Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 47(1), 2020, Pages 42–53
ISSN: 1223-6934

Existence and nonexistence of solution to the discrete fourth-order boundary value problem with parameters

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Abstract. In this paper, we consider the discrete fourth order boundary value problems with three parameters. We apply the direct method of calculus variational and the mountain pass theorem in order to establish the existence of at least one and three nontrivial solutions, also we study the nonexistence of nontrivial solution.

2010 Mathematics Subject Classification. 60J05, 60J20.
Key words and phrases. Discrete fourth order boundary value problems, critical points theory.

1. Introduction

Let $T > 2$ be a positive integer and $[2, T]_{\mathbb{Z}}$ be the discrete interval given by $\{2, 3, 4....., T\}$. We consider the discrete nonlinear fourth order boundary value problems as follows:

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k, u(k)), & k \in [2, T]_Z \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0 \end{cases} \tag{1}$$

where Δ denotes the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^{i+1}u(k) = \Delta(\Delta^i u(k))$, $f : [2, T]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and α, β, λ are parameters satisfying the conditions, $\lambda > 0$ and

$$(C) \quad : \qquad 1 + (T-1)T\alpha_- + T(T-1)^3\beta_- > 0$$

where : $\qquad \alpha_- = min(\alpha, 0) \qquad$ and $\qquad \beta_- = min(\beta, 0)$.

The theory of nonlinear difference equations has been widely used to study discrete models in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. In recent years, a great deal of work has been done in the study of the existence and multiplicity of solutions for discrete boundary value problem. For the background and recent results, we refer the reader to the monographs [1, 2, 3, 4, 5, 17] and the references therein.

We note that problem (1) is the discrete variant of a kind of the problem

$$\begin{cases} u^{(4)}(t) - \alpha u^{(2)}(t) + \beta u(t) = \lambda f(t, u(t)), & t \in (0, 1) \\ u(0) = u'(0) = u'(1) = u^{(3)}(0) = u^{(3)}(1) = 0. \end{cases} \tag{2}$$

The special case of (2) has been studied by many authors using various approaches [10, 11, 20].

Received November 28, 2018. Accepted November 15, 2019.
This paper has been presented at the Conference MOCASIM, Marrakesh, 26-27 November 2018.

In [1, 5, 7, 18, 19], the authors have been investigated the existence and multiplicity of solutions for nonlinear discrete boundary value problems with parameters using variational methods. However, it seems that no similar results in the literatture on the existence and nonexistence of nontrivial solutions for (1) have been obtained.

Motivated by [8, 9], the aim of this work is to give the existence, nonexistence, and multiplicity of nontrivial solutions of (1) by using some basic theorems in critical point theory and variational methods under some conditions imposed on the nonlinear function f.

The rest of this paper is organized as follows, in section 2, we present some preliminary theorems about the critical point theory. In section 3, we introduce the corresponding variational framework of (1) and In section 4, we give the mains results and thier proofs in terms of different values of λ.

2. Preliminaries

In this section, we state some definitions and theorems that will be used below, we can refer to ([6, 12, 13, 14, 15, 16]) for more details.

Definition 2.1. Let E be a real Banach space, D an open subset of E. Suppose that a functional $\varphi : D \longrightarrow \mathbb{R}$ is Fréchet differentiable on D. If $u_0 \in D$ and the Fréchet derivative satisfies $\varphi\prime(u_0) = 0$, then we say that u_0 is a critical point of the functional φ and $\varphi(u_0)$ is a critical value of φ.

Let $C^1(E, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on E.

Definition 2.2. Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$. We say that φ satisfies the Palais-Smale condition ((PS) condition for short) if for every sequence $(u_n) \in E$ such that $\varphi(u_n)$ is bounded and $\varphi\prime(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, there exists a subsequence of (u_n) which is convergent in E.

Theorem 2.1. *Let E be a real Banach space and $\varphi : E \longrightarrow \mathbb{R}$. If φ is weakly lower semi-continuous and coercive, i.e. $\lim\limits_{\|x\|\to+\infty} \varphi(x) = +\infty$, then there exists $x_0 \in E$ such that*

$$\inf_{x\in E} \varphi(x) = \varphi(x_0).$$

Moreover, f $\varphi \in C^1(E, \mathbb{R})$, then x_0 is a critical point of φ i.e. $\varphi\prime(x_0) = 0$.

