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New generalization of Hermite-Hadamard type inequalities via generalized fractional integrals

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Abstract. In this paper we obtain new generalization of Hermite-Hadamard inequalities via generalized fractional integrals defined by Sarikaya and Ertuğral. We establish some midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are convex involving generalized fractional integrals.

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1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[6], [11], [24, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The overall structure of the study takes the form of six sections including introduction. The remainder of this work is organized as follows: we first mention some works which focus on Hermite-Hadamard inequality. In Section 2, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral along with the very first results. In section 3 new Hermite-Hadamard type inequalities for generalized fractional integrals are proved. In Section 4 and Section 5 midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are convex via generalized fractional integrals are presented, respectively. Some conclusions and further directions of research are discussed in Section 6.

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In [7], Dragomir and Agarwal establish the following identity and using this identity, present some bounds for the right hand side of the inequality (1).

Lemma 1.1. *Let $f : I^* \to \mathbb{R}$ be differentiable function on I^*, $a, b \in I^*$ (I^* is interior of I) with $a < b$. If $f' \in L[a,b]$, then we following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt = \frac{b-a}{2}\left[\int_0^1 (1-2t)f'(ta+(1-t)b)dt\right]. \tag{2}$$

In [21], U. S. Kırmacı give the following identity and using this identity, obtain some bounds for the left hand side of the inequality (1).

Lemma 1.2. *Let $f : I^* \to \mathbb{R}$ be differentiable function on I^*, $a, b \in I^*$ (I^* is interior of I) with $a < b$. If $f' \in L[a,b]$, then we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \\ &= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (1-t)\, f'(ta+(1-t)b)dt\right]. \end{aligned} \tag{3}$$

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1). For some examples, please refer to ([1], [3], [4], [6], [26]-[28], [34]).

On the other hand, Sarikaya et al. obtain the Hermite-Hadamard inequality for the Riemann-Lioville fractional integrals in [31]. Sarıkaya and Yıldırım also give the following Hermite-Hadamard type inequality for the Riemann-Lioville fractional integrals in [30].

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{4}$$

Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, please see ([2], [5], [8], [9], [12], [13], [14]-[17], [23], [25], [32], [33], [35]-[38]). For more information about fraction calculus please refer to ([10], [20]).

In this paper, we obtain the new generalized Hermite-Hadamard type inequality for the generalized fractional integrals mentioned in next section.

2. New Generalized Fractional Integral Operators

In this section we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [29].

Let's define a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions :

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a, \tag{5}$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b. \tag{6}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (5) and (6) are mentioned below.

i) If we take $\varphi(t) = t$, the operator (5) and (6) reduce to the Riemann integral as follows:

$$I_{a^+} f(x) = \int_a^x f(t)dt, \quad x > a,$$

$$I_{b^-} f(x) = \int_x^b f(t)dt, \quad x < b.$$

ii) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, the operator (5) and (6) reduce to the Riemann-Liouville fractional integral as follows:

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$, the operator (5) and (6) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I_{a^+,k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$

$$I_{b^-,k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b$$

where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [22].

iv) If we take

$$\varphi(t) = \frac{1}{\Gamma(\alpha)} t (x-t)^s \left(x^{s+1} - t^{s+1}\right)^{\alpha-1}$$

and

$$\varphi(t) = \frac{1}{\Gamma(\alpha)} t (t-x)^s \left(t^{s+1} - x^{s+1}\right)^{\alpha-1},$$

in the operators (5) and (6), respectively, then the (5) and (6) reduce to the Katugampola fractional operators as follows for $\alpha > 0$ and $s \neq -1$ is a real numbers:

$$I^{\alpha}_{a+,s} f(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\alpha-1} t^s f(t) dt, \quad x > a,$$

$$I^{\alpha}_{b-,s} f(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \left(x^{s+1} - t^{s+1}\right)^{\alpha-1} t^s f(t) dt, , \quad x < b$$

are given by Katugampola in [18].

v) If we take $\varphi(t) = t(x-t)^{\alpha-1}$, the operator (5) reduces to the conformable fractional operators as follows:

$$I^{\alpha}_{a} f(x) = \int_a^x t^{\alpha-1} f(t) dt = \int_a^x f(t) d_\alpha t, \quad x > a, \ \alpha \in (0,1)$$

is given by Khalil et.al in [19].

