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Simulation of the transmission line equation by a high-order FDTD method

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Abstract. This note deals with transmission lines equation. The frequency dependent parameters which are often considered constants to ease the numerical implementation, are actually taken for their real physical state. We aim to obtain a physical solution of the equation by a high order scheme. The proposed scheme is an enhanced version of FDTD methods known to be powerful tools for solving equations of this alike. In particular, the scheme is built to deal with oscillations of the load voltage. The oscillations that are a result of the taking-over of the hyperbolic characteristic of the equation are corrected by a high order relaxed FDTD method and a regularizing effect of Runge Kutta. We validate our theoretical results by few numerical simulations.

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Introduction

Below, we present the modelling system of the electrical transmission lines where the mathematical theory grew out of the work of James Clerk Maxwell, Lord Kelvin and Oliver Heaviside. It was in 1885 when Heaviside published the first papers that described his analysis of information propagation in cables which is the modern form of the transmission lines system (TLS) also known as telegraph equation [11].

The equation describes the variation of the voltage V and the current I along an electric cable as a function of time and position. The resistance R and the inductance L represent series impedance along the cable, while the capacitance C and the leakage conductance G form the shunt admittance across the cable. The time-dependent transmission lines governing the line voltages and currents are expressed as

$$\begin{cases} \dfrac{\partial V(z,t)}{\partial z} + L\dfrac{\partial I(z,t)}{\partial t} + RI(z,t) = 0, \\ \dfrac{\partial I(z,t)}{\partial z} + C\dfrac{\partial V(z,t)}{\partial t} + GV(z,t) = 0, \\ V(0,t) = V_s(t),\ V(\mathcal{L},t) = V_l(t), \\ I(0,t) = I_s(t),\ I(\mathcal{L},t) = I_l(t), \end{cases} \tag{1}$$

where the axis Oz corresponds to the direction of the line and $\mathcal{L}$ is the load.

Generally, the solutions of TLS are analytically unattainable, so, attempts to obtain concrete and reliable solutions must resort to numerical methods. Among which,

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asymptotic techniques are known to be very powerful. Yet, the generated solution often fail to acquire properties of the physical phenomena that wraps the transmission lines. Hence, we shall focus on Full Wave techniques known as common and strong tools to simulate electromagnetic problems of this alike. We mention FDTD [14], FEM [8, 20], MoM [7] and TDFD [16].

The parameters L, R, C, and G are often considered frequency-independent constants which leads to a relatively simple implementation (see for instance [28, 24, 31]). But, from a physics point of view, these parameters actually depend to varying degrees on the frequency of excitation of line $w = 2\pi f$, where f is the frequency. This feature makes the implementation of any chosen method relativaly difficult as we are brought to deal with a frequency-domain system instead of the standard time-domain. A universal reference about the matter is the book of Clayton R. Paul [18]; It details the how-about passage from frequency-domain to time-domain and the challenges associated with this transform. We aim to construct efficient schemes to solve the system while keeping in consideration the frequency-dependent parameters.

Through years, many powerful Full Wave techniques are used to simulate the frequency-domain phenomena. In particular, the Finite Difference Time Domain (FDTD) method is most used. It is arguably the simplest, both conceptually and in terms of implementation. However, as with all numerical schemes, it has its share of deficiencies. Once simulated, we immediately notice occurring oscillations at the leading edge of each voltage transition (see Figure 3). A correction must be brought to the method. Motivated by the arguably easy implementation, many researchers invested in constructing enhanced versions of FDTD instead of working on different schemes (see for example [32, 23, 29] and many related works). Likewise, we shall present an effective approach to deal with these oscillations.

The approach is recentely proposed in [2] which is a fourth-order compact scheme by coupling a relaxed FDTD with Runge-kutta of second order ($\beta/RK2$). The first use of the approach included the homogeneous linear transmission line equation where losses were neglected, i.e. $R = G = 0$ and L and C were considered constants. The method succeeded in demolishing oscillations and keeping the physical aspects of the obtained numerical results.

In this note, we extrapolate the method to the nonlinear TLS equation by taking in consideration the frequency dependent parameters mentioned above. We will show that the technique totally demolishes the oscillations that occurs at each leading edge.

To do so, we first incorporate the losses that are related to the electromagnetic phenomena. This relies in studying the Frequency-Domain equation then extract the equivalent Time-Domain form. We shall present a full review on the matter in Section 1. We believe that this is crucial to understanding the following numerical implementation. And while this paper is more interested in the mathematical aspect of the problem, all that is related to the physical phenomena will be extracted from physical refrences. In particular, we adapt the physical simulations in ([18]) and for more visibility to the readers, we shall keep the same notations to ease the comparison between standard FDTD simulated by the author and our proposed enhanced technique.

The paper is organized as follows. In section 1, we give a full review on the how-about passage from Frequency-Domain to Time-Domain equation. Section 2 casts the simulation of the Time-Domain lossy transmission line equation by the

$\beta/RK2$ scheme. In the last section 3, we present numerical simulations to compare the standard FDTD with the proposed method and we verify the correction brought upon the FDTD.

1. Frequency-Domain to Time-Domain transmission line equation

1.1. Incorporation of losses. The real phenomena that wraps the transmission lines is that losses occurs in the conductors and the surrounding medium of the lines. These losses are frequency-dependent. That means that the parameters L, C, R and G are not in fact constants, but depend to varying degrees on the frequency of excitation of line, $w = 2\pi f$. For these imperfect conductors, the currents will be uniformly distributed over the conductor cross sections at low frequencies, but at higher frequencies will, because of skin effect, migrate toward the surfaces of the conductors lying in a thickness on the order of a specific skin depth. Also, the medium surrounding the conductors has an effective conductivity that is primarily due to bound charge in the dielectric and is frequency dependent. Hence, the conductance will be frequency dependent. We refere the readers to [10] for more details.