Theorem 2.2 (Mountain Pass Lemma)**.** *Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$ satisfying the (PS) condition with $\varphi(0) = 0$. Suppose that*
(i) *There exists $\rho > 0$ and α such that $\varphi(u) \geq \alpha$ for all $u \in E$, with $\|u\| = \rho$.*
(ii) *There exists $u_0 \in E$ with $\|u\| \geq \rho$ such that $\varphi(u_0) < 0$.*
Then φ has a critical value $c \geq \alpha$ and $c = \inf\limits_{h\in\Gamma} \max\limits_{s\in[0,1]} \varphi(h(s))$, where

$$\Gamma = \{h \in C([0, 1], E) : h(0) = 0, h(1) = u_0\}.$$

Theorem 2.3. *Let E be a reflexive real Banach space and $\varphi : E \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on E^*, $\psi : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and $\varphi(0) = \psi(0) = 0$.*

Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in E$ with $0 < r < \varphi(\bar{u})$ such that

(i) $\dfrac{\sup\limits_{u \in \varphi^{-1}(]-\infty, r])} \psi(u)}{r} < \dfrac{\psi(\bar{u})}{\varphi(\bar{u})}$,

(ii) *for each* $\lambda \in \Lambda = \left] \dfrac{\varphi(\bar{u})}{\psi(\bar{u})}, \dfrac{r}{\sup\limits_{u \in \varphi^{-1}(]-\infty, r])} \psi(u)} \right[$, $I_\lambda = \varphi - \lambda\psi$ *is coercive.*

Then, for each $\lambda \in \Lambda$, the I_λ has at least three distinct critical points in E.

Proposition 2.4. *Let E be a real reflexive Banach space and E^* be the dual space of E. Suppose that $T : E \to E^*$ is a continuous operator and there exists $\omega > 0$ such that*

$$(Tu - Tv, u - v) \geq \omega \|u - v\|^2 \qquad forall \quad u, v \in E.$$

Then $T : E \to E^*$ is a homeomorphism between E and E^*.

3. Variational framework for the problem (1)

In this section, we introduce the corresponding variational framework for (1). The solutions of BVP (1.1) will be investigated in a space

$$E = \{u : [0, T+2]_Z \to \mathbb{R} : u(0) = \Delta u(0) = \Delta u(T) = 0 = \Delta^3 u(0) = \Delta^3 u(T-1)\},$$

which is a (T-1)-dimensional Hilbert space, see [18] with the inner product

$$(u, v) = \sum_{k=2}^{k=T} u(k)v(k).$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=2}^{k=T} |u(k)|^2 \right)^{\frac{1}{2}}.$$

We will need some preliminary lemmas in order to prove our main results.

Lemma 3.1. *For any $u, v \in E$, we have,*

$$\sum_{k=2}^{k=T} \Delta^4 u(k-2)v(k) = \sum_{k=2}^{k=T+1} \Delta^2 u(k-2)\Delta^2 v(k-2), \tag{3}$$

and

$$\sum_{k=2}^{k=T} \Delta u(k-1)\Delta v(k-1) = -\sum_{k=2}^{k=T} \Delta^2 u(k-1)v(k) \tag{4}$$

Proof. First, we prove (3).For any $u, v \in E$, by the summation by parts formula and the fact that $\Delta v(0) = \Delta v(T) = 0$, it follows that

$$\begin{aligned}\sum_{k=2}^{T+1}\Delta^2u(k-2)\Delta^2v(k-2) &= \Delta^2u(T)\Delta v(T)-\Delta^2u(0)\Delta v(0)-\sum_{k=2}^{T+1}\Delta^3u(k-2)\Delta v(k-1)\\ &= -\sum_{k=2}^{T+1}\Delta^3u(k-2)\Delta v(k-1)\\ &= -\sum_{k=2}^{T}\Delta^3u(k-2)\Delta v(k-1),\end{aligned}$$

on the other hand, by the summation by parts formula and the fact that $\Delta^3u(0)=\Delta^3u(T-1)=0$, we have

$$\sum_{k=2}^{T}\Delta^3u(k-2)\Delta v(k-1)=\Delta^3u(T-1)v(T)-\Delta^3u(0)v(1)-\sum_{k=2}^{T}\Delta^4u(k-2)v(k),$$

So

$$\sum_{k=2}^{T}\Delta^4u(k-2)v(k)=\sum_{k=2}^{T+1}\Delta^2u(k-2)\Delta^2v(k-2),$$

i.e.(3) holds.

