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Multigrid method for a fully nonlinear Black-Scholes equation

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Abstract. In this work, we present a multigrid approach for a fully nonlinear Black-Scholes equation arising in the modeling of markets frictions resulting from transaction costs. We consider a V-cycle method in order to minimize the computational cost of the numerical solution. The purpose of this paper is to show the effectiveness of multigrid approach for solving a fully nonlinear Black-Scholes problem.

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1. Introduction

The celebrated Black-Scholes model gives us an estimate of a European option price. The authors assumed in [5] that the stock value $S(t)$ follows a geometric Brownian motion modelled by

$$dS(t) = \mu S(t) + \sigma S(t) dB(t). \tag{1}$$

Here, μ and σ are the expected instantaneous rate of return and volatility respectively and assumed to be constants, $B(t)$ is the standard Brownian motion. The pricing of derivative products such as options is extremely interesting since they can be used as an insurance from stock fluctuations. Using the non arbitrage principle and Itô's formula we can obtain the Black-Scholes partial differential equation that describes the dynamics of a European option

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad S > 0, \quad t \in (0,T), \tag{2}$$

with the terminal value

$$V(S,T) = (S-K)^+. \tag{3}$$

Where K is the strike price and the unknown V is the European Call option, S is the underlying asset, T is the expiration date of the option and $r \geq 0$ is the risk-free interest rate.

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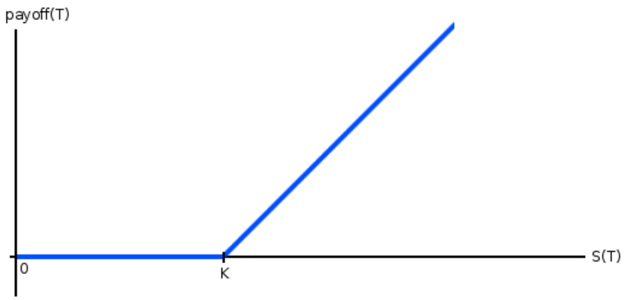

FIGURE 1. The terminal condition of the option price V (Payoff).

Although the Black-Scholes is a very successful pricing model, it is based on unrealistic conditions [2], like the absence of transaction costs in financial markets which is not true in the real world. In fact, in this case the perfect hedging no longer exists [16]. H. Leland [13] had given in his work a discrete approach of frequent revision of the portfolio for option pricing under transaction costs, his approach was later studied numerically in [3, 15]. Another well known model was proposed by G. Barles and H.M. Soner in [4], in their paper, they suggested a nonlinear Black-Scholes extension with a modified volatility to illustrate the impact of transaction costs on the option price. In this work we focus our attention on the Barles' and Soner's model, where the volatility is a function of time, the stock and the second derivative of the option price

$$\tilde{\sigma}(S,t,\frac{\partial^2 V}{\partial S^2})^2 = \sigma^2\left(1+\Psi\left(e^{r(T-t)}a^2S^2\frac{\partial^2 V}{\partial S^2}\right)\right). \quad (4)$$

Here $\tilde{\sigma}$ is the modified volatility proposed in the Barles' and Soner's model, σ is the volatility introduced in the Black-Scholes model, a is a nonnegative parameter that represents transaction costs and Ψ denotes the solution to the nonlinear ordinary differential equation

$$\begin{cases} \Psi'(x) = \dfrac{\Psi(x)+1}{2\sqrt{x\Psi(x)}-x}, & x \neq 0, \\ \Psi(0) = 0. & \end{cases} \quad (5)$$

In Figure 2 we represent the solution of ODE (5) using a fourth order Runge-Kutta method.

This paper is organized as follows. In Section 2, we review the nonlinear deterministic equation for the Barles' and Soner's model. We then give in Section 3 the discrete scheme and boundary conditions. We present the multigrid V-cycle technique for nonlinear problems in Section 4 then numerical results are given in the section thereafter. Finally a conclusion is in order

2. The nonlinear deterministic Black-Scholes equation

One of the successful nonlinear Black-Scholes extensions given in the case when transaction costs occur is the Barles' and Soner's equation [4] which is a fully nonlinear

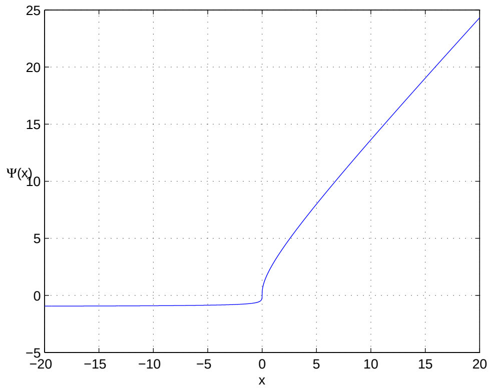

FIGURE 2. Solution Ψ to (5).