By Fourier's transform, the frequency-domain form of equation (1) is then written

$$\frac{\partial \hat{V}}{\partial z}(z,\omega) + R(\omega)\,\hat{I}(z,\omega) + i\omega\,L(\omega)\,\hat{I}(z,\omega) = 0,$$
$$\frac{\partial \hat{I}}{\partial z}(z,\omega) + G(\omega)\,\hat{I}(z,\omega) + i\omega\,C\,\hat{V}(z,\omega) = 0.$$

The inductance L is defined as $L(\omega) = L_e + L_i(\omega)$, where L_e and L_i are the external and internal inductance of the conductor. Remark that L_i depend on the frequency while L_e does not. It's worth mentioning that L_i is negligible compared to L_e. We denote for simplicity $L_e = L$, and so the equation reads

$$\begin{aligned} \frac{\partial \hat{V}}{\partial z}(z,\omega) &= -\,\hat{z}_i(\omega)\hat{I}(z,\omega) - i\omega L\,\hat{I}(z,\omega) \\ \frac{\partial \hat{I}}{\partial z}(z,\omega) &= -\,\hat{y}(\omega)\hat{V}(z,\omega) \end{aligned} \tag{2}$$

where the internal impedance and admittance of the conductors respectively are

$$\hat{z}_i(\omega) = R(\omega) + i\omega L_i(\omega)$$
$$\hat{y}(\omega) = G(\omega) + i\omega C$$

Whereas the main work of this article is more concentrated on the mathematical aspect of the problem, all that is related to the physical phenomena will be extracted from physics references as stated in the introduction. For the computation of the internal impedance and admittance, we refere the readers to ([18], Chapter 2 and 4). We have

$$\hat{z}_i(\omega) = \begin{cases} r_{dc}(1 + i\dfrac{f}{f_0}) & if \;\; f < f_0 \\[2ex] r_{dc}\sqrt{\dfrac{f}{f_0}}(1+i) & if \;\; f > f_0 \end{cases}, \tag{3}$$

$$\hat{y}(\omega) = \sum_{i=1}^{N} (\frac{K_i}{1 + i\omega\tau_i}) i\omega M + i\omega c_{hf}, \tag{4}$$

where r_{dc} is dc resistance, f_0 the break frequency, c_{hf} the high-frequency capacitance, τ_i, K_i and M are coefficient related to the meduim losses. In order to simplify the computational process, we propose The Laplace transform of these equations by replacing s with $i\omega$.

$$\begin{aligned} \frac{\partial \hat{V}}{\partial z}(z, s) &= -\hat{z}_i(s)\hat{I}(z, s) - sL\hat{I}(z, s) \\ \frac{\partial \hat{I}}{\partial z}(z, s) &= -\hat{y}(s)\hat{V}(z, s) \end{aligned} \tag{5}$$

Applying Fourier's inverse transformation, (5) gives convolutions

$$\begin{aligned} \frac{\partial V}{\partial z}(z, t) &= -z_i(t) * I(z, t) - L\frac{\partial I}{\partial t}(z, t) \\ \frac{\partial I}{\partial z}(z, t) &= -y(t) * V(z, t) \end{aligned} \tag{6}$$

1.2. Time-Domain form of the Frequency-dependent parameters. The inclusion of the frequency-domain parameters is easily achieved. However, it presents computational problems as we are brought to transform the frequency dependent parameters into time-domain's and the computation of the required convolution can be quite challenging.

Starting with the admittance, we introduce the first re-form of frequency-dependent to time-domain's.

- **Admittance**: We substitute $i\omega$ by s and we convert to Laplace transform

$$\hat{y}(s) = \sum_{i=1}^{N} (\frac{K_i}{1 + s\tau_i}) Ms + c_{hf}s.$$

A simple inverse transform of $\frac{1}{c+s}$ reads $exp(-c\,t)$ [6]. Hence, the expression of the admittance in time-domain is easily given by

$$y(t) = \sum_{i=1}^{N} \left(M\frac{K_i}{\tau_i} exp(-t/\tau_i) \right) \frac{\partial}{\partial t} + c_{hf}\frac{\partial}{\partial t}. \tag{7}$$

- **Internal impedance**: It is defined as

$$\hat{z}_i(\omega) = \begin{cases} r_{dc}(1 + i\frac{f}{f_0}) & if \;\; f < f_0 \\ r_{dc}\sqrt{\frac{f}{f_0}}(1 + i) & if \;\; f > f_0 \end{cases}, \tag{8}$$

where $\omega = 2\pi f$.

The challenge in this case is that the high-frequency internal impedance varies in function of the square root of frequency. The known inverse Laplace transform of the square root variable reads

$$\frac{1}{\sqrt{s}} \Longleftrightarrow \frac{1}{\sqrt{\pi t}}. \tag{9}$$

We have the following complex relation

$$2i\omega = (1+i)^2\omega.$$

Hence

$$\begin{aligned}(1+i)\sqrt{\pi f} &= (1+i)\sqrt{\frac{\omega}{2}} \\ &= \sqrt{i\omega}.\end{aligned} \tag{10}$$

The two last equations give insight as how to go about obtaining the time-domain functions. First, we need to write the frequency-domain parameter in function of $(1+i)\sqrt{f}$. Then, we transform the result to Laplace domain using (10). We replace $i\omega$ by s, and finally use (9) after multiplying the numerator and denominator by $\sqrt{s}$. Another way of stating this is to write the internal impedance in its Laplace-Domain form and make use of equation (9) as follows:

$$c_1 + c_2\sqrt{s} = c_1 + \frac{c_2}{\sqrt{s}}s \Longleftrightarrow c_1 + \frac{c_2}{\sqrt{\pi t}}\frac{\partial}{\partial t}$$