Next, we show (4). Again, by the summation by parts formula and the fact that $\Delta u(T)=0$ and $v(1)=0$, we have

$$\begin{aligned}\sum_{k=2}^{T}\Delta u(k-1)\Delta v(k-1) &= \Delta u(T)v(T)-\Delta u(1)v(1)-\sum_{k=2}^{T}\Delta^2u(k-1)v(k)\\ &= -\sum_{k=2}^{T}\Delta^2u(k-1)v(k).\end{aligned}$$

This completes the proof of the lemma. □

Definition 3.1. We say that $u\in E$ is a weak solution of problem (1), if for any $v\in E$, we have

$$\sum_{k=2}^{T}\Delta^4u(k-2)v(k)-\alpha\sum_{k=2}^{T}\Delta^2u(k-1)v(k)+\beta\sum_{k=2}^{T}u(k)v(k)=\lambda\sum_{k=2}^{T}f(k,u(k))v(k)$$

We define the energy functional corresponding to (1) by, for $u\in E$

$$I_\lambda(u)=\Phi(u)-\lambda\Psi(u), \tag{5}$$

where

$$\Psi(u)=\sum_{k=2}^{T}F(k,u(k)), \tag{6}$$

$$F(k,x)=\int_0^x f(k,t)dt, k\in[2,T]_{\mathbb{Z}},$$

and

$$\Phi(u)=\frac{1}{2}\left(\sum_{k=2}^{T+1}|\Delta^2u(k-2)|^2+\alpha\sum_{k=2}^{T}|\Delta u(k-1)|^2+\beta\sum_{k=2}^{T}|u(k)|^2\right). \tag{7}$$

Standard argument assure that, with any fixed $\lambda > 0$ the functional I_λ is Fréchet differentiable and the Fréchet derivative of I_λ is given by

$$(I'_\lambda(u), v) = (\Phi'(u), v) - \lambda(\Psi'(u), v), \tag{8}$$

for any $u, v \in E$, where

$$(\Psi'(u), v) = \sum_{k=2}^{T} f(k, u(k))v(k) \tag{9}$$

and

$$(\Phi'(u), v) = \sum_{k=2}^{T+1} \Delta^2 u(k-2)\Delta^2 v(k-2) + \alpha \sum_{k=2}^{T} \Delta u(k-1)\Delta v(k-1) + \beta \sum_{k=2}^{T} u(k)v(k). \tag{10}$$

According to the Definition 3.1 and from equalities (8 - 10), finding solutions to the problem (1) is like looking for the critical points of the functional I_λ, indeed

Lemma 3.2. *If $u \in E$ is a critical point of the functional I_λ then u is a solution of problem (1).*

Proof. Let $u \in E$ is a critical point of the functional I_λ then

$$(I'_\lambda(u), v) = 0, \quad \forall v \in E,$$

so from (8 - 10) and Lemma 3.1, we deduce that

$$\sum_{k=2}^{T} \Delta^4 u(k-2)v(k) - \alpha \sum_{k=2}^{T} \Delta^2 u(k-1)v(k) + \beta \sum_{k=2}^{T} u(k)v(k) - \lambda \sum_{k=2}^{T} f(k, u(k))v(k) = 0,$$

