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$\Delta_\lambda-$ Statistical convergence of order α on time scales

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Abstract. In this study, we introduce the notions $\Delta_\lambda-$statistical convergence of order α (for $\alpha \in (0,1]$) and $\lambda p-$ summability of order α (for $\alpha \in (0,1]$) on an arbitrary time scale. Moreover, some relations about these notions are obtained. We define $\Delta_\lambda-$statistical boundedness of order α (for $\alpha \in (0,1]$) on a time scale. Furthermore, we give connections between $S_{\mathbb{T}}^{(\lambda,\alpha)}(b)$, $S_{\mathbb{T}}^{(\beta,\theta)}(b)$ and $S_{\mathbb{T}}(b)$ for various sequences $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ which are determined in class Λ.

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1. Introduction

Zygmund [35] introduced the idea of statistical convergence in the first edition of his monograph published in Warsaw in 1935. Steinhaus [30] and Fast [13] and later Schoenberg [27] introduced the concept of statistical convergence, independently. Later this concept has been generalized in many directions. Fridy [14], Connor [8], Çolak [10], Maddox [20], Nuray [25], Rath and Tripathy [26], Šalát [31], Moricz [22], Miller [21] and others have studied different properties of space of statistically convergent sequences. Recently, the concept of statistical convergence has been applied to many fields of mathematics and statistics.

The statistical convergence depends on density of subsets of $\mathbb{N}$. The natural density of $K \subset \mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|,$$

where $|\{k \le n : k \in K\}|$ denotes the number of elements of K not exceeding n Niven and Zuckerman [24]. Any finite subset of $\mathbb{N}$ have zero natural density and $\delta(K^c) = 1 - \delta(K)$.

We give some properties related to the concept of natural density as follows:

i) $0 \le \frac{K(n)}{n} \le 1$, so $0 \le \delta(K) \le 1$.

ii) If K is the set of all non- square positive integers, i.e

$$K = \{2,3,5,6,7,8,10,11,12,13,14,15,17,18,...\},$$

then $K(n) = \frac{n-[|\sqrt{n}|]}{n}$, $\frac{K(n)}{n} = 1 - \frac{[|\sqrt{n}|]}{n}$. Obviously, $\delta(K) = 1$. This shows that $\delta(K) = 1$ does not imply that K contains all natural numbers.

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A sequence $x = (x_k)$ of complex numbers is said to be statistically convergent to some number ℓ if, for every positive number ε, $\delta(\{k \in \mathbb{N}\colon |x_k - \ell| \geq \varepsilon\})$ has natural density zero. The number ℓ is called the statistical limit of (x_k) and written as $S - \lim x_k = \ell$, Fridy [14]. We denote the space of all statistically convergent sequences by S.

Fridy and Orhan [15] introduced the idea of statistically bounded as follows:

A sequence $x = (x_k)$ of complex numbers is said to be statistically bounded if there exists some $M \geq 0$ such that

$$\delta(|k \in \mathbb{N} : |x_k| \geq M|) = 0$$

We denote the linear space of all statistically bounded sequences by $S(b)$.

Gadjiev and Orhan [16], took the initiative to introduce the order of statistical convergence and after Çolak [9] continued this work and termed statistical convergence of order α (for $0 < \alpha \leq 1$).

Leindler [19] defined the generalized de la Vallée-Poussin mean as follows:

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. Throughout this study Λ denotes the set of all such sequences. $x = (x_k)$ is said to be $(V, \lambda)-$summable to a number ℓ if $t_n(x) \to \ell$ as $n \to \infty$. $(V, \lambda)-$summability reduces to $(C, 1)$ summability when $\lambda_n = n$ (see Leindler [19]). Later, the notions of $\lambda-$density and $\lambda-$statistical convergence were introduced by Mursaleen [23] as follows: Let $K \subset \mathbb{N}$, $\lambda-$density of K is defined by

$$\delta_\lambda(K) = \lim_{n \to \infty} \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : k \in K\}|.$$

$\delta_\lambda(K)$ reduces to the natural density $\delta(K)$ in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ (see Mursaleen [23]). A sequence (x_k) is said to be $\lambda-$statistically convergent to a number ℓ if

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - \ell| \geq \varepsilon\}| = 0,$$

for each $\varepsilon > 0$ where $I_n = [n - \lambda_n + 1, n]$ (see Mursaleen [23]). Later, generalizing the concept of $\lambda-$statistical convergence, Çolak and Bektaş [11] introduced the concept of $\lambda-$statistical convergence of order α. Bhardwaj and Gupta [4] continued these works and defined statistically bounded of order α and $\lambda-$ statistically bounded of order α (for $0 < \alpha \leq 1$).