The only required task here is to determine the Laplace-Domain form of the impedance. This can be achieved for conductors by adding the dc and high-frequency representations from (8) as

$$\hat{z}_i(\omega) = r_{dc} + r_{dc}(1+i)\sqrt{\frac{f}{f_0}}.$$

For more details on this phase, we refere the readers to [[18], Chapter 4]. And so, converting to Laplace-domain reads

$$\begin{aligned}\hat{z}_i(s) &= r_{dc} + r_{dc}\sqrt{\frac{s}{\pi f_0}} \\ &= r_{dc} + r_{dc}\frac{1}{\sqrt{\pi f_0}}\frac{1}{\sqrt{s}}s\end{aligned}$$

Here $c_1 = r_{dc}$ and $c_2 = r_{dc}\frac{1}{\sqrt{\pi f_0}}$, hence the inverse transform is written

$$z_i(t) = r_{dc} + r_{dc}\frac{1}{\pi\sqrt{f_0}}\frac{1}{\sqrt{t}}\frac{\partial}{\partial t}. \tag{11}$$

1.3. Time-Domain convolutions. The remaining challenge for converting from frequency-domain to time-domain is the how-about inclusion of the convolutions in (6). A valid approach for numerical implementation is needed.

1.3.1. *Time-Domain admittance convolution with the voltage.* According to what precedes and by substituting (7) into the second line of (6), we define the concerned convolution as

$$\begin{aligned}y(t) * V(z,t) &= \sum_{i=1}^{N}\left(M\frac{K_i}{\tau_i}exp(-t/\tau_i)\right) * \frac{\partial V(z,t)}{\partial t} + c_{hf}\frac{\partial V(z,t)}{\partial t} \\ &= \sum_{i=1}^{N}\left(M\frac{K_i}{\tau_i}\int_0^t exp(-\tau/\tau_i)\frac{\partial V(z,t-\tau)}{\partial(t-\tau)}\,\mathrm{d}\tau\right) + c_{hf}\frac{\partial V(z,t)}{\partial t},\end{aligned}$$

where we denote $k(t) = \frac{\partial V(z,t)}{\partial t}$. We divide the time axis into $(n+1)\Delta t$ on which k considered a constant. And so, the discretized convolution can be approximated in the following matter

$$\begin{aligned}\int_0^t exp(-\tau/\tau_i)\, k(t-\tau)\, \mathrm{d}\tau &= \int_0^{(n+1)\Delta t} exp(-\tau/\tau_i)\, k\left((n+1)\Delta t - \tau\right)\, \mathrm{d}\tau \\ &\simeq \sum_{m=0}^{n} k\left((n+1-m)\Delta t\right) \int_{m\Delta t}^{(m+1)\Delta t} e^{-\tau/\tau_i}\, \mathrm{d}\tau \\ &\simeq -\sum_{m=0}^{n} \tau_i\, k^{n+1-m} \left[e^{-(m+1)\Delta t/\tau_i} - e^{-m\Delta t/\tau_i}\right]\end{aligned}$$

where we denote $k(p\Delta t)$ as k^p. The time-domain convolution is then written

$$\begin{aligned}y(t) * V(t) = -\sum_{i=1}^{N} \left(MK_i \sum_{m=0}^{n} k^{n+1-m} \left[e^{-(m+1)\Delta t/\tau_i} - e^{-m\Delta t/\tau_i}\right]\right) + \\ + c_{hf} \frac{\partial V(z,t)}{\partial t}\end{aligned} \tag{12}$$

Remark 1.1. The choice of $n+1$ segments was to ensure the maximum precision of the integral approximation when substituted into equation (19). We make use of all the past values of the voltage.

1.3.2. *Time-Domain Internal-Impedance convolution with the current.* By the time-domain form of the internal impedance established in (11), we have

$$\begin{aligned}z_i(t) * I(z,t) &= r_{dc} I(z,t) + r_{dc} \frac{1}{\pi\sqrt{f_0}} \frac{1}{\sqrt{t}} * \frac{\partial I(z,t)}{\partial t} \\ &= r_{dc} I(z,t) + \frac{r_{dc}}{\pi\sqrt{f_0}} \int_0^t \frac{1}{\sqrt{t}} \frac{\partial I(z,t-\tau)}{\partial (t-\tau)} \mathrm{d}\tau\end{aligned}$$

Similarly, an approximation of the integral is required. We denote $h(t) = \frac{\partial I(z,t)}{\partial t}$ which is also considered a constant over the $(n+1)\Delta t$ segments. The discretized convolution can be approximated as

$$\begin{aligned}\int_0^t \frac{1}{\sqrt{\tau}}\, h(t-\tau)\, \mathrm{d}\tau &\simeq \int_0^{(n+1)\Delta t} \frac{1}{\sqrt{\tau}}\, h\left((n+1)\Delta t - \tau\right)\, \mathrm{d}\tau \\ &\simeq \sum_{m=0}^{n} h\left((n+1-m)\Delta t\right) \int_{m\Delta t}^{(m+1)\Delta t} \frac{1}{\sqrt{\tau}}\, \mathrm{d}\tau \\ &\simeq \sqrt{\Delta t} \sum_{m=0}^{n} h\left((n+1-m)\Delta t\right) \int_{m}^{m+1} \frac{1}{\sqrt{\tau}}\, \mathrm{d}\tau \\ &\simeq \sqrt{\Delta t} \sum_{m=0}^{n} h\left((n+1-m)\Delta t\right)\, 2\left[\sqrt{m+1} - \sqrt{m}\right]\end{aligned}$$