$\forall v \in E$ thus by the arbitrariness of $v \in E$, we have

$$\Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k, u(k)),$$

then $u \in E$ is a solution of the problem (1). □

Lemma 3.3. *For any $u \in E$, we have $\Phi(u) \geq 0$ and*

$$\Phi(u) \geq \frac{1}{2}\mu\|u\|^2, \tag{11}$$

where

$$\mu = T^{-1}(T-1)^{-3}(1 + T(T-1)\alpha_- + T(T-1)^3\beta_-)$$

Proof. Let $u \in E$ and $k \in [2, T]_{\mathbb{Z}}$, note that

$$\Delta u(k-1) = \Delta u(0) + \sum_{i=2}^{k} \Delta^2 u(i-2),$$

in fact that $\Delta u(0) = 0$, then by Hölder's inequality, we have

$$\begin{aligned}|\Delta u(k-1)| \leq \sum_{i=2}^{k} |\Delta^2 u(i-2)| &\leq \sum_{i=2}^{T+1} |\Delta^2 u(i-2)| \\ &\leq \sqrt{T}\Big(\sum_{i=2}^{T+1} |\Delta^2 u(i-2)|^2\Big)^{\frac{1}{2}},\end{aligned}$$

so

$$\sum_{k=2}^{T}|\Delta u(k-1)|^2 \le T(T-1)\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2.$$

Similarly, for any $u \in E$ and $k \in [2,T]_{\mathbb{Z}}$, note that

$$u(k) = u(1) + \sum_{i=2}^{k}\Delta u(i-1),$$

in fact that $u(1) = 0$, then by Hölder's inequality, we have

$$\begin{aligned} |u(k)| \le \sum_{i=2}^{k}|\Delta u(i-1)| &\le \sum_{i=2}^{T}|\Delta u(i-1)| \\ &\le \sqrt{T-1}(\sum_{i=2}^{T}|\Delta u(i-1)|^2)^{\frac{1}{2}}, \end{aligned}$$

then

$$\sum_{k=2}^{T}|u(k)|^2 \le (T-1)^2\sum_{k=2}^{T}|\Delta u(k-1)|^2,$$

so

$$\sum_{k=2}^{T}|u(k)|^2 \le T(T-1)^3\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2,$$

therefore, from (7) and by summation the parts inequalities, we deduce that

$$\begin{aligned} \Phi(u) \ge& \frac{1}{2}(1+T(T-1)\alpha_- + T(T-1)^3\beta_-)\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2 \\ \ge& \frac{1}{2}(1+T(T-1)\alpha_- + T(T-1)^3\beta_-)T^{-1}(T-1)^{-3}\sum_{k=2}^{T}|u(k)|^2 \\ \ge& \frac{1}{2}(1+T(T-1)\alpha_- + T(T-1)^3\beta_-)T^{-1}(T-1)^{-3}\|u\|^2, \end{aligned}$$

then by (1), we deduce that

$$\Phi(u) \ge 0 \quad \text{and} \quad \Phi(u) \ge \frac{1}{2}\mu\|u\|^2.$$

The proof of lemma is completed. $\square$

Lemma 3.4. *For any $u \in E$, we have*

$$\Phi(u) \le \frac{1}{2}\theta\|u\|^2, \tag{12}$$

where

$$\theta = 20 + 4.\alpha_+ + \beta_+, \quad \alpha_+ = max(0,\alpha) \quad \text{and} \quad \beta_+ = max(0,\beta).$$

Proof. Firstly, for any $u \in E$, we have

$$\sum_{k=2}^{T}|\Delta u(k-1)|^2 = \sum_{k=2}^{T}(|u(k)|^2 + |u(k-1)|^2 - 2u(k)u(k-1)),$$

then, the fact that $u(1)=0$ and by Hölder's inequality, we have

$$\sum_{k=2}^{T}|\Delta u(k-1)|^2 \leq \|u\|^2+\|u\|^2+2\|u\|^2=4\|u\|^2,$$

Next, note that

$$\Delta^2 u(k-2)=u(k)-2u(k-1)+u(k-2),$$

then

$$\begin{aligned}\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2 &\leq \sum_{k=2}^{T+1}(|u(k)|^2+4|u(k-1)|^2+|u(k-2)|^2-4u(k)u(k-1)\\ &\quad +2u(k)u(k-2)-4u(k-1)u(k-2))\\ &\leq \sum_{k=2}^{T+1}(|u(k)|^2+4|u(k-1)|^2+|u(k-2)|^2+4|u(k)||u(k-1)|\\ &\quad +2|u(k)||u(k-2)|+4|u(k-1)||u(k-2)|),\end{aligned}$$