equation that reads

$$V_t + \frac{1}{2}\tilde{\sigma}^2\left(S, t, \frac{\partial^2 V}{\partial S^2}\right) S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad S > 0, \quad t \in (0, T), \tag{6}$$

with the terminal condition

$$V(S, T) = (S - K)^+, \tag{7}$$

Equation (6) is supplemented with the boundary conditions

$$\begin{cases} V(t, 0) & = & 0, \quad t \in (0, T), \\ V(t, S) & \sim_\infty & S - Ke^{-r(T-t)}, \quad (S \to +\infty). \end{cases} \tag{8}$$

Equation (6) has been studied theoretically using the stochastic optimal control theory [9]. G. Barles and H.M. Soner have shown the existence and uniqueness of a viscosity solution to (6)-(7)-(8). Another theoretical study for Equation (6) was given by D. Ševčovič in [17] by transforming it into a quasilinear parabolic equation. Recently, a theoretical constructive approach for the Barles' and Soner's equation was given in [1]. Since an exact solution to Equation (6) does not exist many authors studied (6) numerically (see for instance [15, 3, 14, 12, 18, 19]).

2.1. Numerical analysis. Due to the absence of general analytical solution to non-linear Black-Scholes equations, there are various numerical methods for solving non-linear equations for European call options; in our work we use the finite difference technique, we perform the following variable transformation in order to have a forward parabolic equation (see [3, 7])

$$x = \ln\left(\frac{S}{K}\right), \qquad \tau = \frac{1}{2}\sigma^2(T - t), \qquad u(x, \tau) = e^{-x}\frac{V(S, t)}{K}.$$

Plugging these transformations in (6), we get

$$u_\tau - (1 + \Psi(\mathrm{e}^{(D\tau + x)}\, a^2 K(u_x + u_{xx}))(u_x + u_{xx}) - Du_x = 0, \tag{9}$$

where $D = \dfrac{2r}{\sigma^2}$, $x \in \mathbb{R}$ and $0 \leq \tau \leq \dfrac{\sigma^2 T}{2}$. With the initial and boundary conditions

$$u(x,0) = \max(1 - \mathrm{e}^{-x}, 0) \quad (x \in \mathbb{R}), \tag{10}$$

$$u(x,\tau) = 0 \quad (x \to -\infty), \tag{11}$$

$$u(x,\tau) \sim 1 - e^{-D\tau - x} \quad (x \to +\infty). \tag{12}$$

2.2. Finite difference discretization. Black-Scholes equations are defined on an infinite domain $[0, +\infty[$ that becomes $\mathbb{R}$ using the change of variable described above. Therefore, in order to solve it numerically it is necessary to have a bounded domain. Through rigorous error estimates, an optimal size of the domain can be given [11]. Note that artificial boundary conditions can also be introduced for the treatment of the unbounded spatial boundaries of the domain of (9) (see for instance, [8]). Let $x \in [-R, R]$, $R > 0$ and set $\tau \in [0, \dfrac{\sigma^2 T}{2}]$. We denote by dx and dt the spatial and the time step respectively. We set $i \in [-N, N]$ and $n \in [0, M]$ such that $R = Ndx$, $-R = -Ndx$ and $\tilde{T} = \dfrac{\sigma^2 T}{2} = Mdt$. Finally we set U_i^n as an approximation of $u(x_i, \tau_n)$ where $x_i = idx$ and $\tau_n = ndt$. We discretize the initial and the boundary conditions as follows

$$\begin{cases} U_i^0 &= \max(1 - \mathrm{e}^{-idx}, 0), \\ U_{-N}^n &= 0, \\ U_N^n &= 1 - \mathrm{e}^{-Dndt - Ndx}. \end{cases} \tag{13}$$

The problem (9)-(10)-(11)-(12) was studied numerically in [3] using different discretization schemes. In the following, we use a classical difference finite Crank-Nicolson scheme. Replacing all the derivatives in (9) we get,

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{dt} =& \frac{s_i^n}{2}\left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{dx^2} + \frac{U_{i+1}^n - U_{i-1}^n}{2dx}\right) \\ &+ \frac{s_i^n}{2}\left(\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{dx^2} + \frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2dx}\right) \\ &+ (1 + D)\frac{U_{i+1}^n - U_{i-1}^n + U_{i+1}^{n+1} - U_{i-1}^{n+1}}{4dx} \\ &+ \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n + U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{2dx^2}. \end{aligned}$$

