Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 48(2), 2021, Pages 194–205
ISSN: 1223-6934

Signed double Roman domination numbers in digraphs

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Abstract. Let $D = (V, A)$ be a finite simple digraph. A signed double Roman dominating function (SDRD-function) on the digraph D is a function $f : V(D) \to \{-1, 1, 2, 3\}$ satisfying the following conditions: (i) $\sum_{x\in N^-[v]} f(x) \geq 1$ for each $v \in V(D)$, where $N^-[v]$ consist of v and all in-neighbors of v, and (ii) if $f(v) = -1$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor assigned 3, while if $f(v) = 1$, then the vertex v must have at least one in-neighbor assigned 2 or 3. The weight of a SDRD-function f is the value $\sum_{x\in V(D)} f(x)$. The signed double Roman domination number (SDRD-number) $\gamma_{sdR}(D)$ of a digraph D is the minimum weight of a SDRD-function on D. In this paper we study the SDRD-number of digraphs, and we present lower and upper bounds for $\gamma_{sdR}(D)$ in terms of the order, maximum degree and chromatic number of a digraph. In addition, we determine the SDRD-number of some classes of digraphs.

2020 *Mathematics Subject Classification.* Primary 05C20; Secondary 05C69.
Key words and phrases. Digraph, signed double Roman dominating function, signed double Roman domination number, directed graph.

1. Introduction

Let G be a finite and simple graph with vertex set $V(G)$, and let $N_G(v) = N(v)$ be the open neighborhood of the vertex v. A *signed double Roman dominating function* (SDRD-function) on a graph G is defined in [2] as a function $f : V(G) \longrightarrow \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(N[v]) = \sum_{x\in N[v]} f(x) \geq 1$ holds for each vertex $v \in V(G)$. The *signed double Roman domination number* $\gamma_{sdR}(G)$ of G is the minimum weight of a SDRD-function on G. This parameter has been studied in [1, 3, 7, 9]. A $\gamma_{sdR}(G)$-function is a SDRD-function on G of weight $\gamma_{sdR}(G)$. Following the ideas in [2], we study the SDRD-functions on digraphs D.

Suppose D is a finite simple digraph with vertex set $V(D)$ and arc set $A(D)$ (briefly V and A). The *order* and the *size* of D are integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v and we also say that x *dominate* y. For each vertex v, the set of in-neighbors and out-neighbors of v are denoted by $N^-(v) = N^-_D(v)$ and $N^+(v) = N^+_D(v)$, respectively. Assume that $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$. We write $d^+(v) = d^+_D(v)$ for the out-degree of a vertex v and $d^-(v) = d^-_D(v)$ for its in-degree. We denote the *minimum* and *maximum in-degree* and the *minimum* and *maximum*

Received November 12, 2019. Accepted November 3, 2020.

out-degree of D by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. A digraph D is called r-out-regular if $\delta^+(D) = \Delta^+(D) = r$. In addition, suppose $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ and $\Delta = \Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ is the *minimum* and *maximum degree* of D, respectively. A digraph D is called *regular* or r-*regular* if $\delta(D) = \Delta(D) = r$. The *distance* $d_D(u, v)$ from a vertex u to a vertex v is the length of a short directed $u - v$ path in D. For every set $X \subseteq V(D)$, $D[X]$ is the subdigraph induced by X. For a real-valued function $f : V \longrightarrow \mathbb{R}$ the weight of f is $\omega(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we write $f(S) = \sum_{v\in S} f(v)$, so $\omega(f) = f(V)$. Consult Haynes et al. [5] for the notation and terminology which are not defined here.

A *signed double Roman dominating function* (SDRD-function) on a digraph D is a function $f : V \longrightarrow \{-1, 1, 2, 3\}$ such that (i) $f(N^-[w]) = \sum_{x\in N^-[w]} f(x) \geq 1$ for each vertex $w \in V$ and (ii) every vertex u for which $f(u) = -1$ has at least one in-neighbor z with $f(z) = 3$ or to at least two in-neighbor v for which $f(v) = 2$, (iii) every vertex v with $f(v) = 1$ has at least one in-neighbor z with $f(z) \geq 2$. The weight of a SDRD-function f on a digraph D is $\omega(f) = \sum_{v\in V(D)} f(v)$. The *signed double Roman domination number (SDRD-number)* $\gamma_{sdR}(D)$ is the minimum weight of a SDRD-function on D. A $\gamma_{sdR}(D)$-function is a SDRD-function on D of weight $\gamma_{sdR}(D)$.

In this paper we initiate the study of the signed double Roman domination number of digraphs, and we establish lower and upper bounds for $\gamma_{sdR}(D)$ in terms of the order, maximum degree and chromatic number of a directed graph. In addition, we determine the SDRD-number of some classes of digraphs.

The *associated digraph* of a graph G, denoted by $D(G) = G^*$, is defined as a digraph obtained from G if each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, we have the next result.

Remark 1.1. If $D(G)$ is the associated digraph of a graph G, then $\gamma_{sdR}(D(G)) = \gamma_{sdR}(G)$.

In [2], the authors determine the SDRD-number of some classes of graphs including complete graphs, complete bipartite graphs and cycle.