First we will give some information about the time scale.

Time scale calculus was introduced by Hilger in his doctoral dissertion in 1988 (see Hilger [18]). However, similar ideas have been used before and go back at least to the introduction of Riemann-Stieltijes integral which unifies sums and integrals. It gives an efficient tool to unify continuous and discrete problems in one theory. During the years many studies appeared in the time scales theory and its applications. One can see basic calculus of time scales in monographs of Bohner and Peterson [5]. For a time scale $\mathbb{T}$, forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

And, the forward stepsize function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$, where $\inf \phi = \sup \mathbb{T}$ and ϕ is empty set. Let A denotes the family of all left closed and right open intervals of $\mathbb{T}$ of the form $[a, b)_{\mathbb{T}}$. Measure theory on time scales was first constructed by Guseinov [17]. Then further studies were made by Cabada-Vivero [6]. Let $m : A \to [0, \infty)$ be a set function on A such that

$$m\left([a, b)_{\mathbb{T}}\right) = b - a.$$

Then, it is known that m is a countably additive measure on A. Now, the Caratheodory extension of the set function m associated with family A is said to be the Lebesque $\Delta-$measure on $\mathbb{T}$ and is denoted by μ_Δ. In this case, it is known that If $a \in \mathbb{T}-\{\max \mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta-$measurable and $\mu_\Delta(a) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \le b$, then $\mu_\Delta\left((a, b)_{\mathbb{T}}\right) = b - \sigma(a)$. If $a, b \in \mathbb{T}-\{\max \mathbb{T}\}$, $a \le b$; $\mu_\Delta\left((a, b]_{\mathbb{T}}\right) = \sigma(b) - \sigma(a)$ and $\mu_\Delta\left([a, b])_{\mathbb{T}}\right) = \sigma(b) - a$ (see Deniz [12]).

Let $\mathbb{T}$ be a time scale such that $\mathbb{T} \subset [0, \infty)$ and there exits a subset $\{t_k : k_0 \in \mathbb{N}\} \subset \mathbb{T}$ with $0 = t_0 < t_1 < t_2 ...$ and $\lim_{k\to\infty} t_k = \infty$.

In the paper by Batit [3], the following space of continuous functions are defined:

$$\ell_\infty(\mathbb{T}) = \left\{ f/f : \mathbb{T} \to \mathbb{R}, \sup_{t\in\mathbb{T}} |f(t)| < \infty \right\},$$
$$c(\mathbb{T}) = \left\{ f/f : \mathbb{T} \to \mathbb{R}, \lim_{t\to\infty} f(t) < \infty \right\},$$
$$c_0(\mathbb{T}) = \left\{ f/f : \mathbb{T} \to \mathbb{R}, \lim_{t\to\infty} f(t) = 0 \right\}.$$

Now we will give some information about the application of time scale to statistical convergence. Seyyidoglu and Tan [28] gave some new notations such as $\Delta-$convergence, $\Delta-$Cauchy by using $\Delta-$density and investigate their relations. Later, Turan and Duman [32] continued to work on this subject. Turan and Duman [33] applied the time scales to the concept of lacunary statistical convergence. Moreover, Cichon and Yantir [7] applied the time scales for some convergent sets.

First we will give an important definition for our study.

The notions of $\lambda-$density and $\lambda-$statistical convergence on time scales have been introduced by Yilmaz *et al.* [34].