Where s_i^n denotes the discretized nonlinear volatility correction, which is given explicitely in time,

$$s_i^n = \Psi(\mathrm{e}^{(Dndt + x_i)}\, a^2 K(\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{dx^2} + \frac{U_{i+1}^n - U_{i-1}^n}{2dx})). \tag{14}$$

3. Multigrid approach

The aim of this paper is to improve the convergence of the numerical scheme for the Barles' and Soner's model by reducing the CPU time using the multigrid technique introduced by Brandt in [6]. The main idea is to accelerate the convergence of iterative methods by recognizing that low frequencies components of the error become high frequencies on coarser grids, therefore, we can accelerate the convergence of the solution of a fine grid problem by eliminating the high frequencies on coarser grids and then interpolating the corrections back to the fine grid problem [6, 10]. Operators allowing the passage from a fine grid to a coarse grid are called restriction operators and are given by

$$f_j^{2h} = \frac{1}{4}(f_{2j-1}^h + 2f_{2j}^h + f_{2j+1}^h), \quad 1 \leq j \leq \frac{N}{2} - 1,$$

where f_j^h and N are the approximation of the solution and the number of elements on the fine grid respectively. Conversely, in order to pass from the coarse grid to the fine grid we use the prolongation operator which is a linear interpolation that is given by

$$\begin{cases} f_{2j}^h & = & f_j^{2h} \\ f_{2j+1}^h & = & \frac{1}{2}(f_j^{2h} + f_{j+1}^{2h}). \end{cases}, \quad 1 \leq j \leq \frac{N}{2} - 1.$$

3.1. The V-Cycle. There are several multigrid techniques. The 3 main types are the V-cycle, W-cycle and full multigrid cycle. In the following we use the V-cycle method, which consists on solving a system of equations on a sequence of grids of various levels from the finest to the coarsest.

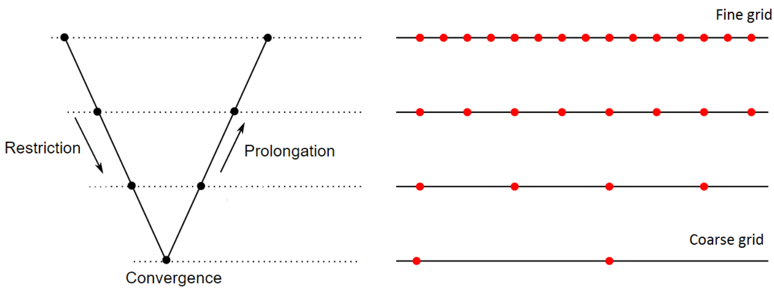

FIGURE 3. Illustration of a V-Cycle.

3.2. The full approximation storage algorithm. Let us suppose that on every grid (Ω_k) $1 \leq k \leq n$ we have an iterative method (Gauss Seidel, Jacobi..) noted G_k for solving the problem $A_k(U_k) = f_k$ where A_k is the nonlinear operator. Let Ω_n be the finest grid and Ω_1 the coarsest. We denote $G_k^{\nu_k}(U_k, f_k)$ the final result after ν_k iterations. In the following, we use the full approximation storage algortihm [10] since it is well established as a fast solver to nonlinear problems.

Algorithm 1 Full Approximation Storage

Presmoothing
Perform nonlinear smoothing on the fine grid problem ν_n times to obtain U^n, $k := n$
Coarse grid correction
(a) Compute $U_k = G_k^{\nu_k}(U_k, f_k)$ ν_k times on Ω_k.
(b) Restrict U_k and the residual $f_k - A_k(U_k)$.
(c) Compute the right hand side $f_{k-1} = \mathcal{R}_{k-1}^k(f_k - A_k(U_k)) + A_{k-1}(\mathcal{R}_{k-1}^k(U_k)))$ and set $U_{k-1} = \mathcal{R}_{k-1}^k(U_k)$, (here $\mathcal{R}_{k-1}^k$ is the restriction operator).
$k := k - 1$.
if $k > 1$ we go to **(a)**, otherwise we are on Ω_1 and we compute then
$U_1 = G_k^{\nu_1}(U_1, f_1)$ ν_1 times.
(d) Compute the correction $e^k = U_k - \mathcal{R}_k^{k+1}(U_{k+1})$.
(e) Correct the solution with the prolongation of e^k: $U_{k+1} = U_{k+1} + \mathcal{P}_k^{k+1}(e^k)$, (here $\mathcal{P}_{k-1}^k$ is the prolongation operator).
$k := k + 1$.
(f) Compute $U_k = G_k^{\mu_k}(U_k, f_k)$ μ_k times on Ω_k.
if $k < n$ we go to **(d)**, otherwise we are on Ω_n.
Postsmoothing
Post-smooth μ_n times and perform a convergence test.