Theorem A. *If $n \neq 4$, then $\gamma_{sdR}(K_n) = 1$ and $\gamma_{sdR}(K_4) = 2$.*

Theorem B. *For $2 \leq m \leq n$,*

$$\gamma_{sdR}(K_{m,n}) = \begin{cases} 3 & \text{if} \quad m = 2 \text{ and } n \geq 3 \\ 4 & \text{if } m \geq 4 \text{ or } m = n = 2 \\ 5 & \text{if } m = 3. \end{cases}$$

Theorem C. *For $n \geq 3$,*

$$\gamma_{sdR}(C_n) = \begin{cases} \frac{n}{3} & if \quad n \equiv 0 \ (mod\ 3) \\ \lceil \frac{n}{3} \rceil + 2 & if \quad n \equiv 1 \ (mod\ 3) \\ \lceil \frac{n}{3} \rceil + 1 & if \quad n \equiv 2 \ (mod\ 3)\ . \end{cases}$$

Using Remark 1.1 and Propositions A, B and C we obtain next result.

Corollary 1.1. (1) *If $n \neq 4$, then $\gamma_{sdR}(K^*_n) = 1$ and $\gamma_{sdR}(K^*_4) = 2$.*

(2) *For* $2 \leq m \leq n$,

$$\gamma_{sdR}(K^*_{m,n}) = \begin{cases} 3 & \textit{if} \quad m = 2 \textit{ and } n \geq 3 \\ 4 & \textit{if } m \geq 4 \textit{ or } m = n = 2 \\ 5 & \textit{if } m = 3. \end{cases}$$

(3) *For* $n \geq 3$,

$$\gamma_{sdR}(C^*_n) = \begin{cases} \frac{n}{3} & \text{if} \quad n \equiv 0 \pmod 3 \\ \lceil \frac{n}{3} \rceil + 2 & \text{if} \quad n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil + 1 & \text{if} \quad n \equiv 2 \pmod 3 \ . \end{cases}$$

A *double Roman dominating function* (DRD-function) on D is defined in [6] as a function $f : V \longrightarrow \{0, 1, 2, 3\}$ such that (i) every vertex u for which $f(u) = 0$ has at least one in-neighbor z with $f(z) = 3$ or to at least two in-neighbor v for which $f(v) = 2$, (ii) every vertex v with $f(v) = 1$ has at least one in-neighbor z with $f(z) \geq 2$. The weight of a DRD-function f on a digraph D is $\omega(f) = \sum_{v \in V(D)} f(v)$. The *double Roman domination number (DRD-number)* $\gamma_{sdR}(D)$ is the minimum weight of an DRD-function on D. A $\gamma_{dR}(D)$-function is a DRD-function on D of weight $\gamma_{dR}(D)$. The proof of the next two results can be found in [6].

Theorem D. *For any digraph D, there is a $\gamma_{dR}(D)$-function such that no vertex needs to be assigned the value 1.*

Theorem E. *For any digraph D,*

$$2\gamma(D) \leq \gamma_{dR}(D) \leq 3\gamma(D).$$

The proof of the following result can be found in Szekeres-Wilf [8].

Theorem F. *For any graph G,*

$$\chi(G) \leq 1 + max\{\delta(H) \mid H \textit{ is a subgraph of } G\}.$$

2. Basic Properties

In this section we investigate basic properties of the SDRD-functions and the SDRD-numbers of digraphs. The definitions immediately lead to our first proposition.

Proposition 2.1. *For any SDRD-function $f = (V_{-1}, V_1, V_2, V_3)$ on a digraph D of order n,*

(a) $|V_{-1}| + |V_1| + |V_2| + |V_3| = n$.
(b) $\omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$.
(c) *$V_2 \cup V_3$ is a dominating set of D. In particular, $|V_2 \cup V_3| \geq \gamma(D)$ where $\gamma(D)$ is the domination number of D.*

Proposition 2.2. *If $f = (V_{-1}, V_1, V_2, V_3)$ is a SDRD-function on a digraph D of order n with maximum out-degree Δ^+ and minimum out-degree δ^+, then*

(i) $(3\Delta^+ + 2)|V_3| + (2\Delta^+ + 1)|V_2| + \Delta^+|V_1| \geq (\delta^+ + 2)|V_{-1}|$.
(ii) $(3\Delta^+ + \delta^+ + 4)|V_3| + (2\Delta^+ + \delta^+ + 3)|V_2| + (\Delta^+ + \delta^+ + 2)|V_1| \geq n(\delta^+ + 2)$.
(iii) $(\Delta^+ + \delta^+ + 2)\omega(f) \geq n(\delta^+ - \Delta^+ + 2) + (\delta^+ - \Delta^+)(2|V_3| + |V_2|)$.
(iv) $\omega(f) \geq n(\delta^+ - 3\Delta^+)/(3\Delta^+ + \delta^+ + 4) + |V_2| + 2|V_3|$

Proof. (i) Proposition 2.1 (a) implies that

$$\begin{aligned}
|V_{-1}|+|V_1|+|V_2|+|V_3| &= n\\
&\le \sum_{v\in V(D)}\sum_{x\in N^-[v]} f(x)\\
&= \sum_{v\in V(D)} (d_D^+(v)+1)f(v)\\
&= \sum_{v\in V_3} 3(d_D^+(v)+1)+\sum_{v\in V_2} 2(d_D^+(v)+1)\\
&\quad +\sum_{v\in V_1}(d_D^+(v)+1)-\sum_{v\in V_{-1}}(d_D^+(v)+1)\\
&\le 3(\Delta^+ +1)|V_3|+2(\Delta^+ +1)|V_2|\\
&\quad +(\Delta^+ +1)|V_1|-(\delta^+ +1)|V_{-1}|.
\end{aligned}$$

This inequality chain leads to the desired bound.