Now let Ω be a $\Delta_\lambda-$ measurable subset of $\mathbb{T}$ and $0 < \alpha \le 1$. Then, $\Omega(t, \lambda)$ is defined by

$$\Omega(t, \lambda) = \left\{ s \in [t - \lambda_t + t_0, t]_{\mathbb{T}} : s \in \Omega \right\},$$

for $t \in \mathbb{T}$. In this case, $(\lambda, \alpha)-$density of Ω on $\mathbb{T}$ is denoted by $\delta_{\mathbb{T}}^{(\lambda,\alpha)}(\Omega)$ and defined as follows:

$$\delta_{\mathbb{T}}^{(\lambda,\alpha)}(\Omega) = \lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}(\Omega(t, \lambda))}{\mu_{\Delta_\lambda^\alpha}\left([t - \lambda_t + t_0, t]_{\mathbb{T}}\right)}, \qquad (1.1)$$

provided that the above limit exists. If $\lambda_t = t$ and $\alpha = 1$ in (1.1), we get classical density of Ω on $\mathbb{T}$ which is denoted by $\delta_{\mathbb{T}}(\Omega)$ and given as follows

$$\delta_{\mathbb{T}}(\Omega) = \lim_{t\to\infty} \frac{\mu_\Delta(\Omega(t))}{\mu_\Delta\left([t_0, t]_{\mathbb{T}}\right)},$$

provided that the right side limit exists (see Seyyidoglu and Tan [28], Turan and Duman [32]).

2. Statistical Convergence

In this section, we introduce the notions $\Delta_\lambda-$statistical convergence of order α (for $\alpha \in (0,1]$) and $\lambda p-$ summability of order α (for $\alpha \in (0,1]$) on an arbitrary time scale, using $\Delta_\lambda-$statistical convergence (Yilmaz *et al.* [34] and Altin [2]) on the time scale. In addition, some relations about these notions are obtained.

Definition 2.1. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$ measurable function and $\alpha \in (0,1]$ be any real number. Then f is $\lambda-$statistically convergent of order α on $\mathbb{T}$ to a number ℓ if

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)} = 0, \tag{2.1}$$

for each $\varepsilon > 0$. In this case, we write $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim_{t\to\infty}(f(t)) = \ell$. The set of all $\Delta_\lambda-$statistically convergent of order α functions on $\mathbb{T}$ is denoted by $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)}$. Similarly, by setting $\lambda_t = t$ and $\alpha = 1$ in (2.1), it turns to classical $\Delta-$statistical convergence on $\mathbb{T}$ as

$$\lim_{t\to\infty} \frac{\mu_{\Delta}\left(s \in [t_0,t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)} = 0,$$

provided that the above limit exists (see Seyyidoglu and Tan [28], Turan and Duman [32]). The set of all $\Delta-$ statistically convergent functions on $\mathbb{T}$ is denoted by $\boldsymbol{s}_{\mathbb{T}}$.

Proposition 2.1. *If $f,g : \mathbb{T} \to \mathbb{R}$ with $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim_{t\to\infty} f(t) = \ell_1$ and $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim_{t\to\infty} g(t) = \ell_2$, $\Delta_\lambda-$ measurable function and the following statements hold:*

i) $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim_{t\to\infty}(f(t)+g(t)) = \ell_1+\ell_2$,

ii) $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim_{t\to\infty} f(t) = \ell$ and $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim_{t\to\infty}(cf(t)) = c\ell \ (c \in \mathbb{R})$.

Theorem 2.2. *Let $0 < \alpha \leq 1$, $\boldsymbol{s}_{\mathbb{T}} \subseteq \boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)}$ if*

$$\lim_{t\to\infty} \inf \frac{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)} > 0 \tag{2.2}$$

Proof. For given $\varepsilon > 0$ we have

$$\mu_{\Delta}\left(s \in [t_0,t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right) \supset \mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right).$$

Therefore,

$$\frac{\mu_{\Delta}\left(s \in [t_0,t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)} \geq \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)}$$

$$= \frac{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)} \frac{1}{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)} \mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right).$$

Hence by using (2.2) and taking the limit as $t \to \infty$, we get $f(s) \to \ell\,(\boldsymbol{s}_{\mathbb{T}})$ implies $f(s) \to \ell\left(\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)}\right)$. □

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$ measurable function, $\lambda \in \Lambda$, $0 < \alpha \leq 1$ and $0 < p < \infty$. We say that f is strongly $\lambda p-$ Cesàro summable of order α on $\mathbb{T}$ if there exists some $\ell \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{\mu^{\alpha}_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|^p \, \Delta s = 0.$$

In this case we write $[W,\lambda,p,\alpha]_{\mathbb{T}}-\lim f(s)=\ell.$ The set of all strongly $\lambda p-$ summable functions of order α on $\mathbb{T}$ will be denoted by $[W,\lambda,\alpha]_{\mathbb{T}}$.