The following algorithm computes the price of a European Call option in the presence of transaction costs using the finite difference scheme described in Section 2.

Algorithm 2 Computation of a European option using the Barles' and Soner's model

Input: $S_{\max}$, σ, K, r, a, T.
-Calculate the volatility correction by solving the ODE (5) using a fourth order Runge-Kutta method.
-Interpolate the solution of (5) using a cubic spline method.
For n=0,. . . , T:
-Initialise V^n and compute the nonlinear volatility correction.
-Compute V^{n+1} using Algorithm 1.
-Copy V^{n+1} in V^n.
end for.

4. Numerical results

4.1. The V-cycle. We start our numerical simulations with a presentation of the efficiency of the multigrid method. All the numerical experiments are performed on uniform grids. We used the following parameters $r = 0.1$, $\sigma = 0.2$, $K = 100$, $T = 1$ (one year), $dx = 0.1$, $dt = 10^{-3}$. We used a V-cycle with 4 levels, we set $\nu_1 = 2$ for pre and post smoothing and $\nu_2 = 3$ for coarse grid corrections. We have compared the total CPU time elapsed with and without using the V-cycle approach. All the numerical experiments were implemented in Mac OS X, 2.7GHz Intel core i5 processor.

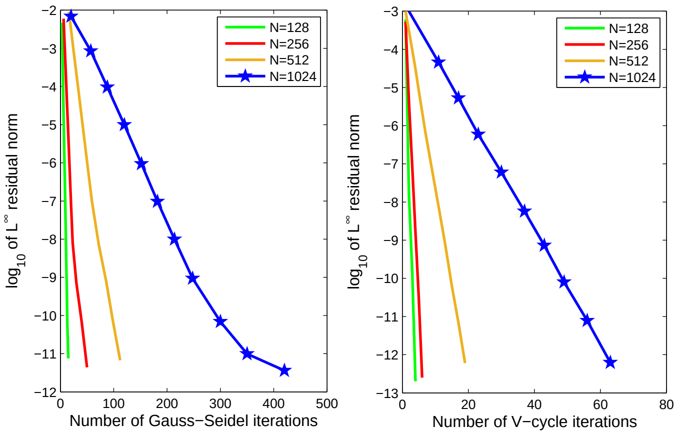

FIGURE 4. Comparison of rate of convergence for the V-cycle method (multigrid) and monogrid using 4 different meshes $(128, 256, 512$ and $1024)$ in the linear case.

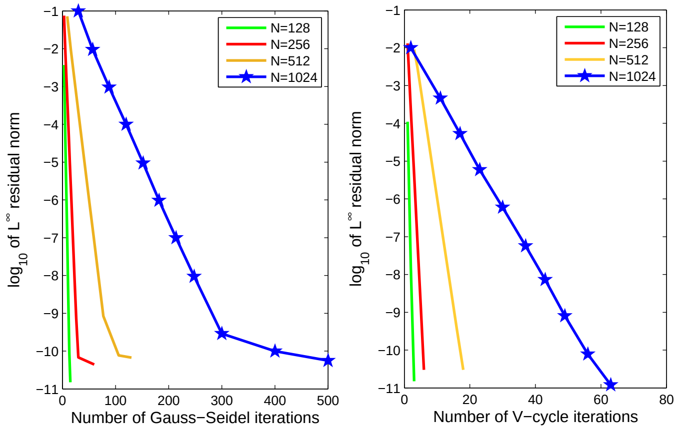

FIGURE 5. Comparison of rate of convergence for the V-cycle method (multigrid) and monogrid using 4 different meshes $(128, 256, 512$ and $1024)$ in the case of the Barles and Soner's model.

