> \varphi(u_n^+) = \varphi_+(u_n^+) \geq \varphi_+(u_1).\end{aligned}$$

namely $\varphi(u_n) > \varphi(u_1)$ for any sufficiently large n. Since $\{u_n\}$ was arbitrary and $u_n \to u_1$ in X, we deduce that u_1 is a local minimizer of φ The same conclusion, with a similar proof, holds for u_2.

Finally, we prove the existence of a function $u_3 \in C^1(\overline{\Omega})\backslash\{0, u_1, u_2\}$ that solves our problem $(S_{p,q})$.

From 3.1 we know that $0, u_1, u_2$ are local minimizers of φ. Without any loss of generality, we may assume that each of them is an isolated critical point of φ. Moreover, we may assume that $\varphi(u_2) \leq \varphi(u_1)$ (the analysis is similar if the opposite inequality holds). Reasoning as [[19], Proposition 5.42], we can find a $\rho \in (0, \|u_1 - u_2\|)$ such that

$$\varphi(u_2) \leq \varphi(u_1) < \inf\{\varphi(u) : u \in \partial B_\rho(u_1)\} := c_\rho. \tag{24}$$

where $\partial B_\rho = \{u \in X : \|u - u_2\| = \rho\}$. Assertion (24) along with the fact that φ satisfies the Palais-Smale condition (see Lemma 2.5) enable us to apply the Mountain-Pass Theorem to φ (see [20]) which yields the existence of $u_3 \in X$ satisfying $\varphi'(u_3) = 0$ and

$$c_\rho \leq \varphi(u_3) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \tag{25}$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_2, \gamma(1) = u_1\}.$$

We see at once that (24) and (25) show that $u_3 \neq u_1$ and $u_3 \neq u_2$. The proof is thus completed once one achieves $u_3 \neq 0$. In order to prove $u_3 \neq 0$ we are going to show that

$$\varphi(u_3) < 0, \tag{26}$$

which is satisfied if there exists a path $\widetilde{\gamma} \in \Gamma$ such that

$$\varphi(\widetilde{\gamma}(t)) < 0, \ \forall\, t \in [0,1]. \tag{27}$$

Let $S = X \cap \partial B_1^{L^p(\partial\Omega)}$, where $\partial B_1^{L^p(\partial\Omega)} = \{u \in L^p(\partial\Omega) : \|u\|_{L^p(\partial\Omega)} = 1\}$, and $S_c = S \cap C^1(\overline{\Omega})$ be equipped with the topologies induced by X and $C^1(\overline{\Omega})$, respectively. Furthermore, we set

$$\Gamma_0 = \{\gamma \in C([-1,1], S_c) : \gamma(-1) = -\phi_{1,q}, \gamma(1) = \phi_{1,q}\},$$

then we have the following variational characterization of $\lambda_{2,q}$ can be represented as follows:

$$\lambda_{2,q} =: \inf_{\gamma \in \Gamma_0} \max_{u \in \gamma([0,1])} \|u\|_{1,q}^q. \tag{28}$$

Suppose that $\mu > 0$ (the reasoning is simpler if $\mu = 0$). Since (28) there exists a $\gamma \in \Gamma_0$ to every $\eta > 0$ such that

$$\max_{t \in [-1,1]} \|\gamma(t)\|_{1,q}^q < \lambda_{2,q} + \frac{\eta}{2\mu}$$

It is well known that S_c is dense in S. Therefore, we can find $\gamma_0 \in \Gamma_0$ and

$$\max_{t \in [-1,1]} \|\gamma(t) - \gamma_0(t)\|_{1,q}^q < \left(\lambda_{2,q} + \frac{\eta}{\mu}\right)^{1/q} - \left(\lambda_{2,q} + \frac{\eta}{2\mu}\right)^{1/q}$$

