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Bifurcation analysis in a 3D symmetric system of interest in fusion plasma physics

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Abstract. In this paper we study the pitchfork and Hopf bifurcations in a 3D dynamical system that models some oscillations of plasma parameters in Tokamaks. The pitchfork bifurcation's parameter is the input power in the system and the Hopf bifurcation is analyzed with respect of the three parameters of the system. In each case necessary and sufficient conditions for the occurrence of the Hopf bifurcation are provided. A special attention is paid to the case when the input power is varied. A case study is performed. It is emphasized the stable periodic orbits collision occurred after Hopf bifurcation, and the appearance of the double-scroll attractor. It is also noticeable the coexistence of three attractors (two stable points and a double-scroll attractor) whose basins of attraction are analyzed.

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1. Introduction

The aim of this paper is to analyze some bifurcations of a 3D-dynamical system which was introduced in [1], in order to model some oscillations of plasma's parameters in Tokamaks (experimental toroidal devices used for obtaining energy through nuclear fusion).

Common properties can be found between the model mentioned before and the Lorenz system [2], which is considered to be a benchmark for chaotic systems theory: it is dissipative and invariant under the change of coordinates $S(x, y, z) = (-x, -y, z)$. It is characterized by its complex dynamics, which is comparable to the one of Lorenz's system [3] (double scroll butterfly-like attractor, coexistence of many attractors in the phase space, bifurcations) even if it can be categorized as a part of Lorenz-like family systems [4] because the non-linear perturbation is not a quadratic polynomial.

But the model has also specific properties, for example the existence of two different time scales. For small values of one of the parameters it is a fast-slow system, with two fast variables and a slow one. Its dynamics explains the oscillations of the sawtooth-type, characteristic for fusion plasma [5].

In this paper we focus on the study of pitchfork and Hopf bifurcations because the formation/destruction of periodic orbits is related with the oscillations observed in experiments. More of this, they represent the first steps in the scenario for passing from regular dynamics to chaotic one.

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The paper is organized as follows: in Section 2 the mathematical model and its general properties are presented; Section 3 is devoted to the study of pitchfork bifurcation and Hopf bifurcation is studied in Section 4. Conclusions are formulated in Section 5.

2. The mathematical model. General properties of the system

The model is based on two equations. The first one express the magnetohydrodynamic (MHD) force balance and the second one comes from the energy conservation (see [1] for details). In the initial model t is the normalized time, $y = y(t)$ is the normalized displacement of the magnetic field due to instability, $z = z(t)$ is the normalized plasma pressure gradient:

$$\begin{cases} y'' = (z-1)\cdot y - \delta \cdot y' \\ z' = \eta \cdot (h - z - y^2 \cdot z) \end{cases} \tag{1}$$

There are three parameters which determine the behavior of the system: $h > 0$, $\delta > 0$, $\eta > 0$. h is the input power in the system, δ and η are related to the dissipation/relaxation of the MHD instability respectively to diffusion phenomena.

The system (1) can be transformed into a system of differential equations of first order, that will be the object of our study:

$$\begin{cases} x' = (z-1)\cdot y - \delta \cdot x \\ y' = x \\ z' = \eta \cdot (h - z - y^2 \cdot z). \end{cases} \tag{2}$$

The general properties of (2), which is generated by the vector field

$$X = ((z-1)\cdot y - \delta \cdot x,\ x,\ \eta \cdot (h - z - y^2 \cdot z)) \tag{3}$$

are similar with those of Lorenz system.

2.1. Dissipativity. The system (2) is dissipative because $div(X) = -(\delta+\eta+\eta\cdot y^2) < 0$.

Because $div(X) < -(\delta+\eta)$ it results that a volume element V_0 is contracted by the flow into a volume element smaller than $V_0 \cdot e^{-(\delta+\eta)t}$ and all bounded trajectories ultimately arrive at an attractor of 0-volume.

The contraction factor is not constant, so the regions where y^2 is large will be contracted faster than the regions where $y^2 \approx 0$.