For simplicity, we write $P(m)$ and h^{n+1-m} instead of $2\left[\sqrt{m+1} - \sqrt{m}\right]$ and $h\left((n+1-m)\Delta t\right)$. The convolution is then defined by

$$z_i(t) * I(z,t) = r_{dc} I(z,t) + \frac{r_{dc}}{\pi\sqrt{f_0}} \sqrt{\Delta t} \sum_{m=0}^{n} h^{n+1-m}\, P(m) \tag{13}$$

2. Simulation by the β-method coupled with second order Runge Kutta

As with all finite difference methods, a grid in space and time has to be set up. The grid is uniform. The position variable z is discretized as Δz and the time variable t is discretized as Δt. The points on the grid can be designated as $z_k = (k-1)\Delta z$ for $k = 1, 2, .., N_z + 1$ and $t_n = (n-1)\Delta t$ for $n = 1, 2, .., N_t$. Additional points are put in place at half space and half time, they can be designated as $z_{k+\frac{1}{2}}$ and $t_{n+\frac{1}{2}}$. We shall compute $V(z,t)$ at the points (z_k, t_n), and $I(z,t)$ at $(z_{k+\frac{1}{2}}, t_{n+\frac{1}{2}})$ i.e. the voltage and currents are computed at offset locations in time and space. We denote,

$$\begin{aligned} V_k^n &= V(z_k, t_n) \\ I_k^n &= I(z_{k+\frac{1}{2}}, t_{n+\frac{1}{2}}) \end{aligned} \tag{14}$$

2.1. Computation of the Voltage at the source and load.

The essential problem in incorporating the terminal conditions is that the FDTD voltages and currents at each end of the line, V_1, I_1 and V_{N_z+1}, I_{N_z}, are not collocated in space or time. Whereas the terminal conditions relate the voltage and current at the same position and at the same time. Take for example the source $z = z_1$ $(k = 1)$, the second line of the system (1) by respecting the notation (14) is written

$$\frac{I_1^n - I(z_1, t_{n+\frac{1}{2}})}{\frac{\Delta z}{2}} + C\frac{V_1^{n+1} - V_1^n}{\Delta t} + GV(z_1, t_{n+\frac{1}{2}}) = 0.$$

We average the voltage value about the discretization point in order to include the G parameter by means of the trapezoidal rule [22]. Hence

$$\frac{I_1^n - I(z_1, t_{n+\frac{1}{2}})}{\frac{\Delta z}{2}} + C\frac{V_1^{n+1} - V_1^n}{\Delta t} + G\frac{V_1^{n+1} + V_1^n}{2} = 0.$$

Similarly the quantity $I(z_1, t_{n+\frac{1}{2}}) = I(0, t_{n+\frac{1}{2}}) = I_s(t_{n+\frac{1}{2}})$ is approached by averaging the source I_s in order to obtain a value located in time as the same time point I_1^n. We obtain

$$\frac{I_1^n - \frac{I_s^{n+1} + I_s^n}{2}}{\frac{\Delta z}{2}} + C\frac{V_1^{n+1} - V_1^n}{\Delta t} + G\frac{V_1^{n+1} + V_1^n}{2} = 0.$$

The equation is then equivalent to

$$(\frac{C\Delta z}{\Delta t} + \frac{G\Delta z}{2})V_1^{n+1} = (\frac{C\Delta z}{\Delta t} - \frac{G\Delta z}{2})V_1^n - 2I_1^n + I_s^{n+1} + I_s^n. \tag{15}$$

In the case of resistive terminations, the terminal characterizations are written in terms of generalized Thevenin's equation as $I_s = \frac{V_s - V_1}{R_s}$, where R_s is the source resistance. And so the property yields

$$V_1^{n+1} = (\frac{CR_s}{F} + \frac{G\Delta z R_s}{2} + 1)^{-1}\left[(\frac{CR_s}{F} - \frac{G\Delta z R_s}{2} - 1)V_1^n - 2R_s I_1^n + V_s^{n+1} + V_s^n\right], \tag{16}$$

where $F = \frac{\Delta t}{\Delta z}$. Similarly, with the terminal characterizations at the load defined with Thevenin's equation as $I_l = \frac{V_{N_z+1} - V_l}{R_L}$, we obtain the recursive equation for the load voltage

$$V_{N_z+1}^{n+1} = (\frac{C}{F}+\frac{G\Delta z R_L}{2}+1)^{-1}\left[(\frac{CR_L}{F}-\frac{G\Delta z R_L}{2}-1)V_{N_z+1}^{n}+2R_L I_{N_z}^{n}+V_L^{n+1}+V_L^{n}\right]. \tag{17}$$

2.2. β-method. The β-method is one of many forms of relaxation techniques. The idea is that given a certain method, we contrive two or more of its different approaches into a controllable scheme by introducing a variable denoted β. We refere the readers to [9] for more details. In our case, the choice of the method is justified in the introduction and shall be the FDTD.