the fact that $u(1)=0$, $u(T+1)=u(T)$ and by Hölder's inequality we deduce that

$$\begin{aligned}\sum_{k=2}^{T+1}|\Delta^2 u(k-2)|^2 &\leq \|u\|^2+|u(T)|^2+4\|u\|^2+\|u\|^2\\ &\quad -|u(T)|^2+8\|u\|^2+4\|u\|^2+2\|u\|^2=20\|u\|^2,\end{aligned}$$

finally, for any $u\in E$, we have

$$\Phi(u)\leq \frac{1}{2}(20+4\alpha_+ +\beta_+)\|u\|^2=\frac{1}{2}\theta\|u\|^2,$$

i.e (12) holds. □

4. Main results and proofs

In this section, we will use the critical point theory to study the existence, nonexistence and multiplicity of nontrivial solutions for the problem (1).

Theorem 4.1. *Assume that the following conditions holds:*

(H0) $f(k,0)\neq 0$ *for at least one* $k\in[2,T]_{\mathbb{Z}}$.

(H1) *There exists* $C>0$ *such that :* $\displaystyle\max_{k\in[2,T]_{\mathbb{Z}}}\limsup_{|x|\to\infty}\frac{F(k,x)}{x^2}<C$.

Then for each $\lambda\in]0,\frac{\mu}{2C}[$ *the problem (1) has at least one nontrivial solution.*

Proof. To prove the theorem, we will use the Theorem 2.1. From $(H1)$, there exist $R>0$ such that for any $k\in[2,T]_{\mathbb{Z}}$, we have

$$F(k,x)\leq Cx^2, \quad \forall |x|>R.$$

Since $x\mapsto F(k,x)-Cx^2$ is continuous on $[-R,R]$ for any $k\in[2,T]_{\mathbb{Z}}$, there exists $C'>0$ such that

$$F(k,x)\leq Cx^2+C', \quad \forall x\in\mathbb{R}.$$

Therefore, by Lemma 3.3 and from (5), we deduce that for any $u\in E$, we have

$$I_\lambda(u)\geq \frac{1}{2}\mu\|u\|^2-\lambda C\sum_{k=2}^{T}|u(k)|^2-(T-1)\lambda C',$$

so

$$I_\lambda(u) \geq (\frac{1}{2}\mu - \lambda C)\|u\|^2 - (T-1)\lambda C',$$

then for all $\lambda \in \left]0, \frac{\mu}{2C}\right[$, we have $\lim\limits_{\|u\| \longrightarrow \infty} I_\lambda(u) = +\infty$, this prove that I_λ is coercive.

Since I_λ is Gâteaux differentiable and continuous, it has by Theorem 2.1 there exist $\tilde{u} \in E$ such that $I'_\lambda(\tilde{u}) = 0$, and by Lemma 3.2, $\tilde{u}$ is a solution of (1). From $(H0)$, it is easy to see that $\tilde{u} \neq 0$. The proof is complete. □

Theorem 4.2. *Assume that the following assumptions holds*

(H2) $f_0 = \max\limits_{k \in [2,T]_{\mathbb{Z}}} \limsup\limits_{x \to 0} \dfrac{f(k,x)}{x} \in [0, +\infty)$.

(H3) *Suppose that there exists $M > 0$ such that*

$$f_\infty = \min_{k \in [2,T]_{\mathbb{Z}}} \liminf_{|x| \to \infty} \frac{f(k,x)}{x} \in (M, +\infty).$$

(H4) $\dfrac{\theta}{M} < \dfrac{\mu}{f_0}$.

Then for each $\lambda \in \left]\frac{\theta}{M}, \frac{\mu}{f_0}\right[$ the (1) has at least one nontrivial solution.

Proof. We will check that the functional I_λ satisfies all the assumptions of theorem 2.2. Obviously, $I_\lambda(0) = 0$ where I_λ is given by (5).

First, we prove that I_λ satisfies the Palais-Smale conditions. Let $\{u_n\}$ be a (PS) sequence of I_λ, that is, $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$.