2.2. Symmetry. The system (2) is invariant under the change of coordinates $S(x, y, z) = (-x, -y, z)$, namely the symmetry with respect to the Oz axis.

The symmetry group is $\mathbf{Z}^2 = \{Id_{\mathbf{R}^3}, S\}$ and we can use some specific techniques to study the bifurcations (see [6], pp 276-288).

The space $\mathbf{R}^3$ is decomposed into a direct sum

$$\mathbf{R}^3 = X^+ \oplus X^-$$

where $S(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X^+$ and $S(\mathbf{x}) = -\mathbf{x}$ for all $\mathbf{x} \in X^-$.

The fixed points subspace of S, namely

$$X^+ = \{\mathbf{x} \in \mathbf{R}^3 \ \mid\ S(\mathbf{x}) = \mathbf{x}\} = \{(0,0,z) \ \mid\ z \in \mathbf{R}\}$$

is an invariant set of the system (2).

2.3. Equilibria and their stability.

Proposition 2.1. *a) Consider the system (2) and assume $h < 1$. The system has one equilibrium point $P_1 = (0, 0, h) \in X^+$ which is a hyperbolic sink.*

b) Assume $h > 1$. The system (2) has three equilibria: $P_1 = (0, 0, h)$ and $P_{2,3} = (0, \pm\sqrt{h-1}, 1)$. Moreover, $P_1 = (0, 0, h) \in X^+$ is a saddle point with two dimensional stable manifold and one dimensional unstable manifold locally characterized

$$W^u_{loc}(0,0,h) = \{(x, \frac{1}{\lambda_3}x + O(x^2 t), h + \frac{\eta h x^2}{2[h-1+\lambda_3(\delta-\eta)]} + O(x^3)), \quad |x| < 1\}$$

and the two S-conjugate equilibria $P_{2,3} = (0, \pm\sqrt{h-1}, 1)$ are stable if

$$\delta h(\delta + \eta h) - 2(h-1) > 0.$$

Proof. The eigenvalues of the Jacobian matrix in $(0, 0, h)$ are $\lambda_1 = -\eta$, $\lambda_2 = -\frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} + h - 1}$ and $\lambda_3 = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + h - 1}$.

a) and b) For $h < 1$ all of them have negative real part (hence P_1 is attractor) and for $h > 1$ two of them are negative (λ_1 and λ_2) and the third one is positive.

b) The characteristic polynomial of the system in P_2 and P_3 is

$$g(\lambda) = \lambda^3 + (\delta + \eta h)\lambda^2 + \delta\eta h\lambda + 2\eta(h-1). \tag{4}$$

Its eigenvalues cannot be computed explicitly, but the parameter's values for marginal stability may be obtained from Routh-Hurwitz criterion: P_2 and P_3 are stable if $a_1 = (\delta + \eta h) > 0$, $a_3 = 2\eta(h-1) > 0$ and $a_1 a_2 - a_3 = \delta\eta h(\delta + \eta h) - 2\eta(h-1) > 0$. The first conditions are trivially satisfied for $h > 1$. Thus the equilibrium points $P_{2,3}$ are stable if the third inequality is satisfied and $h > 1$. Because $\eta > 0$ this leads to (2.1).

The eigenvector corresponding to λ_3 is $\overrightarrow{v}_3 = (1, \frac{1}{\lambda_3}, 0)$.

The local unstable manifold of $P_1 = (0, 0, h)$ is a curve having the equation

$$y = H(x), \quad z = K(x)$$

with the initial conditions

$$H(0) = 0, \quad K(0) = h, \quad H'(0) = \frac{1}{\lambda_3}, \quad K'(0) = 0.$$

If we consider $H(x) = A + Bx + Cx^2 + O(x^3)$ and $K(x) = D + Ex + Fx^2 + O(x^3)$ and we impose the initial conditions, we obtain $A = 0$, $B = \frac{1}{\lambda_3}$, $D = h$ and $E = 0$.