By substituting (14), the discretization by the β-method yields

$$\begin{aligned}\frac{1}{\Delta z}\left(-\frac{\beta}{4}(V_{k+2}^{n+1}-V_{k-1}^{n+1})+\frac{1+\beta}{2}(V_{k+1}^{n+1}-V_{k}^{n+1})\right)+r_{dc}\frac{I_k^{n+1}+I_k^n}{2}+\\ +\frac{\sqrt{\Delta t}\,r_{dc}}{\pi\sqrt{f_0}}\sum_{m=0}^{n}\frac{I_k^{n+1-m}-I_k^{n-m}}{\Delta t}P(m)+L\frac{I_k^{n+1}-I_k^n}{\Delta t}=0.\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{1}{\Delta z}\left(-\frac{\beta}{4}(I_{k+1}^{n}-I_{k-2}^{n})+\frac{1+\beta}{2}(I_{k}^{n}-I_{k-1}^{n})\right)+c_{hf}\frac{V_k^{n+1}-V_k^n}{\Delta t}+\\ -M\sum_{m=0}^{n}\sum_{i=1}^{N}K_i\frac{V_k^{n+1-m}-V_k^{n-m}}{\Delta t}\left(e^{-(m+1)\Delta t/\tau_i}-e^{-m\Delta t/\tau_i}\right)=0.\end{aligned} \tag{19}$$

Solving (18) and (19) gives the recursion relations for any $2\le k\le N_z$.

$$\begin{aligned}I_k^{n+1}=\Bigg[&F_2 I_k^n-\frac{r_{dc}\Delta z}{\pi\sqrt{f_0\Delta t}}\sum_{m=1}^{n}\left(I_k^{n+1-m}-I_k^{n-m}\right)P(m)+\\ &+\left(\frac{\beta}{4}(V_{k+2}^{n+1}-V_{k-1}^{n+1})-\frac{1+\beta}{2}(V_{k+1}^{n+1}-V_{k}^{n+1})\right)\Bigg]/F_1\end{aligned} \tag{20}$$

$$\begin{aligned}V_k^{n+1}=V_k^n+\frac{1}{c_{hf}\,F_3}\Bigg[&M\sum_{m=1}^{n}\sum_{i=1}^{N}K_i\left(V_k^{n+1-m}-V_k^{n-m}\right)\left(e^{-(m+1)\Delta t/\tau_i}-e^{-m\Delta t/\tau_i}\right)\\ &-F\left(\frac{\beta}{4}(I_{k+1}^{n+1}-I_{k-2}^{n+1})-\frac{1+\beta}{2}(I_{k}^{n+1}-I_{k-1}^{n+1})\right)\Bigg]\end{aligned} \tag{21}$$

where

$$\begin{aligned}F_1&=\frac{L}{F}+r_{dc}\frac{\Delta z}{2}+2\frac{r_{dc}\Delta z}{\pi\sqrt{f_0\Delta t}},\\ F_2&=\frac{L}{F}-r_{dc}\frac{\Delta z}{2}+2\frac{r_{dc}\Delta z}{\pi\sqrt{f_0\Delta t}},\\ F_3&=1-\frac{M}{c_{hf}}\sum_{i=1}^{N}K_i\left(e^{-\Delta t/\tau_i}-1\right).\end{aligned}$$

2.3. Recursive convolution. The current and voltage update equations in (20) and (21) call for all the past values of the currents and voltages. A significant storage problem is preventing the execution of the method. We are interested in the same solution proposed by the author in [18], the *recursive convolution*. The justification of our interest relies within on "a see forward" comparison of our relaxed β-scheme and the classic FDTD simulated by the same author. The recursive convolution relies on the property $e^{x+y} = e^x e^y$. We have the following expansion

$$\begin{aligned}
\sum_{m=1}^{n}\sum_{i=1}^{N} K_i \left(V_k^{n+1-m} - V_k^{n-m}\right)&\left(e^{-(m+1)\Delta t/\tau_i} - e^{-m\Delta t/\tau_i}\right) \\
&= \sum_{i=1}^{N} K_i\, e^{-\Delta t/\tau_i} \left(V_k^{n} - V_k^{n-1}\right)\left(e^{-\Delta t/\tau_i} - 1\right) \\
&+ \sum_{i=1}^{N} K_i\, e^{-2\Delta t/\tau_i} \left(V_k^{n-1} - V_k^{n-2}\right)\left(e^{-\Delta t/\tau_i} - 1\right) \\
&+ \sum_{i=1}^{N} K_i\, e^{-3\Delta t/\tau_i} \left(V_k^{n-2} - V_k^{n-3}\right)\left(e^{-\Delta t/\tau_i} - 1\right) \\
&\ \ \vdots
\end{aligned} \tag{22}$$

We define

$$\begin{cases} \varphi_i^n = K_i\, e^{-\Delta t/\tau_i}\left(e^{-\Delta t/\tau_i} - 1\right)\left(V_k^n - V_k^{n-1}\right) + e^{-\Delta t/\tau_i}\varphi_i^{n-1}, \\ \varphi_i^0 = 0. \end{cases}$$

We can easily check that substituting φ into the equation (21) yields the β-scheme for $2 \le k \le N_z$,

$$V_k^{n+1} = V_k^n + \frac{1}{c_{hf}\, F_3}\left[M\sum_{i=1}^{N}\varphi_i^n - F\left(\frac{\beta}{4}(I_{k+1}^{n+1} - I_{k-2}^{n+1}) - \frac{1+\beta}{2}(I_k^{n+1} - I_{k-1}^{n+1})\right)\right]. \tag{23}$$

Remark that only one additional past value I_k^{n-1} and V_k^{n-1} must be retained in (23) instead of n values in (21).

The convolution of current requires an additional effort. In equation (20), we can notice the absence of the exponential property on which the recursive convolution is based. A simple solution to this problem is to approximate the functional $P(m) = 2\left(\sqrt{m+1} - \sqrt{m}\right)$ with exponential terms.

To do so, many methods are available. In particular, *Prony*'s method [30], the Matrix Pencil [13] and the work of G. Beylkin and L. Monzón [3] are often considered strong and reliable techniques. We shall use *Prony* to ensure the same precision order (fourth order) of the proposed scheme. More details are presented below.