Sarıkaya and Ertuğral also establish the following Hermite-Hadamard inequality for the generalized fractional integral operators:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities for fractional integral operators hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Lambda(1)} \left[{}_{a+}I_\varphi f(b) + {}_{b-} I_\varphi f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{7}$$

where the mapping $\Lambda : [0,1] \to \mathbb{R}$ is defined by

$$\Lambda(x) = \int_0^x \frac{\varphi((b-a)t)}{t} dt.$$

3. Hermite-Hadamard Type Inequalities for Generalized Fractional Integral Operators

In this section, we will present a theorem for Hermite-Hadamard type inequalities with generalized fractional integral operators which is the generalization of previous work.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then we have the following inequalities for generalized fractional integral operators:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Psi(1)} \left[{}_{\left(\frac{a+b}{2}\right)+}I_\varphi f(b) + {}_{\left(\frac{a+b}{2}\right)-} I_\varphi f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{8}$$

where the mapping $\Psi : [0,1] \to \mathbb{R}$ is defined by

$$\Psi(x) = \int_0^x \frac{\varphi\left(\frac{b-a}{2}t\right)}{t} dt.$$

Proof. Since f is a convex function on $[a,b]$, we have for $x,y\in[a,b]$

$$f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2}.$$

For $x=\frac{t}{2}a+\frac{2-t}{2}b$ and $y=\frac{2-t}{2}a+\frac{t}{2}b$, we obtain

$$2f\left(\frac{a+b}{2}\right)\le f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right). \tag{9}$$

Multiplying both sides of (9) by $\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)&\int_0^1\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}dt\\&\le\int_0^1\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt.\end{aligned}$$

For $u=\frac{t}{2}a+\frac{2-t}{2}b$ and $v=\frac{2-t}{2}a+\frac{t}{2}b$, we obtain

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)\Psi(1)dt&\le\int_{\frac{a+b}{2}}^{b}\frac{\varphi(b-u)}{b-u}f(u)\,du+\int_a^{\frac{a+b}{2}}\frac{\varphi(v-a)}{v-a}f(v)\,dv\\&=\left[{}_{\left(\frac{a+b}{2}\right)^+}I_\varphi f(b)+{}_{\left(\frac{a+b}{2}\right)^-}I_\varphi f(a)\right]\end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality (8), we first note that if f is a convex function, it yields

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\le\frac{t}{2}f(a)+\frac{2-t}{2}f(b)$$

and

$$f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\le\frac{2-t}{2}f(a)+\frac{t}{2}f(b).$$

By adding these inequalities together, one has the following inequality:

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\le f(a)+f(b). \tag{10}$$

Then multiplying both sides of (10) by $\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}\int_0^1\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\\le[f(a)+f(b)]\int_0^1\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}dt.\end{aligned}$$

That is,

$$\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\leq\Psi(1)\left[f(a)+f(b)\right].$$

Hence, the proof is completed. ☐

Remark 3.1. Under assumption of Theorem 3.1 with $\varphi(t)=t$, then inequalities 8 reduce to the inequalities (1).

Remark 3.2. Under assumption of Theorem 3.1 with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then inequalities 8 reduce to the inequalities (4).

Remark 3.3. Under assumption of Theorem 3.1 with $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then Theorem 3.1 reduces to Theorem 1.1 in [8].

4. Midpoint Type Inequalities for Differentiable Functions with Generalized Fractional Integral Operators

In this section, firstly we need to give a lemma for differentiable functions which will help us to prove our main theorems. Then, we present some midpoint type inequalities which are the generalization of those given in earlier works.

Lemma 4.1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $f'\in L[a,b]$, then we have the following identity for generalized fractional integral operators:*

$$\begin{aligned}&\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&\quad=\frac{b-a}{4\Psi(1)}\left[\int_0^1\Psi(t)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1\Psi(t)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right]\end{aligned}\tag{11}$$

where the mapping $\Psi(t)$ is defined as in Theorem 3.1.