(ii) Using Proposition 2.1 (a) and Part (i), we arrive at (ii).

(iii) This part can be obtained from Proposition 2.1 and Part (ii) as follows

$$\begin{aligned}
(\Delta^+ +\delta^+ +2)\omega(f) &= (\Delta^+ +\delta^+ +2)(4|V_3|+3|V_2|+2|V_1|-n)\\
&\ge 2n(\delta^+ +2)-4|V_3|(1+2\Delta^+)-2|V_2|(\Delta^+ +1)\\
&\quad +(\Delta^+ +\delta^+ +2)(2|V_3|+|V_2|-n)\\
&= n(\delta^+ -\Delta^+ +2)+(\delta^+ -\Delta^+)(2|V_3|+|V_2|).
\end{aligned}$$

(iv) The inequality chain in the proof of Part (i) and Proposition 2.1 (a) implies

$$\begin{aligned}
n &\le 3(\Delta^+ +1)|V_1\cup V_2\cup V_3|-(\delta^+ +1)|V_{-1}|\\
&= 3(\Delta^+ +1)|V_1\cup V_2\cup V_3|-(\delta^+ +1)(n-|V_1\cup V_2\cup V_3|)\\
&= (3\Delta^+ +\delta^+ +4)-n(\delta^+ +1)
\end{aligned}$$

and so

$$|V_1\cup V_2\cup V_3|\ge \frac{n(\delta^+ +2)}{3\Delta^+ +\delta^+ +4}.$$

Applying above inequality and Proposition 2.1, we get

$$\begin{aligned}
\omega(f) &= 2|V_1\cup V_2\cup V_3|-n+|V_2|+2|V_3|\\
&\ge \frac{2n(\delta^+ +2)}{3\Delta^+ +\delta^+ +4}-n+|V_2|+2|V_3|\\
&= \frac{n(\delta^+ -3\Delta^+)}{3\Delta^+ +\delta^+ +4}+|V_2|+2|V_3|
\end{aligned}$$

and the proof is complete. □

Corollary 2.3. For any r-out-regular digraph D of order n with $r\ge 1$, $\gamma_{sdR}(D)\ge n/(r+1)$.

Applying Corollary 2.3 and Observation 1.1, we obtain the next known result.

Corollary 2.4. (Ahangar et al. [1]) For any r-regular graph G of order n with $r\ge 1$, $\gamma_{sdR}(G)\ge n/(r+1)$.

If D is not out-regular, then we can get the next lower bound on the SDRD-number.

Corollary 2.5. If D is a digraph of order n with minimum out-degree δ^+, maximum out-degree Δ^+ and $\delta^+ < \Delta^+$, then

$$\gamma_{sdR}(D) \geq \Big(\frac{3\Delta^+\delta^+ - 3(\Delta^+)^2 + 3\delta^+ + \Delta^+ + 4}{(\Delta^+ + 1)(3\Delta^+ + \delta^+ + 4)}\Big)n.$$

Proof. The result follows by multiplying both sides of the inequality in Proposition 2.2 (iv) and adding the resulting inequality to the inequality in Proposition 2.2 (iii). □

Since $\Delta^+(D(G)) = \Delta(G)$ and $\delta^+(D(G)) = \delta(G)$, Observation 1.1 and Corollary 2.5 leads to the next known result.

Corollary 2.6. [1] If G is a graph of order n, minimum degree δ and maximum degree Δ where $\delta < \Delta$, then

$$\gamma_{sdR}(D) \geq \Big(\frac{3\Delta\delta - 3(\Delta)^2 + 3\delta + \Delta + 4}{(\Delta + 1)(3\Delta + \delta + 4)}\Big)n.$$

3. Bounds on the signed double Roman domination number

In this section we present sharp bounds on the signed double Roman domination number of digraphs. We start with a simple but sharp upper bound on the SDRD-number of a digraph.

Proposition 3.1. For any non-empty digraph D of order n, $\gamma_{sdR}(D) \leq 2n$. The equality holds if and only if D is the disjoint union of isolated vertices.

Proof. Obviously the function f defined on D by $f(x) = 2$ for each $x \in V(D)$, is a SDRD-function on D yielding $\gamma_{sdR}(D) \leq 2n$.

If D is disjoint union of isolated vertices, then by definition of SDRD-function we have $\gamma_{sdR}(D) = 2n$.

Conversely, assume that $\gamma_{sdR}(D) = 2n$. If $u \to v$ is an arc in G, then the function $f : V(D) \longrightarrow \{-1, 1, 2, 3\}$ defined by $f(u) = 3, f(v) = -1$ and $f(x) = 2$ for $x \in V(D) - \{u, v\}$, is a SDRD-function on D of weight $2n - 2$ which is a contradiction. Thus D is disjoint union of isolated vertices. □

The bound in Proposition 3.1 can be improved if $\delta^-(D) \geq 1$.