Figures 4 and 5 represent a comparison of rate of convergence for both the V-cycle and monogrid (standard Gauss-Seidel method) in the linear and nonlinear case respectively. In Figure 4 the error reaches the tolerance 10^{-11} after 100 iterations for $N = 512$ and 430 iterations for $N = 1024$ for the standard Gauss-Seidel method in contrast to the multigrid method where the error reaches 10^{-12} after only 19 V-cycle iterations for $N = 512$ and 64 iterations for $N = 1024$. In figure 5 the error reaches 10^{-10} after more than 420 iteration in monogrid for $N = 1024$, while it took only 63 iterations for the error to reach 10^{-11} using the V-cycle for the same mesh ($N = 1024$). Table 1 shows the time elapsed by the monogrid (standard Gauss-Seidel

TABLE 1. Comparison of the elapsed CPU time for the multigrid method for different discretization steps for both the linear case and the Barles' and Soner's model.

Mesh size	Linear case		Barles' and Soner's model	
	one grid	Multigrid	one grid	Multigrid
$N = 512$	9	1	9	1
$N = 1024$	15.9	1	11.2	1
$N = 2048$	26	1	22.3	1

method) and the multigrid method for different discretization steps. It is apparent that multigrid method gives better results. We scale the CPU time with the multigrid method. Here 1 represents the calculation time needed for the multigrid method.

4.2. Back to the Barles' and Soner's equation. We end our numerical results with an analysis of the effect of transaction costs on the option price. In Figure 6 we show the influence of transaction costs on the numerical solution, we observe that the option price increases by increasing the parameter a. Note that the choice of the constant a has a significant impact on the stability of the Crank-Nicolson scheme, in fact, for $a = 0.5$, adjusting the time step was needed to overcome the instability of the scheme.

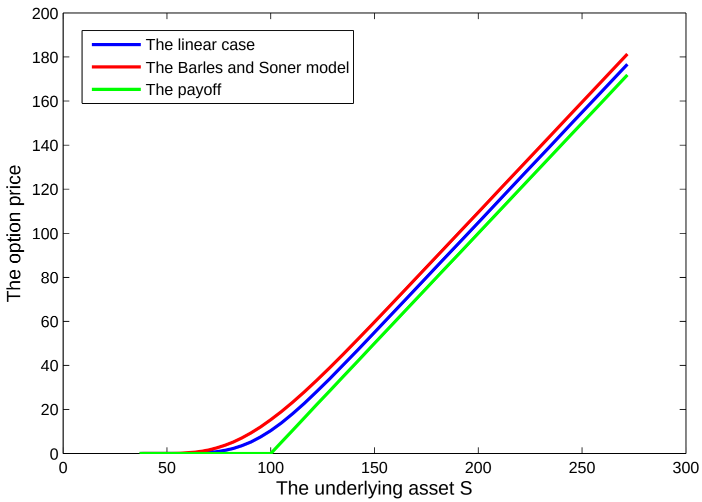

FIGURE 7. Value of the European option in the linear and nonlinear cases.

Figure (7) represents the value of the option price in the case of the Barles' and Soner's model ($a = 0.02$) and in the linear case. As we can notice, there is a price

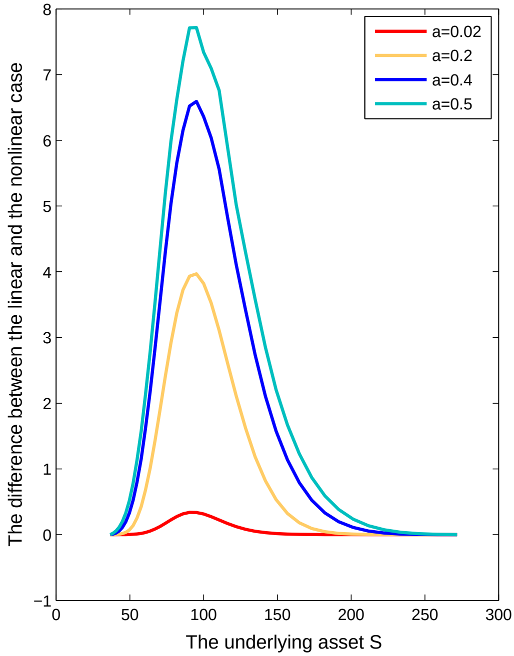

FIGURE 6. Comparison of the difference between the linear and the nonlinear case for differents transaction costs parameters a.

deviation between the linear and the nonlinear model. This is an expected result, in fact, the option price when transaction costs occur is more expensive than the option price in the linear case since it implies additional costs.

5. Conclusion

In this paper, we examined the effectiveness of the multigrid method on a fully nonlinear Black-Scholes model. The numerical results showed a substantially improvement of the computational cost of the numerical solution in contrast to the standard iterative method on a fine grid.

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