Lemma 2.3. [See Yilmaz et al. [34]] *Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta_\lambda-$ measurable function and let*

$$\Omega(t,\lambda)=\{s\in[t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)-\ell|\geq\varepsilon\}$$

for $\varepsilon>0$. In this case, we have

$$\mu_{\Delta_\lambda}(\Omega(t,\lambda)) \leq \frac{1}{\varepsilon}\int\limits_{\Omega(t,\lambda)} |f(s)-\ell|\,\Delta s \leq \frac{1}{\varepsilon}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|\,\Delta s.$$

Theorem 2.4. *Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta_\lambda-$ measurable function, $\lambda\in\Lambda$, $\ell\in\mathbb{R}$, $\alpha\in(0,1]$ and $0<p<\infty$. If f is strongly $\lambda p-$ summable of order α to ℓ, then $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)}-\lim_{t\to\infty}(f(t))=\ell$.*

Proof. Let f is strongly $\lambda p-$ summable of order α to ℓ. For given $\varepsilon>0$, let $\Omega(t,\lambda)=\{s\in[t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)-\ell|\geq\varepsilon\}$. Then, it follows from Lemma 2.3 that

$$\varepsilon^p \mu_{\Delta_\lambda}(\Omega(t,\lambda)) \leq \int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|^p\,\Delta s.$$

Dividing both sides of the last equality by $\mu^{\alpha}_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})$ and taking limit as $t\to\infty$, we obtain that

$$\lim_{t\to\infty}\frac{\mu_{\Delta_\lambda}(\Omega(t,\lambda))}{\mu^{\alpha}_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})} \leq \frac{1}{\varepsilon^p}\lim_{t\to\infty}\frac{1}{\mu^{\alpha}_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|^p\,\Delta s = 0,$$

which yields that $s_{\mathbb{T}}^{(\lambda,\alpha)}-\lim_{t\to\infty}(f(t))=\ell$. □

Theorem 2.5. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_{\lambda(t)}}\leq\mu_{\Delta_{\beta(t)}}$ for all $t\in\mathbb{T}$ and $0<\alpha\leq\theta\leq1$.*
i) If

$$\lim_{t\to\infty}\inf\frac{\mu^{\alpha}_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})}{\mu^{\theta}_{\Delta_\beta}([t-\beta_t+t_0,t]_{\mathbb{T}})}>0 \tag{2.5}$$

then $\boldsymbol{s}_{\mathbb{T}}^{(\beta,\theta)}\subseteq\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)}$.
ii) If

$$\lim_{t\to\infty}\frac{\mu^{\alpha}_{\Delta_\lambda}([t-\lambda_t+t_0,t]_{\mathbb{T}})}{\mu^{\theta}_{\Delta_\beta}([t-\beta_t+t_0,t]_{\mathbb{T}})}=1 \text{ and } \lim_{t\to\infty}\frac{\mu_{\Delta_\beta}([t-\lambda_t+t_0,t]_{\mathbb{T}})}{\mu^{\theta}_{\Delta_\beta}([t-\beta_t+t_0,t]_{\mathbb{T}})}=1 \tag{2.6}$$

then $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)}\subseteq\boldsymbol{s}_{\mathbb{T}}^{(\beta,\theta)}$.

Proof. i) Suppose that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and let (2.5) be satisfied. Then $I_t \subset J_t$ and so that for $\varepsilon > 0$, we have

$$\mu_{\Delta_\beta}\left(s \in [t-\beta_t+t_0, t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right) \geq \mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0, t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)$$

Therefore, we have

$$\frac{\mu_{\Delta_\beta}\left(s \in [t-j_t+t_0, t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} \geq \frac{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0, t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} \frac{1}{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0, t]_{\mathbb{T}}\right)}$$
$$\times\ \mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0, t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right).$$

for all $t \in \mathbb{T}$,where $J_t = [t-\beta_t+t_0, t]$. Hence by using (2.5) and taking the limit as $t \to \infty$, we get $\boldsymbol{s}_{\mathbb{T}}^{(\beta,\theta)} \subseteq \boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)}$.