Theorem 3.2. If D is a digraph of order n with minimum in-degree $\delta^- \geq 1$, then

$$\gamma_{sdR}(D) \leq 2n - 3\left\lceil \frac{\delta^-}{2} \right\rceil + 1.$$

Proof. Assume that $t = \lceil \frac{\delta^-}{2} \rceil$. It follows from

$$n \cdot \Delta^+(D) \geq \sum_{x \in V(D)} d^+(x) = \sum_{x \in V(D)} d^-(x) \geq n \cdot \delta^-(D),$$

that $\Delta^+(D) \geq \delta^-(D) \geq t$. Suppose $u \in V(D)$ is a vertex with out-degree $\Delta^+(D)$, and let $B = \{w_1, w_2, \ldots, w_t\}$ be a set of t out-neighbor of u. Define the function f on $V(D)$ by $f(u) = 3, f(x) = -1$ for $x \in B$ and $f(x) = 2$ for $x \in V(D) - (B \cup \{u\})$. Then for each vertex $z \in V(D)$ we have

$$f(N^-[z]) \geq -t + 2(\delta^- + 1 - t) = 2\delta^- - 3t + 2 \geq 1$$

and so f is a SDRD-function with weight $3-t+2(n-t-1)=2n-3t+1$. This implies that $\gamma_{sdR}(D)\le 2n-3\lceil\frac{\delta^-}{2}\rceil+1$. □

Proposition 3.3. For any digraph D of order n,

$$\gamma_{sdR}(D)\ge 2+\Delta^-(D)-n.$$

Moreover, this bound is sharp.

Proof. Assume $v\in V(D)$ is a vertex with in-degree $\Delta^-(D)$, and f is a $\gamma_{sdR}(D)$-function. By definition we have

$$\begin{aligned}\gamma_{sdR}(D) &= \sum_{x\in N^-[v]} f(x)+\sum_{x\in V(D)\setminus N^-[v]} f(x)\\ &\ge 1+\sum_{x\in V(D)\setminus N^-[v]} f(x)\\ &\ge 1-(n-(\Delta^-(D)+1))\\ &= 2+\Delta^-(D)-n\end{aligned}$$

as desired.

To show the sharpness, let n,t be integers such that $n\ge 3$ and $2t+2\le n-1$, and let $K_{1,n-1}$ be a star centered at u with leaves $u_1,u_2,\ldots,u_{n-1}$. Assume D_t be a digraph obtained from $K_{1,n-1}$ by orienting the edges from u into leaves and adding arcs (u_i,u_1) for $2\le i\le 2t+2$. Define g on $V(D_t)$ by $g(u)=3$ and $g(u_1)=g(u_2)=\cdots=g(u_t)=1$ and $g(x)=-1$ otherwise. One can see that g is a SDRD-function on D_t with weight $4+2t-n=2+\Delta^-(D_t)-n$ implying that $\gamma_{sdR}(D_t)\le 2+\Delta^-(D_t)-n$ and so $\gamma_{sdR}(D_t)=2+\Delta^-(D_t)-n$. Therefore the bound of Proposition 3.3 is sharp for $\Delta^-(D)$ even.

Assume now that $t\ge 2$ be an integer with $2t\le n-1$ and let D_{2t} be the digraph obtained from $K_{1,n-1}$ by orienting the edges from u into leaves and adding arcs (u_i,u_1) for $2\le i\le 2t+1$. Define h on $V(D_{2t})$ by $h(u)=3, h(u_1)=2$ and $h(u_2)=h(u_3)=\ldots=h(u_{t-1})=1$ and $h(x)=-1$ otherwise. Clearly h is a SDRD-function on D_{2t} with weight $3+2t-n=2+\Delta^-(D_{2t})-n$. It follows that $\gamma_{sdR}(H_{2t})=2+\Delta^-(H_{2t})-n$ by Proposition 3.3. Hence the bound of Proposition 3.3 is sharp for $\Delta^-(D)$ odd too. □

Theorem 3.4. *For any digraph D of order $n\ge 2$,*

$$\gamma_{sdR}(D)\ge 4-n.$$

The equality holds if and only if $1\le\Delta^-(D)\le 2$ and $\Delta^+(D)=n-1$.

Proof. Consider a $\gamma_{sdR}(D)$-function f. If $f(u)\ge 1$ for each $u\in V(D)$, than we have $\gamma_{sdR}(D)\ge n+1>4-n$ as desired. Hence we assume that $f(w)=-1$ for at least one vertex $w\in V(D)$. By definition there exists a vertex $v\in N^-(w)$ such that $f(w)=3$ or there exist two vertices $u,v\in N^-(w)$ such that $f(v)=f(u)=2$. This implies that $\gamma_{sdR}(D)\ge 4-n$ as desired.

If D is a digraph with $1\le\Delta^-(D)\le 2$ and $\Delta^+(D)=n-1$ and $w\in V(D)$ is a vertex with out-degree Δ^+, then the function f defined on D by $f(w)=3$ and $f(x)=-1$ for each $x\in V(D)\setminus\{w\}$, is a SDRD-function on D weight of $4-n$. Hence $\gamma_{sdR}(D)\le 4-n$ and thus $\gamma_{sdR}(D)=4-n$.