Proof. Integrating by parts gives

$$\begin{aligned}I_1&=\int_0^1\Psi(t)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\\&=-\frac{2}{b-a}\Psi(t)f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\Bigg|_0^1+\frac{2}{b-a}\int_0^1\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\\&=-\frac{2}{b-a}\Psi(1)f\left(\frac{a+b}{2}\right)+\frac{2}{b-a}\ {}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)\end{aligned}\tag{12}$$

and similarly we get

$$I_2=\int_0^1\Psi(t)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt=\frac{2}{b-a}\Psi(1)f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\ {}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a).\tag{13}$$

By subtracting equation (13) from (12), we have

$$\frac{b-a}{4\Psi(1)}\left(I_1-I_2\right)=-\Psi(1)f\left(\frac{a+b}{2}\right)+\frac{1}{2\Psi(1)}\left[\ {}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+\ {}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right].$$

By re-arranging the last equality above, we get the desired result. $\square$

Remark 4.1. Under assumption of Lemma 4.1 with $\varphi(t)=t$, then the equality (11) reduces to the equality (3).

Remark 4.2. Under assumption of Lemma 4.1 with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then Lemma 4.1 reduces to Lemma 3 in [30].

Remark 4.3. Under assumption of Lemma 4.1 with $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then Lemma 4.1 reduces to Lemma 3.1 in [8].

Theorem 4.2. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $|f'|$ is convex function, then we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\qquad\le\frac{b-a}{4\Psi(1)}\left(\int_0^1|\Psi(t)|\,dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\end{aligned}$$

where the mapping $\Psi(t)$ is defined as in Theorem 3.1.

Proof. From Lemma 4.1, by using the convexity of $|f'|$, we have

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{b-a}{4\Psi(1)}\left[\int_0^1|\Psi(t)|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1|\Psi(t)|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right]\\&\le\frac{b-a}{4\Psi(1)}\left[\int_0^1|\Psi(t)|\left[\frac{t}{2}\left|f'(a)\right|+\frac{2-t}{2}\left|f'(b)\right|\right]dt\right.\\&\quad\left.+\int_0^1|\Psi(t)|\left[\frac{2-t}{2}\left|f'(a)\right|+\frac{t}{2}\left|f'(b)\right|\right]dt\right]\\&=\frac{b-a}{4\Psi(1)}\left(\int_0^1|\Psi(t)|\,dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\end{aligned}$$

This completes the proof. $\square$

Remark 4.4. Under assumption of Theorem 4.2 with $\varphi(t)=t$, then Theorem 4.2 reduces to Theorem 2.2 in [21].

Remark 4.5. Under assumption of Theorem 4.2 with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then Theorem 4.2 reduces to Theorem 5 (for $q=1$) in [30].

Remark 4.6. Under assumption of Theorem 4.2 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then Theorem 4.2 reduces to Theorem 3.1 (for $q=1$) in [8].

Corollary 4.3. *Under assumption of Theorem 4.2 with* $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, *then for* $A = \frac{1-\alpha}{\alpha}\frac{b-a}{2}$ *we have the following inequality*

$$\left|\frac{1-\alpha}{2[1-\exp\{-A\}]}\left[\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b) + \mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{2}\frac{A+\exp\{-A\}-1}{A(1-\exp\{-A\})}\left[\frac{|f'(a)|+|f'(b)|}{2}\right].$$

Theorem 4.4. *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable function on* (a,b) *with* $a<b$. *If* $|f'|^q$, $q>1$, *is convex function, then we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b) + {}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\ &\le \frac{b-a}{4\Psi(1)}\left(\int_0^1 |\Psi(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\ &\le \frac{b-a}{2^{\frac{2}{q}}\Psi(1)}\left(\int_0^1 |\Psi(t)|^p\,dt\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|] \qquad (14)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and the mapping* Ψ *is defined as in Theorem 3.1.*

Proof. Taking modulus of (11) and using the well-known Hölder inequality, we obtain

$$\begin{aligned}&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b) + {}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\ &\le \frac{b-a}{4\Psi(1)}\left[\int_0^1 |\Psi(t)|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt + \int_0^1 |\Psi(t)|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right] \\ &\le \frac{b-a}{4\Psi(1)}\left(\int_0^1 |\Psi(t)|^p\,dt\right)^{\frac{1}{p}} \\ &\quad\times\left[\left(\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \qquad (15)\end{aligned}$$

Since $|f'|^q$, $q>1$, is convex, we have

$$\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt \le \int_0^1\left[\frac{t}{2}|f'(a)|^q + \frac{2-t}{2}|f'(b)|^q\right]dt = \frac{|f'(a)|^q+3|f'(b)|^q}{4} \qquad (16)$$

and similarly

$$\int_0^1 \left| f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right|^q dt \le \frac{3\left|f'(a)\right|^q + \left|f'(b)\right|^q}{4}. \tag{17}$$

By substituting inequalities (16) and (17) into (15), we obtain the first inequalty in (14).