2.4. Prony's method. Prony's method for representing a function $P(m)$ approximates it as the sum of N exponential functions. As explained above, we use this method to make existence to the required exponential terms in the internal-impedance convolution with the current. The function $P(m)$ is approached as follows

$$P(m) = \sum_{j=1}^{N} a_j\, e^{m b_j}. \tag{24}$$

To respect the precision order of the scheme (see Proposition 2.1), no less than a fourth order approximation is acceptable. By fixing N to 11, an appropriate approximation to $P(m)$ is obtained (see figure 1). In fact, for 100 samples, the error is of order 4. The computed coefficients a_j and b_j in (24) can be found on Table 1. We note that a different version from the standard in [30] was used. The factorization of a certain system uses a *modified Gram-Schmidt*. This allows us to achieve a fourth-order precision for 11 terms instead of 17 terms for the standard QR factorization used in the reference.

i	a_i	b_i
1	0.154602	-0.004705
2	0.153681	-0.042713
3	0.151819	-0.120744
4	0.149014	-0.243123
5	0.145335	-0.417252
6	0.140981	-0.655239
7	0.136386	-0.977444
8	0.132494	-1.420951
9	0.131724	-2.064312
10	0.143502	-3.131904
11	0.560456	-6.791631

TABLE 1. Prony's coefficients for $N = 11$.

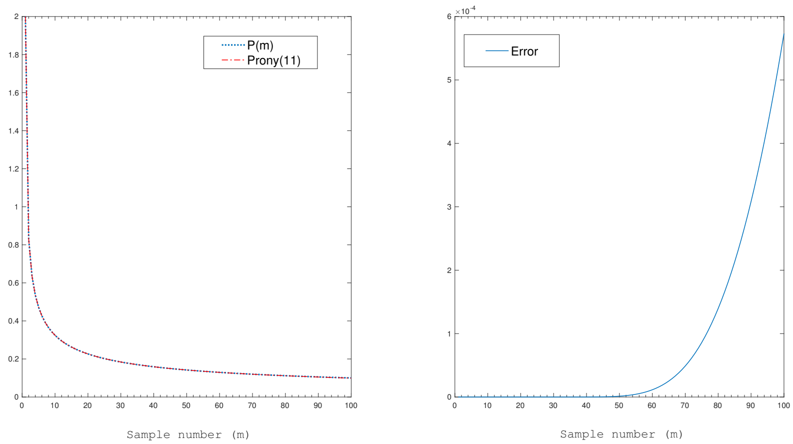


FIGURE 1. $P(m)$ vs *Prony* (11).

Using Prony's, we update the β-scheme of the current including the frequency-dependent parameters for $2 \leq k \leq N_z$ in a similar fashion as

$$I_k^{n+1} = \left[F_2 I_k^n - \frac{r_{dc}\Delta z}{\pi\sqrt{f_0 \Delta t}} \sum_{i=1}^{N} \Psi_i^n - \left(\frac{\beta}{4}(V_{k+2}^{n+1} - V_{k-1}^{n+1}) - \frac{1+\beta}{2}(V_{k+1}^{n+1} - V_k^{n+1}) \right) \right] / F_1, \tag{25}$$

where

$$\begin{cases} \Psi_i^n = a_i e^{b_i} \left(I_k^n - I_k^{n-1} \right) + e^{b_i} \Psi_i^{n-1}, \\ \Psi_i^0 = 0. \end{cases}$$

2.5. Principal of the β-method coupled with second order Runge Kutta. By introducing the classical β-method, we can ensure the stability and the high order compact of the scheme. Yet, the β-scheme is a relaxation technique of the FDTD, hence oscillations shall remain as we compute the load voltage at the near end of the line. To eliminate the ringing, we take advantage of the regularizing effect of Runge Kutta.

The proposed algorithm is obtained by applying second order Runge Kutta to the equations (23) and (25). We start with incorporating the terminal conditions that causes a problem; the voltages and currents at are not collocated in space or time at each end of the line; (see Subsection 2.1). The second iteration of space ($k = 2$) is approximated using the standard FDTD. The application of the proposed method is narrowed to $3 \le k \le N_z$.

• $k = 1$

$$V_1^{n+1} = \left(\frac{CR_s}{F} + \frac{G\Delta z R_s}{2} + 1\right)^{-1} \left[(\frac{CR_s}{F} - \frac{G\Delta z R_s}{2} - 1)V_1^n - 2R_s I_1^n + V_s^{n+1} + V_s^n\right]$$

$$I_1^{n+1} = \left(\frac{L}{F} + \frac{R\Delta z}{2}\right)^{-1} \left((\frac{L}{F} - \frac{R\Delta z}{2})I_1^n - V_1^{n+1} + V_1^n\right)$$

•• $k = 2$

$$V_2^{n+1} = \left(\frac{C}{F} + \frac{G\Delta z}{2}\right)^{-1} \left((\frac{C}{F} - \frac{G\Delta z}{2})V_2^n - I_2^{n+1} + I_2^n\right)$$

$$I_2^{n+1} = \left(\frac{L}{F} + \frac{R\Delta z}{2}\right)^{-1} \left((\frac{L}{F} - \frac{R\Delta z}{2})I_2^n - \frac{V_2^{n+1} - V_1^n}{2}\right)$$