Conversely, let $\gamma_{sdR}(D)=4-n$. Suppose $f=(V_{-1},V_1,V_2,V_3)$ is a $\gamma_{sdR}(D)$-function. If $V_{-1}=\emptyset$, then $\omega(f)\ge n+1$ and this leads to a contradiction. Therefore

$V_{-1} \neq \emptyset$. We first show that $\Delta^-(D) \leq 2$. On the contrary, let $\Delta^-(D) \geq 3$ and let $w \in V(D)$ be a vertex of in-degree Δ^- and $N^-(w) = \{w_1, w_2, \ldots, w_{\Delta^-(D)}\}$. If $f(w) = -1$, then there exist two vertices w_i and w_j in $N^-(w)$ such that $f(w_i) + f(w_j) \geq 4$ and this implies that

$$\gamma_{sdR}(D) = f(w_i) + f(w_j) + f(w) + \sum_{x \in V(D) \setminus \{w_i, w_j, w\}} f(x) \geq 3 - (n-3) = 6 - n,$$

contradicting the assumption. If $f(w) = 1$, then for some $1 \leq i \leq t$, $f(w_i) \geq 2$ and we have

$$\gamma_{sdR}(D) = f(w_i) + f(w) + \sum_{x \in V(D) \setminus \{w_i, w\}} f(x) \geq 3 - (n-2) = 5 - n,$$

contradicting the assumption $\gamma_{sdR}(D) = 4 - n$. Therefore $f(w) \geq 2$. Then we must have $f(w_i) \geq 1$ for some $1 \leq i \leq t$ implying that

$$\gamma_{sdR}(D) = \sum_{x \in V(D)} f(x) = f(w_i) + f(w) + \sum_{x \in V(D) - \{w_i, w\}} f(x) \geq 3 - (n-2) = 5 - n,$$

a contradiction with $\gamma_{sdR}(D) = 4 - n$. On the other hand, it follows from $4 - n < 2n$ and Proposition 3.1 that $1 \leq \Delta^-(D)$. Thus $1 \leq \Delta^-(D) \leq 2$.

Next, we show that $\Delta^+(D) = n - 1$. On the contrary, assume that $\Delta^+(D) < n - 1$. Suppose w is a vertex in $V_2 \cup V_3$ and let $u \in V(D) \setminus N^+(w)$. If $f(u) = -1$, then u has an in-neighbor v with $f(v) \geq 2$. Since $v \neq w$, $f(v) + f(u) + f(w) \geq 3$, we have

$$\gamma_{sdR}(D) = f(u) + f(v) + f(w) + \sum_{x \in V(D) \setminus \{v, w, u\}} f(x) \geq 3 - (n-3) = 6 - n$$

which is a contradiction. If $f(u) \geq 1$, then

$$\gamma_{sdR}(D) = f(u) + f(w) + \sum_{x \in V(D) \setminus \{w, u\}} f(x) \geq 3 - (n-2) = 5 - n$$

a contradiction again. Therefore $\Delta^+(D) = n - 1$. □

Proposition 3.5. For any digraph D of order n with $\Delta^+(D) \geq 2$,

$$\gamma_{sdR}(D) \geq \frac{(2 - \Delta^+)n}{\Delta^+} + \frac{2\Delta^+ - 2}{\Delta^+}\gamma(D).$$

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}(D)$-function. Assume that $S = V_{-1} \cap N^+(V_3)$ and $T = V_{-1} \setminus S$. Since each vertex in V_3 dominate at most Δ^+ of S, we have $|S| \leq \Delta^+|V_3|$. Also, since each vertex in V_2 dominate at most Δ^+ vertices of T and since each vertex in T has at least two in-neighbors in V_2, we get $2|T| \leq |E(V_2, T)| \leq \Delta^+|V_2|$ implying that $|T| \leq \frac{\Delta^+}{2}|V_2|$. Thus $|V_{-1}| = |S| + |T| \leq \Delta^+|V_3| + \frac{\Delta^+}{2}|V_2|$. Thus

$$\begin{aligned}
\Delta^+\gamma_{sdR}(D) &= \Delta^+(|V_1| + 2|V_2| + 3|V_3| - |V_{-1}|) \\
&= \Delta^+(|V_1| + |V_2| + |V_3|) + 2\Delta^+|V_3| + \Delta^+|V_2| - \Delta^+|V_{-1}| \\
&\geq \Delta^+(|V_1| + |V_2| + |V_3|) + (2 - \Delta^+)|V_{-1}| \\
&= (2\Delta^+ - 2)(|V_1| + |V_2| + |V_3|) + (2 - \Delta^+)n \\
&\geq (2\Delta^+ - 2)\gamma(D) + (2 - \Delta^+)n,
\end{aligned}$$

(since $V_2 \cup V_3$ is a dominating set of D) and this leads to the desired bound. □

Next we present a lower bound in terms of the order and the domination number. We start with a lemma.

Lemma 3.6. For any digraph D of order $n \geq 2$, $\gamma_{dR}(D) - \gamma_{sdR}(D) + \gamma(D) \leq n$.

Proof. Consider a $\gamma_{sdR}(D)$-function $f = (V_{-1}, V_1, V_2, V_3)$. Note that $\gamma_{sdR}(D) = \omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$ and $\gamma(D) \leq |V_2| + |V_3|$ because $V_2 \cup V_3$ dominate D. Clearly the function g defined on $V(D)$ by $g(x) = 0$ if $x \in V_{-1}$ and $g(x) = f(x)$ otherwise, is a DRDF on D, and so $\gamma_{dR}(D) \leq |V_1| + 2|V_2| + 3|V_3| = \gamma_{sdR}(D) + |V_{-1}|$. This implies that

$$\gamma_{dR}(D) \leq \gamma_{sdR}(D) + n - \gamma(D) - |V_1| \leq \gamma_{sdR}(D) + n - \gamma(D),$$

as desired. □

Theorem 3.7. For any digraph D of order $n \geq 2$, $\gamma_{sdR}(D) \geq 3\gamma(D) - n$.