Introducing the expressions of y and z in the first equation of (2) we obtain

$$x' = (K(x) - 1)H(x) - \delta x.$$

From the equality of the free terms and the coefficients of x the third equation of (2), i.e. $K'(x)x' = \eta(h - K(x) - H^2(x)K(x))$, we obtain

$$\begin{aligned} C &= 0 \\ F &= \frac{\eta h}{2\,[h - 1 + \lambda_3(\delta - \eta)]} \end{aligned}$$

It means that $y = \frac{1}{\lambda_3}x + O(x^2)$ and $z = h + \frac{\eta h x^2}{2[h-1+\lambda_3(\delta-\eta)]} + O(x^3)$. $\square$

3. Bifurcations of the equilibrium point $P_1 = (0, 0, h)$ (pitchfork bifurcation)

Proposition 3.1. *Suppose that $h = 1$. The system (2) undergoes a supercritical pitchfork bifurcation at $P_1 = (0, 0, 1)$. More precisely if $h < 1$ there is an unique equilibrium point $P_1 = (0, 0, h)$ which is stable and for $h > 1$ there are three equilibria: $P_1 = (0, 0, h)$ (which is unstable) and $P_{2,3} = (0, \pm\sqrt{h-1}, 1)$ which are stable near $h = 1$.*

Proof. The Jacobi matrix $J(0,0,1) = \begin{pmatrix} -\delta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\eta \end{pmatrix}$ has the eigenvalues $\lambda_1 = -\eta$, $\lambda_2 = -\delta$, $\lambda_3 = 0$.

The eigenvector of λ_3 is $v_3 = (0, y, 0) \in X^-$. In this case the center manifold W_h^c intersects the fixed subspace X^+ only in $P_1 = (0, 0, 1)$ and, for small enough values of $h - 1$ the restriction of the system to W_h^c is locally equivalent with the normal form $\dot{\xi} = \beta\xi \pm \xi^3$ (Theorem 7.7, pp 281, [6]).

A pitchfork bifurcation occurs in $h = 1$. For $h > 1$ the stable equilibrium becomes unstable and a pair of S-conjugate equilibria, $P_{2,3} = (0, \pm\sqrt{h-1}, 1)$, is formed.

The equilibria $P_{2,3}$ are stable if $s(h) = \delta h(\delta + \eta h) - 2(h - 1) > 0$. Because $s(1) = \delta^2 + \eta > 0$ it results that $s(h) > 0$ for h in a neighborhood of 1, i.e. $P_{2,3}$ are stable near $h = 1$.

In Figure 1 we can see the bifurcation diagram of the system (2) for $\delta = 0.3$, $\eta = 1$. The bifurcation parameter is h. The pitchfork bifurcation, which occurs in $h = 1$ is labelled "A", the continuous lines represent the stable equilibrium points and the dashed ones correspond to repelling equilibrium points. □

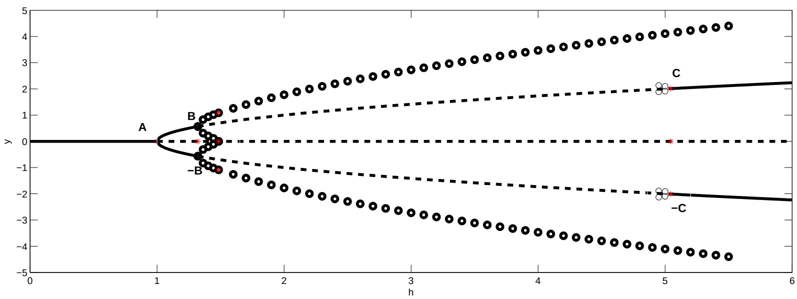

FIGURE 1. Bifurcation diagram of system (2) : $\delta = 0.3$, $\eta = 1$.

4. Bifurcations of the equilibrium points $P_{2,3} = (0, \pm\sqrt{h-1}, 1)$ (Hopf bifurcation)

To study in detail Hopf's bifurcation we need to use Hopf Theorem ([7], pp 150-156). This implies the existence of a single bifurcation parameter, but in our case we have three such parameters. In that case we can suppose, because of system's structure, that two parameters are fixed and one is varying. Moreover, we should not choose between the twin equilibrium points, as their properties are identical.