••• $k = 3 : N_z$

• `First phase RK2: Auxiliary values of voltage and current by` β`-scheme`

$$(V_k^n)^* = V_k^n + \frac{1}{c_{hf}\, F_3}\left[M\sum_{i=1}^{N}\varphi_i^n - F\left(\frac{\beta}{8}(I_{k+1}^n - I_{k-2}^n) - \frac{1+\beta}{4}(I_k^n - I_{k-1}^n)\right)\right]$$

$$(I_k^n)^* = \left[F_2 I_k^n - \frac{r_{dc}\Delta z}{\pi\sqrt{f_0\Delta t}}\sum_{m=1}^{n}\Psi_i^n + \left(\frac{\beta}{8}((V_{k+2}^n)^* - (V_{k-1}^n)^* - \frac{1+\beta}{4}((V_{k+1}^n)^* - (V_k^n)^*)\right)\right]/F_1$$

• `Second Phase of RK2: voltage and current updates`

$$V_k^{n+1} = V_k^n + \frac{1}{c_{hf}\, F_3}\left[M\sum_{i=1}^{N}\varphi_i^n - F\left(\frac{\beta}{4}((I_{k+1}^n)^* - (I_{k-2}^n)^*) - \frac{1+\beta}{2}((I_k^n)^* - (I_{k-1}^n)^*)\right)\right]$$

$$I_k^{n+1} = \left[F_2 I_k^n - \frac{r_{dc}\Delta z}{\pi\sqrt{f_0\Delta t}}\sum_{m=1}^{n}\Psi_i^n + \left(\frac{\beta}{4}(V_{k+2}^{n+1} - V_{k-1}^{n+1}) - \frac{1+\beta}{2}(V_{k+1}^{n+1} - V_k^{n+1})\right)\right]/F_1$$

• k=N_z+1:

$$V_{N_z+1}^{n+1} = (\frac{C}{F} + \frac{G\Delta z R_L}{2} + 1)^{-1}\left[(\frac{CR_L}{F} - \frac{G\Delta z R_L}{2} - 1)V_{N_z+1}^n + 2R_L I_{N_z}^n + V_L^{n+1} + V_L^n\right]$$

2.6. A hint about stability & precision order. For the case of an ideal transmission, that is when the leakage conductance G and the resistance R are neglected and the capacitance and inductance are considered frequency-independent parameters, we have the following proposition concerning the stability and the precision of the scheme.

Proposition 2.1. *Stability and Order.*
The scheme is stable under the CFL condition:

$$\nu\frac{\Delta t}{\Delta z} \leq \min_{\theta\in[0,2\pi]} \frac{4}{\mid 2(1+\beta)sin(\theta) - \beta sin(2\theta) \mid},$$

where $\nu = \frac{1}{\sqrt{LC}}$ *is the velocity of the propagation.*

The order the β*-scheme is always of order 2 in space and time. Moreover, if* $\beta < \frac{1}{3}$ *and* $\frac{\Delta t}{\Delta z} = 2\sqrt{LC(1-3\beta)}$*, the order increases to four in space and in time.*

Proof. See [2]. □

The demonstration was conducted by the authors in [2] where they established the CFL condition using Von Neumann's property and a Matlab formal calculus. The key to the fourth order are the following properties which are direct results of the lossless transmission line equations

$$C\frac{\partial^3 V}{\partial^3 t} = -\frac{1}{LC}\frac{\partial^3 I}{\partial^3 z} \quad and \quad L\frac{\partial^3 I}{\partial^3 t} = -\frac{1}{LC}\frac{\partial^3 V}{\partial^3 z}.$$

For the case of lossy transmission lines and as explained in [18], the losses barely affect the stability and precision of the scheme. In fact, by all simulated methods by the author and many related works, the condition of stability for lossy and nonlossy lines are always identical. This was also confirmed for our proposed method (see Figures 4 and 5).

3. Numerical simulations

To simulate the case of losses and to compare the proposed method, we consider an experiment conducted by [18]. A two-conductor line is specified by a total length of $\mathcal{L} = 20cm$ with resistive loads $R_s = R_L = 50\,\Omega$. The capacitance and inductance are respectively $C = 88.2488\,pF/m$ and $L = 0.805969\,\mu H/m$. The line then has a velocity of propagation $\nu = 1.18573 \times 10^8\,m/s$. The break frequency f_0 is of value $393.06\,Mhz$ and the dc resistance is equal to $86.207\,\Omega/m$. The source V_s is a ramp function rising from 0 to 1 V with a rise time of $t_0 = 50\,ps$. The one-way transit time is computed to be $1.6867\,ns$. We divide the total line length $\mathcal{L}$ into N_z sections and the final solution time T_f into N_t intervals where $T_f = 10ns$.

However, we note that dielectric losses were not included in this experiment, which means the leakage conductance G was neglected.

Figure 2 illustrates the simulation by the FDTD of lossy and nonlossy lines where lossy results are obtained using Prony's coefficients presented in the table 24. The stability condition for the nonlossy lines by FDTD is easily established by Von Neumann's as $\frac{\Delta t}{\Delta z} \leq \frac{1}{\nu}$ ([12]). It's also worth mentioning that when applied to the nonlossy lines, the FDTD is arguably the best. In fact, under the *magic time step* (*mts*) condition $\Delta t = \frac{\Delta z}{\nu}$, the FDTD acquires an infinite order precision hence generates a solution with no **approximation error**. The existence of the *mts* is related to the ideal transmission property $\frac{\partial^j I(z,t)}{\partial z^j} = -L^{\frac{j-1}{2}} C^{\frac{j+1}{2}} \frac{\partial^j V(z,t)}{\partial t^j}$ for all odd j.