ii) Let f be a $\Delta_\lambda-$ measurable function and , $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim f(s) = \ell$ and (2.6) be satisfied. Since $I_t \subset J_t$ and all $t \in \mathbb{T}$, we can write

$$\begin{aligned}
\frac{\mu_{\Delta_\beta}\left(s \in [t-\beta_t+t_0, t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} &= \frac{\mu_{\Delta_\beta}\left(t-\beta_t+t_0 \leq s \leq t-\lambda_t : |f(s)-\ell| \geq \varepsilon\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} \\
&+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0, t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} \\
&\leq \frac{\mu^\theta_{\Delta_\beta}\left(s \in [t-\beta_t+t_0, t]_{\mathbb{T}}\right) - \mu^\alpha_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0, t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} \\
&+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0, t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} \\
&\leq \left(1 - \frac{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0, t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)}\right) \\
&+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0, t]_{\mathbb{T}} : |f(s)-\ell| \geq \varepsilon\right)}{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0, t]_{\mathbb{T}}\right)}
\end{aligned}$$

for all $t \in \mathbb{T}$. Since

$$\lim_{t\to\infty} \frac{\mu^\alpha_{\Delta_\lambda}\left([t-\lambda_t+t_0, t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} = 1 \text{ and } \lim_{t\to\infty} \frac{\mu_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} = 1$$

by ii), the term in above inequality tends to 0 as $t \to \infty$. Furthermore, since $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} - \lim f(s) = \ell$, the second term of the right hand side of the above inequailty goes to 0 as $t \to \infty$.

$$\frac{\mu_{\Delta_\beta}\left(t-\beta_t+t_0 \leq s \leq t : |f(s)-\ell| \geq \varepsilon\right)}{\mu^\theta_{\Delta_\beta}\left([t-\beta_t+t_0, t]_{\mathbb{T}}\right)} \to 0$$

as $t \to \infty$. Therefore $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} \subseteq \boldsymbol{s}_{\mathbb{T}}^{(\beta,\theta)}$. □

From Theorem 2.5, we have the following result.

Corollary 2.6. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$. If (2.6) holds, then $\boldsymbol{s}_{\mathbb{T}}^{(\lambda,\alpha)} = \boldsymbol{s}_{\mathbb{T}}^{(\beta,\theta)}$.*

Theorem 2.7. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$. Then we get:*
i) *If (2.5) holds then $[W, \beta, \alpha]_{\mathbb{T}} \subseteq [W, \lambda, \theta]_{\mathbb{T}}$.*
ii) *If (2.6) holds then $\ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}} \subseteq [W, \beta, \theta]_{\mathbb{T}}$.*

Proof. i) Suppose that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$. Then $I_t \subset J_t$ for all $t \in \mathbb{T}$ so that we may write

$$\frac{1}{\mu_{\Delta_\beta^\theta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|\,\Delta s \geq \frac{1}{\mu_{\Delta_\lambda^\alpha}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|\,\Delta s$$

for all $t \in \mathbb{T}$. This gives that

$$\frac{1}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|\,\Delta s \geq \frac{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \frac{1}{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|\,\Delta s.$$

Then taking limit $t \to \infty$ in the last inequality and using (2.5) we obtain $[W, \beta, \alpha]_{\mathbb{T}} \subseteq [W, \lambda, \theta]_{\mathbb{T}}$.

ii) Let $f \in \ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}}$ and suppose that (2.6) holds. Since $f \in \ell_\infty(\mathbb{T})$ then there exits some a positive number M such that $|f(s)| \leq M$ for all $s \in \mathbb{T}$ and also now, since $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ and so that $\frac{1}{\mu_{\Delta_{\beta(t)}}} \leq \frac{1}{\mu_{\Delta_{\lambda(t)}}}$ and $I_t \subset J_t$ for all $t \in \mathbb{T}$, we may write

$$\begin{aligned}
&\frac{1}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}} |f(s)-\ell|\,\Delta s \\
&\leq \frac{1}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{J_t/I_t} |f(s)-\ell|\,\Delta s + \frac{1}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{I_t} |f(s)-\ell|\,\Delta s \\
&\leq \left(\frac{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right) - \mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta^\theta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\right) M \\
&\quad + \frac{1}{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{I_t} |f(s)-\ell|\,\Delta s
\end{aligned}$$

for all $t \in \mathbb{T}$. Since $\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}^\alpha\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}^\theta\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = 1$ by (2.6) the first term and since $f \in [W, \lambda, \alpha]_{\mathbb{T}}$ the second term of right hand side of above inequality tend to 0 as $t \to \infty$. This implies that $\ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}} \subseteq [W, \beta, \theta]_{\mathbb{T}}$ and so that $\ell_\infty(\mathbb{T}) \cap [W, \lambda, \alpha]_{\mathbb{T}} \subseteq \ell_\infty(\mathbb{T}) \cap [W, \beta, \theta]_{\mathbb{T}}$. □

From Theorem 2.7 we have the following result.