Proof. By Theorem E and Lemma 3.6, we have

$$\begin{aligned} \gamma_{sdR}(D) &\geq \gamma_{dR}(D) + \gamma(D) - n \\ &\geq 3\gamma(D) - n. \end{aligned}$$

□

For any digraph D, the *complement* $\overline{D}$ of D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u and v, $(u, v) \in \overline{D}$ if and only if $(u, v) \notin D$. Next we present a lower bound on the sum $\gamma_{sdR}(D) + \gamma_{sdR}(\overline{D})$ for r-regular digraphs.

Theorem 3.8. Let D be an r-regular digraph of order n. Then

$$\gamma_{sdR}(D) + \gamma_{sdR}(\overline{D}) \geq \frac{4n}{n+1}.$$

If n is even, then $\gamma_{sdR}(D) + \gamma_{sdR}(\overline{D}) \geq \frac{4(n+1)}{n+2}$.

Proof. Since D is r-regular, its complement $\overline{D}$ is $(n - r - 1)$-regular. Corollary 2.3 implies that

$$\gamma_{sdR}(D) + \gamma_{sdR}(\overline{D}) \geq n\Big(\frac{1}{r+1} + \frac{1}{n-r}\Big).$$

Since the function $g(x) = \frac{1}{(x+1)} + \frac{1}{n-x}$ takes its minimum at $\frac{n-1}{2}$ for $1 \leq x \leq n-1$, we obtain

$$\gamma_{sdR}(D) + \gamma_{sdR}(\overline{D}) \geq n\Big(\frac{2}{n+1} + \frac{2}{n+1}\Big) = \frac{4n}{n+1}$$

as desired. For even n, the function g takes its minimum at $r = x = (n-2)/2$ or $r = x = n/2$, because r is an integer and we have

$$\gamma_{sdR}(D) + \gamma_{sdR}(\overline{D}) \geq n\Big(\frac{1}{r+1} + \frac{1}{n-r}\Big) \geq n\Big(\frac{2}{n} + \frac{2}{n+2}\Big) = \frac{4(n+1)}{n+2}.$$

□

4. A lower bound in terms of chromatic number

In this section we establish a sharp lower bounds on SDRD-number in terms of the order, the maximum degree and the chromatic number of D. To this end, we first determine SDRD-number of an oriented cycles.

Proposition 4.1. *Let $\overrightarrow{C_n} = v_1v_2\ldots v_nv_1$ be an oriented cycle of order $n \geq 2$. Then $\gamma_{sdR}(\overrightarrow{C_n}) = n$ when n is even and $\gamma_{sdR}(\overrightarrow{C_n}) = n+1$ when n is odd.*

Proof. Suppose first that n is even. Clearly, the function $g : V(\overrightarrow{C_n}) \longrightarrow \{-1, 1, 2, 3\}$ defined by $g(v_{2i-1}) = 3$ and $g(v_{2i}) = -1$ for $1 \leq i \leq \frac{n}{2}$ is a SDRD-function on C_n of weight n and so $\gamma_{sdR}(C_n) \leq n$. To prove the inverse inequality, Suppose $f = (V_{-1}, V_1, V_2, V_3)$ is a $\gamma_{sdR}(C_n)$-function. If $V_{-1} = \emptyset$, then the result is immediate. Suppose that $V_{-1} \neq \emptyset$ and let without loss of generality that $V_{-1} = \{v_{i_1}, \ldots, v_{i_t}\}$. By definition we must have $\{v_{i_1-1}, \ldots, v_{i_t-1}\} \subseteq V_3$. It follows that

$$\gamma_{sdR}(\overrightarrow{C_n}) = \sum_{j=1}^{t}(f(v_{i_j}) + f(v_{i_j-1})) + \sum_{x\in V(\overrightarrow{C_n})\setminus\{v_{i_j}, v_{i_{j-1}} | 1\leq j\leq t\}} f(x) \geq n.$$

Thus $\gamma_{sdR}(\overrightarrow{C_n}) = n$ in this case.

Next let n be odd. Clearly, the function g defined on $V(\overrightarrow{C_n})$ by $g(v_{2i-1}) = 3, g(v_{2i}) = -1$ for $1 \leq i \leq \frac{n-1}{2}$ and $g(v_n) = 2$, is a SDRD-function on C_n of weight $n+1$ and so $\gamma_{sdR}(\overrightarrow{C_n}) \leq n+1$. To prove the inverse inequality, let f be a $\gamma_{sdR}(\overrightarrow{C_n})$-function. Similar as above we can see that $|V_3| \geq |V_{-1}|$ and $|V_2| \geq |V_1|$. Since n is odd, we must have $|V_3| > |V_{-1}|$ or $|V_2| > |V_1|$ and this implies that $\gamma_{sdR}(\overrightarrow{C_n}) \geq n+1$. Therefore $\gamma_{sdR}(C_n) = n+1$ when n is odd. □

Applying a similar argument, we can see that if $\overrightarrow{P_n}$ is a directed cycle, then $\gamma_{sdR}(\overrightarrow{P_n}) = n$ when n is even and $\gamma_{sdR}(\overrightarrow{P_n}) = n+1$ when n is odd.

The proof of next result is essentially similar to the proof of Theorem 3 in [4].