At the Hopf bifurcation points the characteristic polynomial (4) has a simple pair of purely imaginary eigenvalues $\pm i\beta$ and the non-vanishing real value α. To locate the Hopf points, we thus try to satisfy the condition

$$g(\lambda) = (\lambda - \alpha)(\lambda - i\beta)(\lambda + i\beta) \quad \text{with } \alpha, \beta \neq 0. \tag{5}$$

Comparison of the coefficients of (4) and (5) leads to

$$\alpha = -(\delta + \eta h), \qquad \beta^2 = \delta\eta h, \qquad -\alpha\beta^2 = \delta\eta h(\delta + \eta h) = 2\eta(h-1).$$

Because $\eta > 0$, the last equation reads

$$\delta h(\delta + \eta h) - 2(h-1) = 0. \tag{6}$$

If (6) is fulfilled, the eigenvalues are $\lambda_{1,2} = \pm i\sqrt{\delta\eta h} \overset{notation}{=} \pm i\omega$ and $\lambda_3 = -(\delta + \eta h)$.

Two nondegeneracy conditions must be satisfied in order to have a Hopf bifurcation:

- the transversality condition ($\frac{dRe(\lambda)}{dh}(h_1) \neq 0$, respectively $\frac{dRe(\lambda)}{dh}(\delta_1) \neq 0$ or $\frac{dRe(\lambda)}{dh}(\eta_1) \neq 0$) depends on which parameter is considered as bifurcation-parameter (see the next subsections).

- the first Lyapunov coefficient in the Hopf bifurcation point is not null.

The computation of the first Lyapunov coefficient can be done without specifying which is the bifurcation parameter. In order to do this we used the projection method for computing the central manifold ([6] pp 171-181, [8] pp 193-201). The detailed computations are presented in Annex A.

Proposition 4.1. *The first Lyapunov coefficient in the equilibrium points $P_{2,3}$ when the condition (6) is fulfilled is*

$$l_1(\delta, \eta, h) = \frac{A_1\omega^8 + A_2\omega^6 + A_3\omega^4 + A_4\omega^2 + A_5}{4\delta\omega^3((\eta h + \delta)^2 + \omega^2)((\eta h + \delta)^2 + 4\omega^2)(h-1)^2}$$

where

$$\begin{aligned}
A_1 &= -5h^2 + 50h - 52\\
A_2 &= \delta^2(5h^2 + 20h - 32) - \eta^2 h^2(h^2 - 8h + 8) - 10h^2 - 27h + 53\\
A_3 &= \delta^4(h^2 + 2h - 4) - \delta^2(7h^2 + h - 12) - 16\delta\eta + \\
&\quad +\eta^2 h(h-1)(17h^2 - 50h + 28) + 6h^2 + 10h - 38\\
A_4 &= \delta^4(1-h) - 4\delta^3\eta + 2\delta^2(h-2) + 22\delta\eta - 2\eta^2 h(7h-12)(h-1)^2\\
A_5 &= 2\delta^3 h.
\end{aligned}$$

4.1. Hopf bifurcation with respect to η. We consider fixed $\delta > 0$, $h > 1$ and $\eta > 0$ as bifurcation parameter.

Proposition 4.2. *The system (2) undergoes a Hopf bifurcation of the equilibrium points $P_{2,3} = (0 \pm \sqrt{h-1}, 1)$ in*

$$\eta_H = \frac{2(h-1) - \delta^2 h}{\delta h^2} \tag{7}$$

iff $\delta^2 < 2(h-1)/h$ and $l_1(\delta, \eta_H, h) \neq 0$.