Corollary 2.8. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$. If (2.6) holds then $\ell_\infty(\mathbb{T}) \cap [W,\lambda,\alpha]_{\mathbb{T}} = \ell_\infty(\mathbb{T}) \cap [W,\beta,\theta]_{\mathbb{T}}$.*

3. Statistical boundedness

In this section, we use the concepts of statistical boundedness on time scales defined by Seyyidoğlu and Tan [29] and $\Delta_\lambda-$statistical boundedness defined by Altin *et al.* [1], we define $\Delta_\lambda-$statistical boundedness of order α (for $0 < \alpha \leq 1$) on time scale. Furthermore, the connections between $S_{\mathbb{T}}^{(\lambda,\alpha)}(b)$, $\boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b)$ and $\boldsymbol{S}_{\mathbb{T}}(b)$ for various sequences are determined class Λ, according to in $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$.

Definition 3.1. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$measurable function and $\alpha \in (0,1]$. f is $\Delta_\lambda-$statistically bounded of order α on $\mathbb{T}$ if there exists some $M > 0$ such that

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\right)}{\mu_{\Delta_\lambda}^{\alpha}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)} = 0, \tag{3.1}$$

for $t \in \mathbb{T}$. The set of all $\Delta_\lambda-$statistically bounded functions of order α on $\mathbb{T}$ is denoted by $S_{\mathbb{T}}^{(\lambda,\alpha)}(b)$. If $\alpha = 1$ and $\lambda_t = t$, then we obtain $S_{\mathbb{T}}(b)$ which is the set of all $\Delta-$statistically bounded functions [29].

Theorem 3.1. *Let $\mu_{\Delta_{\lambda(t)}}, \mu_{\Delta_{\beta(t)}} \in \Lambda$ such that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq \theta \leq 1$.*
i) If

$$\liminf_{t\to\infty} \frac{\mu_{\Delta_{\lambda(t)}}^{\alpha}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_{\beta(t)}}^{\theta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} > 0, \tag{3.2}$$

then $\boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b) \subset \boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b)$.
ii) If

$$\lim_{t\to\infty} \frac{\mu_{\Delta_{\lambda(t)}}^{\alpha}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_{\beta(t)}}^{\theta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = 1, \tag{3.3}$$

then $\boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b) \subseteq \boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b)$.
iii) If

$$\lim_{t\to\infty} \frac{\mu_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_{\beta(t)}}^{\theta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = 1,$$

then $\boldsymbol{S}_{\mathbb{T}}^{(\lambda,\alpha)}(b) = \boldsymbol{S}_{\mathbb{T}}^{(\beta,\theta)}(b)$.

Proof. i) Suppose that $\mu_{\Delta_{\lambda(t)}} \leq \mu_{\Delta_{\beta(t)}}$ for all $t \in \mathbb{T}$ and (3.3) is satisfied. Then, $I_t \subset J_t$ and so for $\varepsilon > 0$, we have

$$\begin{aligned}\mu_{\Delta_{\beta(t)}}&\left(\{s \in [t-\beta_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\right) \\ &\geq \mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right).\end{aligned}$$

Therefore,

$$\frac{\mu_{\Delta_{\beta(t)}}\left(\{s \in [t-j_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$
$$\geq \frac{\mu^{\alpha}_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)\mu^{\alpha}_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}$$
$$\times \mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right),$$

for $\forall t \in \mathbb{T}$ where $J_t = [t-\beta_t+t_0,t]$. Hence by using (3.3) and taking the limit as $t \to \infty$, we get $\boldsymbol{S}^{(\beta,\theta)}_{\mathbb{T}}(b) \subseteq \boldsymbol{S}^{(\lambda,\alpha)}_{\mathbb{T}}(b)$.