Theorem 4.2. *If D is a connected digraph of order $n \geq 3$ and k is a nonnegative integer such that $\delta^+(D) \geq k$, then*

$$\gamma_{sdR}(D) \geq \chi(G) + \left\lceil \frac{3}{2}(k - \Delta(G)) \right\rceil + 3 - n$$

where G is the underlying graph of D. This bound is sharp for oriented K_2.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}(D)$-function. First assume that $\Delta(G) = 1$. Then $\gamma_{sdR}(D) = n$ (note that $G = K_2$), $k = 0$ and $\chi(G) = 2$. Thus $\chi(G) + \lceil\frac{3}{2}(k - \Delta(G))\rceil + 3 - n \leq 5 - n \leq n = \gamma_{sdR}(D)$. Suppose now that $\Delta(G) = 2$. Then G is a path or a cycle and $k \leq 1$. Therefore $\chi(G) \leq 3$ and the equality holds if and only if G is an odd cycle. If G is not an odd cycle, then $\chi(G) = 2$ and the result obtain from Theorem 3.4. Suppose G is an odd cycle. If $k = 0$, then the result follows by Theorem 3.4. If $k = 1$, then D is a directed cycle of order odd and by Proposition 4.1 we have $\gamma_{sdR}(D) = n+1$. Hence $\chi(G) + \lceil\frac{3}{2}(k - \Delta(G))\rceil + 3 - n \leq 4 - n < n + 1 = \gamma_{sdR}(D)$.

Finally, assume that $\Delta(G) \geq 3$. Suppose $\mu = \frac{3\Delta(G)-3k-2}{4}$. We show that $k \leq \Delta(G) - 2$. Suppose, to the contrary, that $k \geq \Delta(G) - 1$. Since $k \leq d^+(x)$ and $d^-(x) + d^+(x) \leq \Delta(G)$ for each $x \in V(D)$, we have $d^-(x) \leq 1$ for for each $x \in V(D)$.

But then $\Delta(G) - 1 \leq \frac{1}{n}\sum_{v\in V} d^+(x) = \frac{1}{n}\sum_{v\in V} d^-(x) \leq 1$ and this leads to a contradiction. Hence $k \leq \Delta(G) - 2$ and so $\mu \geq 1$. For each $x \in V_{-1}$,

$$|E(V_{-1}, x)| \leq 3|E(V_3, x)| + 2|E(V_2, x)| + |E(V_1, x)| - 2$$

and so

$$\begin{aligned}\Delta(G) \geq \deg(x) &= |E(V_{-1}, x)| + |E(V_3, x)| + |E(V_2, x)| + |E(V_1, x)| + d^+(x)\\ &\geq |E(V_3, x)| + \frac{2|E(V_2, x)|}{3} + \frac{|E(V_1, x)|}{3} + |E(V_{-1}, x)| + k\\ &\geq \frac{4|E(V_{-1}, x)|}{3} + k + \frac{2}{3}\end{aligned}$$

which implies that $|E(V_{-1}, x)| \leq \frac{3\Delta(G)-3k-2}{4} = \mu$. Assume $H = D[V_{-1}]$ is the subdigraph induced by V_{-1} and let $H' = G[V_{-1}]$ be the underlying graph of H. Let H_1 be an induced subdigraph of H. Then $d^-(x) \leq |E(V_{-1}, x)| \leq \mu$ for each $x \in H_1$, and hence $\Sigma_{x\in V(H_1)} d^+(x) = \Sigma_{x\in V(H_1)} d^-(x) \leq \mu|V(H_1)|$. Hence there exists a vertex $x \in V(H_1)$ such that $d^+(x) \leq \mu$. It follows that $\delta(H_1') \leq 2\mu$, where H_1' is the underlying graph of H_1. We conclude from Proposition F that

$$\begin{aligned}\chi(H') &\leq 1 + \max\{\delta(H'') \mid H'' \text{ is a subgraph of } H'\}\\ &\leq 1 + 2\mu.\end{aligned}$$

Since $|V_1| + |V_2| + |V_3| = n - |V_{-1}| \leq n + \gamma_{sdR}(D) - 3$, we have

$$\begin{aligned}\chi(G) &\leq \chi(G[V_{-1}]) + \chi(G[V_1 \cup V_2 \cup V_3])\}\\ &\leq 2\mu + 1 + |V_1| + |V_2| + |V_3|\\ &\leq 2\mu + n + \gamma_{sdR}(D) - 2.\end{aligned}$$

Thus $\gamma_{sdR}(D) \geq \chi(G) + \frac{3}{2}(k - \Delta(G)) + 3 - n$, as desired. □

5. Two families of tournaments

A *tournament* is a digraph in which for every pair u, v of different vertices, either $(u, v) \in A(D)$ or $(u, v) \in A(D)$, but not both. In this section we determine the exact value of the SDRD-number of two families of tournaments.

The *acyclic tournament* $AT(n)$ with n with $V(AT(n)) = \{u_1, u_2, \ldots, u_n\}$ and an arcs goes from u_i into u_j whenever $i < j$.

Let n be an odd positive integer such that $n = 2r + 1$ with a positive integer r. We define the *circulant tournament* $CT(n)$ with n vertices as follows. The vertex set of $CT(n)$ is $V(CT(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$. For each i, the arcs going from u_i to the vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken module n.

Proposition 5.1. *If $n \neq 4$, then $\gamma_{sdR}(AT(n)) = 1$ and $\gamma_{sdR}(AT(4)) = 2$.*

Proof. It is easy to see that $\gamma_{sdR}(AT(4)) = 2$. Let $n \neq 4$ and let f be a $\gamma_{sdR}(AT(n))$-function. Since $V(AT(n)) = N^-[u_n]$, we have $\gamma_{sdR}(AT(n)) = f(N^-[u_n]) \geq 1$.