This evidently forces

$$\max_{t \in [-1,1]} \|\gamma_0(t)\|_{1,q}^q < \lambda_{2,q} + \frac{\eta}{\mu}. \tag{29}$$

Owing to the compactness of $\gamma_0([-1,1])$ in $C^1(\overline{\Omega})$ we obtain c_7 satisfying

$$\|\gamma_0(t)\|_{1,q}^q \leq c_7, \ t \in [-1,1], \tag{30}$$

as well as $\epsilon_0 > 0$ such that

$$\epsilon_0 \max_{x \in \overline{\Omega}} |u(x)| \leq \delta, \text{for all } x \in \Omega \text{ and all } u \in \gamma_0([-1,1]). \tag{31}$$

Pick $\eta < c_5 - \mu\lambda_{2,q}$, with c_5 given by (14). Since $q < p$, there exists $\varepsilon_1 > 0$ fulfilling

$$\varepsilon^{p-q}c_7 + \mu\lambda_{2,q} + \eta - c_5 < 0 \ \forall\ \varepsilon \in (0, \varepsilon_1). \tag{32}$$

The function $t \mapsto \gamma_0(t)$, $t \in [-1, 1]$, is a continuous path in S_c, which joins $-\phi_{1,q}$ and $\phi_{1,q}$. Moreover, if $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ then (29), (30), (31) and (14) entail

$$\begin{aligned}\varphi(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p}\|\gamma_0(t)\|_{1,p}^p + \mu\frac{\varepsilon^q}{q}\|\gamma_0(t)\|_{1,q}^q - \int_{\partial\Omega} F(x, \varepsilon\gamma_0(t)(x))d\sigma \\ &\leq \frac{\varepsilon^p}{p}c_7 + \mu\frac{\varepsilon^q}{q}\left(\lambda_{2,q} + \frac{\eta}{\mu}\right) - \frac{\varepsilon^q}{q}c_5 \\ &= \frac{\varepsilon^q}{q}(\varepsilon^{p-q}c_7 + \mu\lambda_{2,q} + \eta - c_5) < 0 \ \forall t \in [-1, 1]. \end{aligned} \tag{33}$$

Now, write $a := \varphi_+(u_1)$. Because (16) on has $a < 0$. we may suppose $K(\varphi_+) = \{0, u_1\}$, otherwise the conclusion follows. Hence, no critical value of φ_+ lies in $(a, 0)$ while $K_a(\varphi_+) = \{u_1\}$.

Thanks to the second deformation lemma [[16], Theorem 5.1.33], there exists a continuous function $h : [0, 1] \times (\varphi_+^0 \backslash \{0\}) \to \varphi_+^0$ satisfying

$$h(0, u) = u, \ h(1, u) = u_0, \text{ and } \varphi_+(h(t, u)) \leq \varphi_+(u)$$

for all $(t, u) \in [0, 1] \times (\varphi_+^0 \backslash \{0\})$. Let $\gamma_+(t) := h(t, \varepsilon\phi_{1,q})^+$, $t \in [0, 1]$. then $\gamma_+(0) = \varepsilon\phi_{1,q}$, $\gamma_+(0) = u_1$, as well as

$$\varphi(\gamma_+(t)) = \varphi_+(\gamma_+(t)) \leq \varphi_+(h(t, \varepsilon\phi_{1,q})) < 0 \text{ in}[0, 1]. \tag{34}$$

In a similar way, but with φ_- in place of φ_+, we can construct a continuous function $\gamma_- : [0, 1] \to X$ such that $\gamma_-(0) = u_1$, $\gamma_-(1) = -\phi_{1,q}$, and

$$\varphi(\gamma_-(t)) < 0 \ \forall t \in [0, 1]. \tag{35}$$

Concatenating γ_-, $\varepsilon\gamma_0$, and γ_+ one obtains a path $\widehat{\gamma} \in \Gamma$ which, in view of (33)-(35), fulfils (27). This shows (26), whence $u_3 \neq 0$. □

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