Proof. From (6) it results $\eta_H = \frac{2(h-1)-\delta^2 h}{\delta h^2} > 0$. In this case the equilibrium point P_2 has the eigenvalues $\lambda_{1,2} = \pm i\beta(\eta_H)$, $\lambda_3 = -(\delta + \eta_H h)$, where $\beta(\eta_H) = \sqrt{\delta\eta_H h}$. We denote $\lambda = \alpha(\eta) + i \cdot \beta(\eta)$, we differentiate the equality $\lambda^3 + (\delta + \eta h)\lambda^2 + \delta\eta h\lambda + 2\eta(h-1) = 0$ and we obtain

$$\frac{d\alpha}{d\eta}(\eta_H) = \frac{-\delta\eta_H h^2}{2\left[\delta\eta_H h + (\delta + \eta_H h)^2\right]} \neq 0.$$

The transversality condition, essential in the Hopf theorem is fulfilled. Moreover, the periodic orbits appear for $\eta < \eta_H$ (bellow η_H) because $\frac{d\alpha}{d\eta}(\eta_H) < 0$. □

Remark 4.1. From Hopf theorem also results that the period T_{η_H} of the periodic solutions for vanishing amplitude is $T_{\eta_H} = \frac{2\pi}{\sqrt{\delta\eta_H h}}$.

4.2. Hopf bifurcation with respect to δ. We consider fixed $\eta > 0$, $h > 1$ and $\delta > 0$ as bifurcation parameter.

Proposition 4.3. *The system (2) undergoes a Hopf bifurcation of the equilibrium points $P_{2,3} = (0 \pm \sqrt{h-1}, 1)$ in*

$$\delta_H = \frac{-\eta h + \sqrt{\eta^2 h^2 + 8h(h-1)}}{2}$$

iff $l_1(\delta_H, \eta, h) \neq 0$. The periodic orbit is formed for $\delta < \delta_H$. It is stable if $l_1(\delta_H, \eta, h) > 0$ and repelling if $l_1(\delta_H, \eta, h) < 0$.

Proof. The equation (6) has one positive solution

$$\delta_H = \frac{-\eta h + \sqrt{\eta^2 h^2 + 8h(h-1)}}{2}.$$

In this case the equilibrium point P_2 has the eigenvalues $\lambda_{1,2} = \pm i\beta(\delta_H)$, $\lambda_3 = -(\delta_H + \eta h)$, where $\beta(\delta_H) = \sqrt{\delta_H \eta h}$. We denote $\lambda = \alpha(\delta) + \beta(\delta)$, we differentiate the equality $\lambda^3 + (\delta + \eta h)\lambda^2 + \delta\eta h\lambda + 2\eta(h-1) = 0$ and we obtain

$$\frac{d\alpha}{d\delta}(\delta_H) = -\frac{\eta h(2\delta_H + \eta h)}{2\left[\delta_H \eta h + (\delta_H + \eta h)^2\right]} \neq 0.$$

The transversality condition, essential in the Hopf theorem is fulfilled. Moreover, the periodic orbits appear for $\delta < \delta_H$ (bellow δ_H) because $\frac{d\alpha}{d\delta}(\delta_H) < 0$. □

From Hopf theorem also results that the period T_{δ_H} of the periodic solutions for vanishing amplitude is $T_{\delta_H} = \frac{2\pi}{\sqrt{\delta_H \eta h}}$.

4.3. Hopf bifurcation with respect to h. We consider fixed $\delta > 0$, $\eta > 0$ and h as bifurcation parameter.

This is the most important case, because h is the bifurcation parameter for the pitchfork bifurcation. It is also interesting from a practical point of view because in real Tokamak devices the input power in the system (i.e. h in our notation) can be modified in order to obtain different experimental results.

Proposition 4.4. *The system (2) undergoes a Hopf bifurcation of the equilibrium points* $P_{2,3} = (0, \pm\sqrt{h-1}, 1)$ *in*

$$h_1 = \frac{(2-\delta^2) - \sqrt{(2-\delta^2)^2 - 8\delta\eta}}{2\delta\eta} \text{ and } \quad h_2 = \frac{(2-\delta^2) + \sqrt{(2-\delta^2)^2 - 8\delta\eta}}{2\delta\eta} \tag{8}$$

iff
a) $0 < \delta < \sqrt{2}$, $\eta < \frac{(2-\delta^2)^2}{8\delta}$
b) $\delta\eta h_1^2 \neq 2$, *respectively* $\delta\eta h_2^2 \neq 2$
c) $l_1(\delta, \eta, h_1) \neq 0$, *respectively* $l_1(\delta, \eta, h_2) \neq 0$.