To prove the inverse inequality, we define a SDRD-function f on $V(AT(n))$ of weight 1 as follows: if n is odd, then we define

$$f(u_1) = 3, f(u_2) = f(u_3) = -1 \ \text{ and } \ f(u_i) = (-1)^i \ \text{ for } \ 4 \leq i \leq n$$

and if $n \geq 6$ is even, then we define

$$f(u_1) = 3, f(u_2) = 2, f(u_3) = f(u_4) = f(u_5) = f(u_n) = -1 \text{ and}$$
$$f(u_i) = (-1)^i \text{ for } 6 \leq i \leq n-1.$$

It is easy to see that f is a SDRD-function on $AT(n)$ with $\omega(f) = 1$. Therefore $\gamma_{sdR}(AT(n)) \leq 1$ and so $\gamma_{sdR}(AT(n)) = 1$. This complete the proof. □

Proposition 5.2. *Let $n = 2r + 1$ where r is a positive integer. If $r \neq 1$, then $\gamma_{sdR}(CT(n)) = 3$ and $\gamma_{sdR}(CT(3)) = 4$.*

Proof. By Proposition 4.1, we have $\gamma_{sdR}(CT(3)) = 4$. Suppose $r \geq 2$ and $f = (V_{-1}, V_1, V_2, V_3)$ is a $\gamma_{sdR}(CT(n))$-function. If $V_{-1} = \emptyset$, then obviously $\omega(f) \geq n+2 > 3$. Assume that $V_{-1} \neq \emptyset$ and let without loss of generality that $u_0 \in V_{-1}$. As f is a SDRD-function, we have

$$\omega(f) = f(N^-[u_0]) + f(N^-[u_r]) - f(u_0) \geq 3.$$

This implies that $\gamma_{sdR}(CT(3)) \geq 3$.

To prove the inverse inequality, we define a SDRD-function f on $V(CT(n))$ of weight 3 as follows: if $r = 2$ then we define $f(u_0) = f(u_3) = 3$ and $f(x) = -1$ otherwise; if $r = 4$ then we define $f(u_3) = f(u_4) = f(u_7) = f(u_8) = 2$ and $f(x) = -1$ otherwise; if $r \geq 6$ is even, then we define $f(u_{\frac{r+2}{2}}) = f(u_{\frac{r+4}{2}}) = f(u_{\frac{3r+2}{2}}) = f(u_{\frac{3r+4}{2}}) = 2, f(u_i) = -1$ for $0 \leq i \leq \frac{r}{2}$ or $r + 1 \leq i \leq r + \frac{r}{2}$ and $f(x) = 1$ otherwise; and if r is odd, then we define $f(u_0) = f(u_{r+1}) = 3, f(u_1) = f(u_{r+2}) = 2, f(u_2) = f(u_3) = f(u_4) = f(u_r) = f(u_{r+3}) = f(u_{r+4}) = -1, f(u_i) = (-1)^{i+1}$ for $4 \leq i \leq r-1$ and $f(u_{r+i}) = (-1)^i$ for $5 \leq i \leq r$. Obviously, f is a SDRD-function on $CT(n)$ with $\omega(f) = 3$. Hence $\gamma_{sdR}(CT(n)) \leq 3$ and thus $\gamma_{sdR}(CT(n)) = 3$. This complete the proof. □

Conclusion

The main objective of this paper is to study the signed double Roman domination number of a digraph D. We first investigate basic properties of signed double Roman domination number and then we establish sharp bounds on the signed double Roman domination number and determine signed double Roman domination number of acyclic tournament and circulant tournament. Analogous work can be carried out for other digraph parameters such as signed total double Roman domination number.

References

[1] H. Abdollahzadeh Ahangarand, M. Chellali, and S.M. Sheikholeslami, Signed double Roman domination in graphs, *Discrete Appl. Math.* **257** (2019), 1–11.

[2] H. Abdollahzadeh Ahangarand, M. Chellali, and S.M. Sheikholeslami, Signed double Roman domination of graphs, *Filomat* **33** (2019), 121–134.

[3] J. Amjadi, H. Yang, S. Nazari-Moghaddam, S.M. Sheikholeslami, and Z. Shao, Signed double Roman k-domination in graphs, *Australas. J. Combin.* **72** (2018), 82-–105.

[4] H. Karami, A. Khodkar, and S.M. Sheikholeslami, Lower bounds on signed domination number of a digraph, *Discrete Math.* **309** (2009), 2567–2570.

[5] T.W. Haynes and S.T. Slater PJ, *Fundamental of Dimination in Graphs*, Marcel Dekker, Inc., New York,(1998).

[6] G. Hao, X. Chen, and L. Volkmann, *Double Roman domination in digraphs*, Bull. Malayas. Math. Sci. Soc. **42** (2019), 1907-1920.

[7] L. Shahbazi, H. Abdollahzadeh Ahangar, R. khoeilar, and S.M. Sheikholeslami, Bounds on signed total double Roman domination, *Commun. Comb. Optim.* **5** (2020), 191–206.

[8] G. Szekeres and H.S. Wilf, *An inequality for the chromatic number of a graph*, J. Combin. Theory **4** (1963), 1-3.

[9] H. Yang, P. Wu, S. Nazari-Moghaddam, S.M. Sheikholeslami, X. Zhang, Z. Shao, and Y.Y. Tang, Bounds for signed double Roman k-domination in trees, *RAIRO - Operations Research* **53** (2019), 627–643.

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