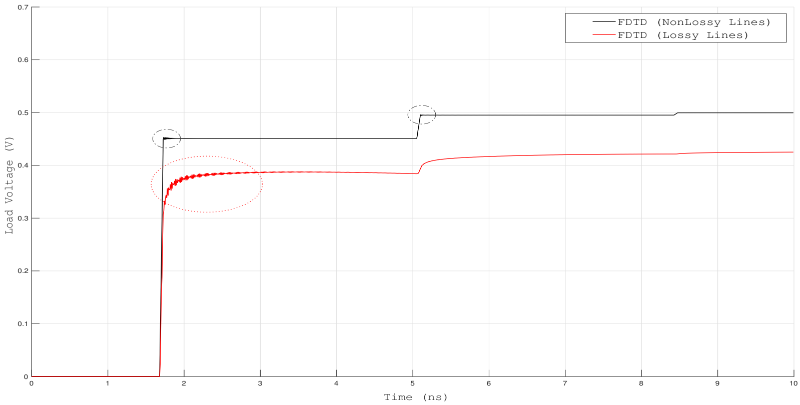
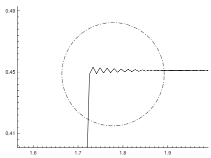
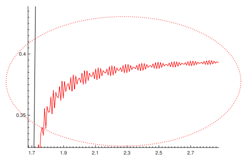


FIGURE 2. Simulation of the computation of load voltage by the FDTD for lossy and nonlossy line.



(A) Nonlossy line



(B) Lossy line

FIGURE 3. Leading edge of the load voltage by FDTD for nonlossy line (A) and lossy line (B).

For nonlossy lines, we observe ringing on the leading edge of the first and second transitions of the load voltage (Figure 3a). This is unexpected since the FDTD was supposed to deliver a solution with no approximation error. The explanation to this, is that for $N_z = 215$, the *mts* matches exactly $N_t = \frac{N_z}{\mathcal{L}} \times T_f \times \nu = 1274,65975$. Hence, the approximation we made by taking $N_t = 1275$ disturbs the infinite order precision. This shows how sensible the FDTD is. As for the case of nonlossy lines, the oscillations in Figure 3b are natural since the existence of *mts* is restricted to the ideal transmission lines. And without an infinite precision, the hyperbolic characteristic of the equation takes over and yields approximation errors in form of oscillations.

Figure 4 is the first tentative to enhance the FDTD. It is a comparison of the FDTD and the β-method. For $\beta = \frac{1}{3}\left(1 - (\frac{T_f}{2\mathcal{L}\,\nu})^2\right)$, $N_z = 215$ and $N_t = 1275$ the Proposition 2.1 is ensured. Notice that the amplitude of oscillations has decreased due to the relatively high precision of the β-method. The oscillations however are not entirely demolished.

Figure 5 illustrates the proposed method. We can see that the oscillations are entirely demolished by the regularizing effect of Runge Kutta. A side effect however, is the deviation of the load voltage around the rise time. This is clearly negligible since the generated solution converges to the same limit as the FDTD.

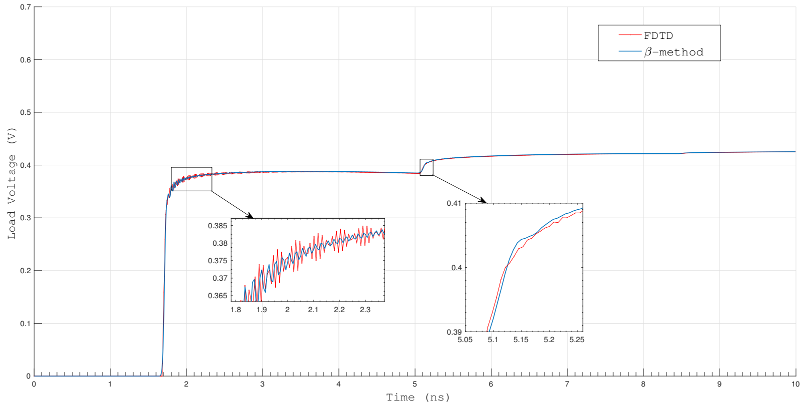


FIGURE 4. Illustration of the load voltage by FDTD and β-method

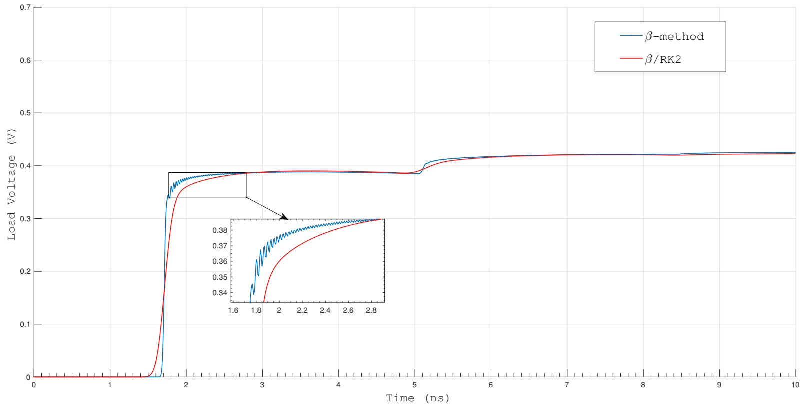


FIGURE 5. Illustration of the load voltage by classic β and coupled $\beta/RK2$ method.

Conclusion

By taking in consideration the physical phenomena that wraps the transmission lines equation, we have proposed a recent approach to deal with oscillations that occurs at each leading edge of the load voltage. Generally, FDTD methods are powerful tools to deal with such equations. However, we have shown that it also has its share of deficiencies. The proposed method, the $\beta/RK2$, was built to enhance the FDTD. It efficiently removed all kind of ringing while generating a solution that converges to the desired value of the voltage. We also recall that our investment in correcting the FDTD relies within the relative simplicity of its numerical implementation.

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