We fixed $\lambda \in]\frac{\theta}{M}, \frac{\mu}{f_0}[$. From $(H3)$, there exist $R > 0$ such that for any $k \in [2,T]_{\mathbb{Z}}$

$$\frac{f(k,x)}{x} \geq M, \quad \forall |x| > R,$$

then, we have

$$(\forall x > R) \quad : \quad f(k,x) \geq Mx \quad \text{and} \quad (\forall x < -R) \quad : \quad f(k,x) \leq Mx.$$

This, together with the continuity of $x \mapsto f(k,x) - Mx$, on $[-R,R]$ for any $k \in [2,T]_{\mathbb{Z}}$, implies that there exist $M' > 0$ such that

$$f(k,x) \leq Mx + M', \quad \forall x \in (-\infty, 0], k \in [2,T]_{\mathbb{Z}},$$

and

$$f(k,x) \geq Mx - M', \quad \forall x \in [0, +\infty) \quad , k \in [2,T]_{\mathbb{Z}}.$$

Hence, for $k \in [2,T]_{\mathbb{Z}}$, we have

$$\forall x \geq 0 \quad : \quad F(k,x) = \int_0^x f(k,t)dt \geq \frac{1}{2}Mx^2 - M'x,$$

and

$$\forall x \leq 0 \quad : \quad F(k,x) = -\int_x^0 f(k,t)dt \geq \frac{1}{2}Mx^2 + M'x = \frac{1}{2}Mx^2 - M'|x|.$$

Then for $x \in \mathrm{R}$ and $k \in [2,T]_{\mathrm{Z}}$, we have

$$F(k,x) \geq \frac{1}{2}Mx^2 - M'|x|.$$

Therefore, by Lemma 3.4 and from (5), we have for $u \in E$

$$I_\lambda(u_n) \leq \frac{1}{2}\theta\|u_n\|^2 - \frac{1}{2}\lambda M\|u_n\|^2 + \lambda M'\sqrt{T-1}\|u_n\|,$$

so

$$I_\lambda(u_n) \leq \frac{1}{2}(\theta - \lambda M)\|u_n\|^2 + \lambda M'\sqrt{T-1}\|u_n\|, \tag{13}$$

since $\lambda > \dfrac{\theta}{M}$ and $I_\lambda(u_n)$ is bounded then $\{u_n\}$ is bounded. In view of the fact that E is a finite dimension space, we deduce that I_λ satisfies the (PS) condition.

In the other hand, since $\lambda \in \left]\dfrac{\theta}{M}, \dfrac{\mu}{f_0}\right[$ and from (H_2), $\exists \rho > 0$, such that for $|x| < \rho$ and $k \in [2,T]_{\mathrm{Z}}$

$$|f(k,x)| \leq \frac{\mu}{\lambda}(1-\nu)|x|$$

where $\nu = \dfrac{\lambda\rho}{\mu} \in (0,1)$, so for $|x| < \rho$ and $k \in [2,T]_{\mathrm{Z}}$, we have

$$F(k,x) \leq \frac{\mu}{2\lambda}(1-\nu)\rho^2.$$

Let

$$B_\rho = \{u \in E, \|u\| < \rho\} \quad \textit{and} \quad \eta = \frac{1}{2}\mu\nu\rho^2,$$

then for $u \in \partial B_\rho = \{u \in E, \|u\| = \rho\}$, we have

$$I_\lambda(u) \geq \frac{1}{2}\mu\|u\|^2 - \frac{1}{2}\mu(1-\nu)\rho^2 = \frac{1}{2}\mu\rho^2 - \frac{1}{2}\mu(1-\nu)\rho^2 = \eta,$$

this implies that the condition (I_1) of Theorem 2.2 is satisfied.

Next, using (13) then for $u \in E$, we have

$$I_\lambda(u) \leq \frac{1}{2}(\theta - \lambda M)\|u\|^2 + \lambda M'\sqrt{T-1}\|u\|.$$