ii) Let f be a $\Delta_\lambda-$ measurable and $\boldsymbol{s}^{(\lambda,\alpha)}_{\mathbb{T}} - \lim f(s) = \ell$. Since $I_t \subset J_t$, we can write

$$\frac{\mu_{\Delta_{\beta(t)}}\left(\{s \in [t-\beta_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = \frac{\mu_{\Delta_{\beta(t)}}\left(\{t-\beta_t+t_0 \leq s \leq t-\lambda_t : |f(s)| \geq M\}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$
$$+\frac{\mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$
$$\leq \frac{\mu^{\theta}_{\Delta_{\beta(t)}}\left(s \in [t-\beta_t+t_0,t]_{\mathbb{T}}\right) - \mu^{\alpha}_{\Delta_{\lambda(t)}}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$
$$+\frac{\mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}$$
$$\leq \left(1-\frac{\mu^{\alpha}_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\right)$$
$$+\frac{\mu_{\Delta_{\lambda(t)}}\left(\{s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : |f(s)| \geq M\}\right)}{\mu^{\alpha}_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}$$

for all $t \in \mathbb{T}$. Since $\lim_{t\to\infty} \frac{\mu^{\alpha}_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu^{\theta}_{\Delta_{\beta(t)}}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = 1$ by ii), the term in above inequality tends to 0 as $t \to \infty$. Furthermore, since $\boldsymbol{s}^{(\lambda,\alpha)}_{\mathbb{T}} - \lim f(s) = \ell$, the second term of the right hand side of the above inequality goes to 0 as $t \to \infty$. Therefore $\boldsymbol{S}^{(\lambda,\alpha)}_{\mathbb{T}}(b) \subseteq \boldsymbol{S}^{(\beta,\theta)}_{\mathbb{T}}(b)$.

iii) The proof is clear. □

Theorem 3.2. *Let $\mu_{\Delta_{\lambda(t)}} \in \Lambda$ for all $t \in \mathbb{T}$ and $0 < \alpha \leq 1$.*
i) If

$$\liminf_{t\to\infty} \frac{\mu^{\alpha}_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)} > 0,$$

then $S_{\mathbb{T}}(b) \subset \boldsymbol{S}^{(\lambda,\alpha)}_{\mathbb{T}}(b)$.
ii) If

$$\lim_{n\to\infty} \frac{\mu^{\alpha}_{\Delta_{\lambda(t)}}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)} = 1,$$

then $\boldsymbol{S}^{(\lambda,\alpha)}_{\mathbb{T}}(b) \subset \boldsymbol{S}_{\mathbb{T}}(b)$ and hence $\boldsymbol{S}^{(\lambda,\alpha)}_{\mathbb{T}}(b) = \boldsymbol{S}_{\mathbb{T}}(b)$.