Remark 4.2. If $\delta\eta h_1^2 > 2$ (respectively $\delta\eta h_1^2 < 2$) the Hopf bifurcation is above (respectively bellow) h_1, i.e. the periodic orbits exist for $h > h_1$ (respectively for $h < h_1$). Similar results for h_2.

Proof. The Hopf bifurcation may occur if the condition (6) is fulfilled.

The solutions of the equation $f(h) = \delta h(\delta + \eta h) - 2(h-1) = 0$ are $h_{1,2} = \frac{(2-\delta^2) \pm \sqrt{(2-\delta^2)^2 - 8\delta\eta}}{2\delta\eta}$. In order to have a Hopf bifurcation we must impose that at least one solution is larger than 1.

Because $f(1) = \delta^2 + \delta\eta > 0$ we can not be in the situation $h_1 < 1 < h_2$ so we must impose $1 < h_1 \leq h_2$.

This is equivalent to the system $\begin{cases} (2-\delta^2)^2 - 8\delta\eta \geqslant 0 \\ -\frac{\delta^2 - 2}{\delta\eta} > 1 \end{cases}$ i.e.

$$\begin{cases} \eta \leq \frac{(2-\delta^2)^2}{8\delta} = \frac{2-\delta^2}{2\delta} \cdot \frac{2-\delta^2}{4} < \frac{2-\delta^2}{2\delta} \cdot \frac{2}{4} \\ \eta < \frac{2-\delta^2}{\delta} \end{cases}$$

It results that $1 < h_1 \leq h_2$ iff $0 < \delta < \sqrt{2}$ and $\eta < \frac{(2-\delta^2)^2}{8\delta}$.

If $h_1 > 1$ is a solution of the equation $\delta h(\delta + \eta h) - 2(h-1) = 0$, the equilibrium point P_2 has the eigenvalues $\lambda_{1,2} = \alpha(h_1) \pm i \cdot \beta(h_1)$, $\lambda_3 = -(\delta + \eta h_1)$, where $\alpha(h_1) = 0$ and $\beta(h_1) = \sqrt{\delta\eta h_1}$. We denote $\lambda = \alpha(h) + i \cdot \beta(h)$, we differentiate the equality $\lambda^3 + (\delta + \eta h)\lambda^2 + \delta\eta h\lambda + 2\eta(h-1) = 0$ and we obtain

$$\frac{d\alpha}{dh}(h_1) = \frac{\eta(2 - \delta\eta h_1^2)}{2h_1\left[\delta\eta h_1 + (\delta + \eta h_1)^2\right]}.$$

The transversality condition, essential in the Hopf theorem is fulfilled iff $\delta\eta h_1^2 \neq 2$. □

Remark 4.3. From Hopf theorem also results that the period T_i of the periodic solutions for vanishing amplitude is $T_i = \frac{2\pi}{\sqrt{\delta\eta h_i}}$, $i \in \{1, 2\}$.

The following case study is edifying.

For $\delta = 0.3$ and $\eta = 1$, we obtains $h_1 = 1.3213$ and $h_2 = 5.0453$. The transversality conditions are fulfilled: $\frac{d\alpha}{dh}(h_1) = 0.1846 > 0$, so the periodic orbits exist for $h > h_1$ and $\frac{d\alpha}{dh}(h_2) = -0.0186$, so the periodic orbits exist for $h < h_2$. Because $l1(h_1) = -1.2111 < 0$ it results that the bifurcation is supercritical and a unique stable limit cycle bifurcates from the equilibrium point for $h > h_1$ (see [6] pp 179). We have $l1(h_2) = 0.1088 > 0$ so the bifurcation is subcritical, which means that a periodic orbit (that is unstable in our case) exists for $h < h_2$ becomes smaller and smaller when h approaches h_2 and disappears for $h = h_2$.