Put $\bar{u} = \{\bar{u(k)}\} \in E$ such that $\bar{u}(k) = (\sqrt{T-1})^{-1}$, then for any $t > 0$, we have

$$I_\lambda(t\bar{u}) \leq \frac{1}{2}(\theta - \lambda M)t^2\|\bar{u}\|^2 + \lambda M'\sqrt{T-1}\|\bar{u}\| = \frac{1}{2}(\theta - \lambda M)t^2 + \lambda M'(T-1),$$

since $\lambda > \dfrac{\theta}{M}$, then $I_\lambda(t\bar{u}) \longrightarrow -\infty$ as $t \longrightarrow +\infty$, consequently there exists a sufficiently large $t_0 > \rho$ such that $u_0 = t_0\bar{u} \notin \bar{B_\rho}$ and $I_\lambda(u_0) < 0$. This implies that the condition (I_2) of Theorem 2.2 is satisfied. Therefore, the functional I_λ has a critical value $c^* > 0$, that is, $\exists u^* \in E$ such that $I'_\lambda(u^*) = 0$ and $I_\lambda(u^*) = c^* > 0$. Since $I_\lambda(0) = 0$ then $u^* \neq 0$. □

Example 4.1. We consider the problem (1) with $\alpha = 1$, $\beta = 1$ and $T = 10$. For $k \in [2,10]_{\mathrm{Z}}$ and $x \in \mathrm{R}$, let

$$f(k,x) = p(k)\frac{x^2}{1+x^2} + (10^{-2}\mu - \theta - 1)\frac{x}{1+|x|} + (\theta+1)x,$$

where $p(k) : [2,10]_{\mathrm{Z}} \longrightarrow \mathrm{R}$ is one polynomial and μ, θ are given by (11) and (12) respectively.

It is easy to verify that $f_\infty = \theta + 1 \in (M, +\infty)$, with $M = \theta$, $f_0 = 10^{-2}\mu \in [0, +\infty)$ and $\dfrac{\theta}{M} = 1 < \dfrac{\mu}{f_0} = 100$. Hence by Theorem 4.2, when $\lambda \in (1, 100)$ the problem (1) has at least one nontrivial solution.

Theorem 4.3. *Assume that there exists $a > 0$ and $b > 0$ with $a < b\sqrt{T-1}$ and the following conditions hold,*

(H5) $\quad \forall x \in \mathrm{R}$ *and* $k \in [2, T]_{\mathrm{Z}} \quad : F(k, x) \leq 1 + |x|$.

(H6) $\quad \displaystyle\sum_{k=2}^{T} F(k, b) > 0.$

(H7) $\quad \displaystyle\frac{\mu a^2}{b^2\gamma}\sum_{k=2}^{T} F(k, b) > (T-1) \max_{(k,x)\in[2,T]_{\mathrm{Z}}\times[-a,a]} F(k, x).$

Where : $\gamma = 10 + 3\alpha_+ + (T-1)\beta_+$

Then for each $\lambda \in \Lambda = \left]\dfrac{\gamma b^2}{2\displaystyle\sum_{k=2}^{T} F(k, b)}, \dfrac{\mu a^2}{(T-1)\displaystyle\max_{(k,x)\in[2,T]_{\mathrm{Z}}\times[-a,a]} F(k, x)}\right[$ *the problem (1) has at least three distinct solutions in E.*

Proof. The functional Φ given by (7) is continuously Gâteaux differentiable and by Lemma 3.3, we deduce that Φ is coercive, also we have the regularity assumptions required on Φ and Ψ.

Firstly, we put $r = \dfrac{1}{2}\mu a^2$ and pick $\bar{u} \in E$ defined as for $k \in [2, T]_{\mathrm{Z}} \quad : \bar{u}(k) = b$. Using Lemma 3.3 with $u = \bar{u}$ and $a < b\sqrt{T-1}$, we obtain

$$\Phi(\bar{u}) \geq \frac{1}{2}\mu\|\bar{u}\|^2 = \frac{1}{2}\mu(T-1)b^2 > \frac{1}{2}\mu a^2 = r.$$

Taking into the fact that, for any $k \in [2, T]_{\mathrm{Z}}$,

$$|u(k)| \leq \|u\| \leq \sqrt{\frac{2.\Phi(u)}{\mu}},$$

we have

$$\Phi^{-1}_{((-\infty,r])} \subseteq \{u \in E : |u(k)| \leq a, k \in [2, T]_{\mathrm{Z}}\},$$

then

$$\sup_{u\in\Phi^{-1}_{((-\infty,r])}} \Psi(u) = \sup_{u\in\Phi^{-1}_{((-\infty,r])}} \sum_{k=2}^{T} F(k, u(k)) \leq (T-1) \max_{(k,x)\in[2,T]_{\mathrm{Z}}\times[-a,a]} F(k, x).$$