Proof. The proof is obvious. □

References

[1] Y. Altin, B.N. Er, and E. Yilmaz, Δ_λ-statistical boundedness on time scales, *Comm. Statist. Theory Methods* **50** (3) (2021), no. 3, 738–746. DOI:10.1080/03610926.2019.1640880
[2] Y. Altin, Some results on $\lambda-$statistical convergence on time scales, *Maejo Int. J. Sci. Technol.* **11** (2017), no. 1, 90–96.
[3] O. Batit, Function spaces and their dual spaces on time scales, *Int. J. Difference Equ.* **2** (2007), no. 1, 13–23.
[4] V.K. Bhardwaj and S. Gupta, On some generalizations of statistical boundedness, *J. Inequal. Appl.* **2014** (2014), Article 12. DOI:10.1186/1029-242X-2014-12
[5] M. Bohner and M. Peterson, *Dynamic equations on time scales, an introduction with applications*, Birkhauser, Boston, 2001.
[6] A. Cabada and D.R. Vivero, Expression of the Lebesque $\Delta-$ integral on time scales as a usual Lebesque integral; application to the calculus of $\Delta-$antiderivates, *Math. Comput. Modelling* **43** (2006), no. 1-2,194–207. DOI:10.1016/j.mcm.2005.09.028
[7] M. Cichon and A. Yantir, On the convergence of sets and the approximation property for dynamic equations on time scales, *Mathematica Aeterna* **5** (2015), no. 5, 883-904.
[8] J.S. Connor, The statistical and strong $p-$Cesàro convergence of sequences, *Analysis* **8** (1988), 47–63. DOI:10.1524/anly.1988.8.12.47
[9] R. Çolak, Statistical convergence order α, *Modern Methods in Analysis and Its Applications*, Anamaya Pub., New Delhi (2010), 121–129.
[10] R. Çolak, On λ-statistical convergence, *Proceedings of Conference on Summability and Applications*, 2011, İstanbul, Turkey, 4–5.
[11] R. Çolak and Ç.A. Bektaş, $\lambda-$statistical convergence of α, *Acta Math. Sci. Ser. B* (Engl. Ed.) **31** (2011), no. 3, 953–959. DOI:10.1016/S0252-9602(11)60288-9
[12] A. Deniz, Measure theory on time scales, Master of Science, Izmir Institute of Technology, Izmir, Turkey, 2007.
[13] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), 241–244.
[14] J. Fridy, On statistical convergence, *Analysis* **5** (1985), 301–313. DOI:10.1524/anly.1985.5.4.301
[15] J.A. Fridy and C. Orhan, Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.* **125** (1997), no. 12, 3625–3631. DOI:10.1090/S0002-9939-97-04000-8
[16] A.D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* **32** (2002), no. 1, 129–138. DOI:10.1216/rmjm/1030539612
[17] G.Sh. Guseinov, Integration on time scales, *J. Math. Anal. Appl.* **285** (2003), no. 1, 107–127. DOI:10.1016/S0022-247X(03)00361-5
[18] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), no. 1-2, 18–56. DOI:10.1007/BF03323153
[19] L. Leindler, Über die verallgemeinerte de la Vallée-Poussinsche Summierbarkeit allgemeiner Orthogonalreihen, (German) *Acta Math. Acad. Sci. Hungar.* **16** (1965), 375–387. DOI:10.1007/BF01904844
[20] I.J. Maddox, Statistical convergence in a locally convex space, *Math. Proc. Cambridge Philos. Soc.* **104** (1988), no. 1, 141–145. DOI:10.1017/S0305004100065312
[21] H.I. Miller, A measure theoretical subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1811–1819. DOI:10.1090/S0002-9947-1995-1260176-6
[22] F. Moricz, Statistical limits of measurable functions, *Analysis* **24** (2004), 207–219. DOI:10.1524/anly.2004.24.1.1
[23] M. Mursaleen, $\lambda-$statistical convergence, *Math. Slovaca* **50** (2000), no. 1, 111–115.
[24] I. Niven and H.S. Zuckerman, *An introduction to the theory of numbers*, John Wiley and Sons, Inc. New York, 1960.
[25] F. Nuray, λ-strongly summable and λ-statistically convergent functions, *Iran. J. Sci. Technol. Trans. A Sci.* **34** (2010), no. 4, 335–338.
[26] D. Rath and B.C. Tripathy, On statistically convergent and statistically Cauchy sequences, *Indian J. Pure Appl. Math.* **25** (1994), no. 4, 381–386.

[27] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* **66** (1959), 361–375. DOI:10.2307/2308747
[28] M.S. Seyyidoglu and N.Ö. Tan, A note on statistical convergence on time scale, *J. Inequal. Appl.* **2012** (2012), Article 219. DOI:10.1186/1029-242X-2012-219
[29] M.S. Seyyidoglu and N.Ö. Tan, On a generalization of statistical cluster and limit points. Advances in Difference Equations 2015 (2015), Article 55. DOI:10.1186/s13662-015-0395-9
[30] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloquium Math.* **2** (1951), 73–74.
[31] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca* **30** (1980), no. 2, 139–150.
[32] C. Turan and O. Duman, Statistical convergence on time scales and its characterizations, In: (Anastassiou, G., Duman, O. eds.) *Advances in Applied Mathematics and Approximation Theory*, Springer Proceedings in Mathematics & Statistics **41** (2013), 57–71, Springer, New York, NY. DOI:10.1007/978-1-4614-6393-1_3
[33] C. Turan and O. Duman, Fundamental properties of statistical convergence and Lacunary statistical convergence on time scales, *Filomat* **31** (2017), no. 14, 4455–4467. DOI:10.2298/FIL1714455T
[34] E. Yilmaz, Y. Altin, and H. Koyunbakan, λ-Statistically Convergence on Time Scales, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **23** (2016), no. 1, 23 69–78.
[35] A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge UK. 1979.

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