The results are confirmed by Figure 1. There the twin Hopf bifurcations points corresponding to h_1 and h_2 are denoted by B and $-B$, respectively by C and $-C$. A stable periodic orbit of period $T(h)$ is formed when h increases and becomes larger than h_1, respectively decreases and becomes smaller than h_2. The circles in Figure 1 indicates the extremum values of $y(t)$, for $0 \leq t \leq T(h)$ versus h. When $h > h_1$ increases, the periodic orbits approach and numerical simulations show that they collide in P_1 at $h_0 \approx 1.4849$ as it is presented in Figure 2.

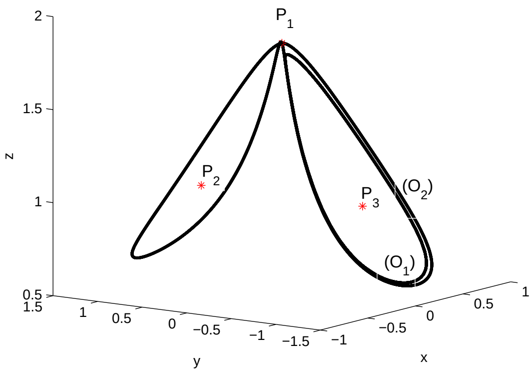

FIGURE 2. Periodic orbits (O_2) of the system (2) for $\delta = 0.3$, $\eta = 1$, $h_0 = 1.4849$ collide in the equilibrium point P_1. The orbit (O_1) is obtained for $\delta = 0.3$, $\eta = 1$, $h_0 = 1.4840$.

Details about this phenomenon are presented in Figure 3 : the periods of the separated twin periodic orbits are slowly increasing for $h \in [1.4820, 1.4849]$ and the orbits get closer and closer. In Figure 3 the periods are represented through the curve (P_1). For $h >= 1.4849$ a double scroll attracting set is formed through the collision of

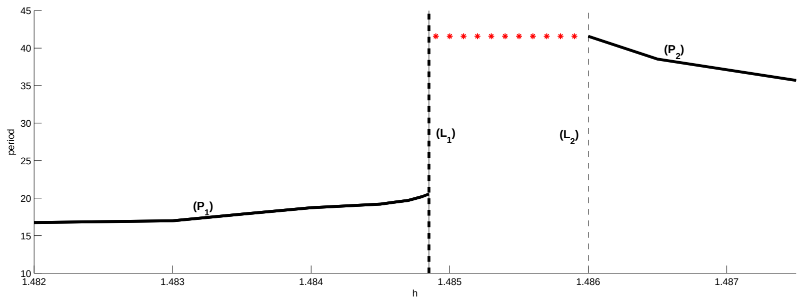

FIGURE 3. Periods of periodic attracting orbits for $\delta = 0.3, \eta = 1, h \in [1.4820, 1.4859]$.

these periodic orbits. Due to the complexity of the basins of attraction of each orbit, the dynamics on this double scroll attracting set is not periodic if $h \in [1.4849, 1.4859]$, the two wings of the attracting set being occupied in an irregular way as shown in Figure 4.

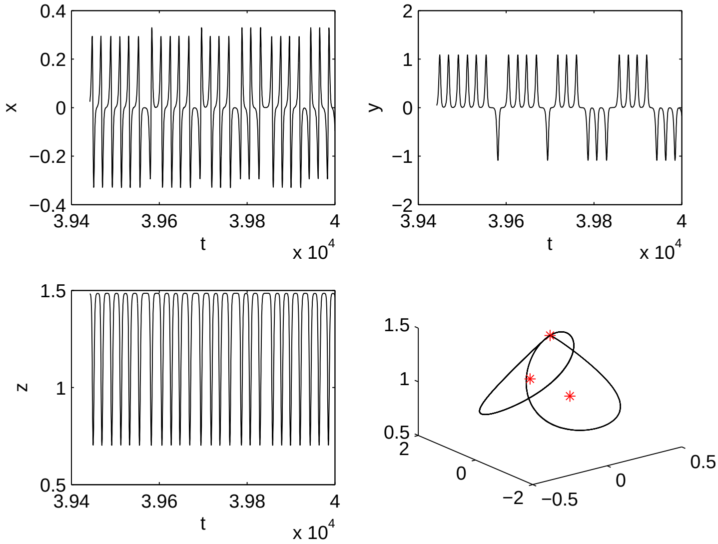

FIGURE 4. Non-periodic behavior of double-scroll attractor for $\delta = 0.3,\ \eta = 1,\ h = 1.4851$.