Therefore, from $(H7)$, we have

$$\sup_{u\in\Phi^{-1}_{((-\infty,r])}} \Psi(u) < \frac{\mu a^2 \displaystyle\sum_{k=2}^{T} F(k, b)}{b^2\gamma} = r\frac{2\Psi(\bar{u})}{\gamma b^2}$$

it is easy to verify that

$$\Phi(\bar{u}) = \frac{1}{2}(10 + 3\alpha + (T-1)\beta)b^2 \leq \frac{1}{2}\gamma b^2.$$

Then

$$\frac{\sup\limits_{u\in\Phi^{-1}_{((-\infty,r])}} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})},$$

this implies that the assumption (i) of Theorem 2.3 is verified.

In other hand from $(H5)$, (5) and by Lemma 3.3, we deduce that for any $u \in E$,

$$I_\lambda(u) \geq \frac{1}{2}\mu\|u\|^2 - \lambda\sum_{k=2}^{T}(1+|u(k)|) \geq \frac{1}{2}\mu\|u\|^2 - \lambda(T-1) - \lambda\|u\|\sqrt{T-1},$$

then I_λ is coercive, this imply that the assumption (ii) of Theorem 2.3 is verified.

It is clear that $\phi(0) = \psi(0) = 0$, therefore the problem (1) has at least three distinct solutions in E. $\square$

Theorem 4.4. *Assume that there exists $B > 0$ such that*

$(H8)$ $\quad \max\limits_{k\in[2,T]_{\mathrm{Z}}} (f^0(k), f^\infty(k)) < B,$ *where*

$$f^0(k) = \lim_{x\to 0}\frac{f(k,x)}{x} \quad \textit{and} \quad f^\infty(k) = \lim_{|x|\to+\infty}\frac{f(k,x)}{x}.$$

Then there exist $\lambda_0 > 0$ such that for any $\lambda \in]0,\lambda_0[$ the problem (1) has no nontrivial solution.

Proof. From $(H8)$, there exists $R > r > 0$ such that for any $k \in [2,T]_{\mathrm{Z}}$, we have

$$xf(k,x) \leq Bx^2, \quad |x| < r \quad or \quad |x| > R,$$

since for any $k \in [2,T]_{\mathrm{Z}}$, $x \mapsto \dfrac{f(k,x)}{x}$, is continuous on $[-R,-r]\cup[r,R]$ then there exist $B' > 0$ such that

$$xf(k,x) \leq B'x^2, \qquad \forall x \in [-R,-r]\cup[r,R], k \in [2,T]_{\mathrm{Z}}.$$

So for $B_0 = max(B,B')$, $k \in [2,T]_{\mathrm{Z}}$ and $x \in \mathrm{R}$, we have

$$xf(k,x) \leq B_0x^2.$$

Therefore for any $u \in E$, we prove that

$$(\Psi'(u),u) \leq B_0\|u\|^2.$$

In the other hand, for $u \in E$, we have $(\Phi'(u),u) = 2\Phi(u)$ then by (7) we obtain $u \in E$

$$(I'_\lambda(u),u) \geq \mu\|u\|^2 - \lambda B_0\|u\|^2 = (\mu - \lambda B_0)\|u\|^2, \quad \forall u \in E.$$

Pick $\lambda_0 = \dfrac{\mu}{B_0} > 0$ then we deduce that, for all $\lambda \in]0,\lambda_0[$ and $u \in E, u \neq 0$

$$(I'_\lambda(u),u) > 0,$$

which implies that $I'_\lambda(u) \neq 0$ for $u \in E, u \neq 0$. Then the functional I_λ does not admit critical point $u \neq 0$, this complete the proof. $\square$

Remark 4.1. It is easy to see that, if for $(k,x) \in [2,T]_{\mathrm{Z}} \times \mathrm{R}$: $xf(k,x) \leq 0$ the problem (1) has no nontrivial solution for any $\lambda \in]0,+\infty[$.

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