In Figure 3 the line (L_1) point out the formation of the double scroll attracting limit and the line (L_2) shows the limit for recovering periodicity. For $h >= 1.4860$ the dynamics becomes periodic and the period is practically the double of the period of the twin periodic orbit. These periods are represented in Figure 3 through the curve (P_2).

Increasing h, one can observe that the period is decreasing. In the same conditions the length of the attracting periodic orbit is increasing, which means that the points are moving faster on the orbit. This result is confirmed by the computation of the mean velocity for increasing values of h.

The double scroll periodic orbit goes away from P_1 when h increases.

For $h > h_2$ three attractors coexist: the twin stable equilibrium points $P_{2,3}$ and the double scroll attractor.

In order to understand this phenomenon, various values of $h \in [h_2, 8.5]$ were considered and the intersection of the line $x = 0$, $z = 1$ with the basins of attraction of $P_{2,3}$ and with the double scroll attractor was plotted in Figure 5.

This line was chosen because it contains the equilibrium points $P_{2,3}$ for all $h > 1$. The curves (P_2) and (P_3) indicate the position of the corresponding equilibrium points; $(B\sup P_2)$ and $(B\inf P_2)$, respectively $(B\sup P_3)$ and $(B\inf P_3)$ represent the borders of the basin of attraction of P_2, respectively P_3. Only the points situated between them are attracted by P_2, respectively P_3. The intersections of the double

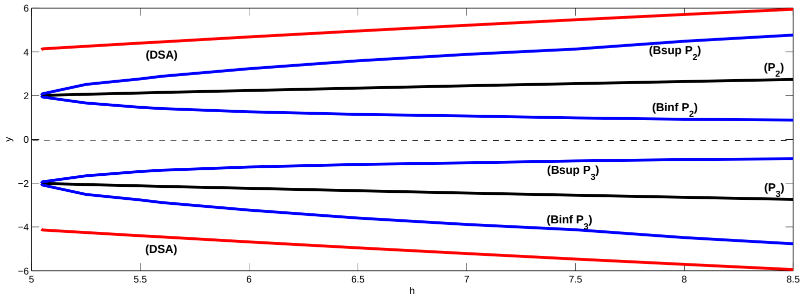

FIGURE 5. The intersection of the line $x = 0$, $z = 1$ with the basins of attraction of $P_{2,3}$ and with the double scroll attractor for $\delta = 0.3,\ \eta = 1,\ h \in [h_2, 8.5]$.

scroll attractor with the line $x = 0$, $z = 1$ are represented by the curves (DSA). It can be observed that the size of the basins of attraction of the twin equilibria increases, when h increases, but the dominant attractor is the double-scroll one.

5. Conclusions

The pitchfork and Hopf bifurcations were study for a 3D dynamical system modelling oscillations of plasma parameters in Tokamaks. The pitchfork bifurcation was analyzed in respect with the input power in the system and the Hopf bifurcation was studied with respect of each of the three parameters of the system. In the case when the input power is varied and the other two parameters (related to the dissipation/relaxation of the MHD instability respectively to diffusion phenomena) are fixed, a new bifurcation was point out. This bifurcation is obtained through the collision of two twin stable periodic orbits. As a result, a double scroll-attractor is formed. For some values of the input power, tree attractors (two stable points and a double-scroll attractor) coexist, the double-scroll attractor being dominant. In a small range of the input power, near the collision limit, the dynamics on the double scroll attracting set is not periodic, the two wings of the attracting set being occupied in an irregular way. After this, the dynamic become periodic. The periodic behavior of the mathematical model reflect the oscillations of normalized displacement of magnetic field and of the normalized pressure gradient in the physical system.

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