For the proof of second inequality, let $a_1 = \left|f'(a)\right|^q$, $b_1 = 3\left|f'(b)\right|^q$, $a_2 = 3\left|f'(a)\right|^q$ and $b_2 = \left|f'(b)\right|^q$. Using the fact that

$$\sum_{k=1}^{n} (a_k + b_k)^s \le \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s, \ 0 \le s < 1 \tag{18}$$

and $1 + 3^{\frac{1}{q}} \le 4$ then the desired result can be obtained straightforwardly. □

Remark 4.7. Under assumption of Theorem 4.4 with $\varphi(t) = t$, then Theorem 4.4 reduces to combining of Theorem 2.3 and Theorem 2.4 in [21].

Remark 4.8. Under assumption of Theorem 4.4 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then Theorem 4.4 reduces to Theorem 6 in [30].

Remark 4.9. Under assumption of Theorem 4.4 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then Theorem 4.4 reduces to Theorem 3.2 in [8].

Corollary 4.5. *Under assumption of Theorem 4.4 with $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, then for $A = \frac{1-\alpha}{\alpha}\frac{b-a}{2}$ we have the following inequality for the fractional integrals with exponential kernel*

$$\begin{aligned}
&\left|\frac{1-\alpha}{2\left[1-\exp\{-A\}\right]}\left[\mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)+} f(b) + \mathcal{I}^\alpha_{\left(\frac{a+b}{2}\right)-} f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{b-a}{4\left[1-\exp\{-A\}\right]}\left(\int_0^1 (1-\exp\{-At\})^p\, dt\right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\frac{\left|f'(a)\right|^q + 3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'(a)\right|^q + \left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\le \frac{b-a}{2^{\frac{2}{q}}\left[1-\exp\{-A\}\right]}\left(\int_0^1 (1-\exp\{-At\})^p\, dt\right)^{\frac{1}{p}} \left[\left|f'(a)\right| + \left|f'(b)\right|\right].
\end{aligned}$$

Theorem 4.6. *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$. If $\left|f'\right|^q$, $q \ge 1$, is a convex function, then we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_\varphi f(b) + {}_{\left(\frac{a+b}{2}\right)-}I_\varphi f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{b-a}{2^{2+\frac{1}{q}}\Psi(1)}\left(\int_0^1 \left|\Psi(t)\right| dt\right)^{1-\frac{1}{q}} \left[\left(B_1\left|f'(a)\right|^q + B_2\left|f'(b)\right|^q\right)^{\frac{1}{q}} + \left(B_2\left|f'(a)\right|^q + B_1\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where the mapping $\Psi(t)$ is defined as in Theorem 3.1 and the constants B_1 and B_2 are defined by

$$B_1=\int_0^1 |\Psi(t)|\, tdt \text{ and } B_2=\int_0^1 |\Psi(t)|\,(2-t)\,dt.$$

Proof. The case of $q=1$ is obvious from Theorem 4.2.

For $q>1$ we proceed as follows. Taking modulus of (11) and using well-known power mean inequality, we obtain

$$\begin{aligned}
&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\Psi(1)}\left[\int_0^1|\Psi(t)|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1|\Psi(t)|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right] \\
&\leq \frac{b-a}{4\Psi(1)}\left(\int_0^1|\Psi(t)|\,dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1|\Psi(t)|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1|\Psi(t)|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|f'|^q$ is convex, we have

$$\begin{aligned}
&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\Psi(1)}\left(\int_0^1|\Psi(t)|\,dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1|\Psi(t))|\left[\frac{t}{2}\left|f'(a)\right|^q+\frac{2-t}{2}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1|\Psi(t)|\left[\frac{2-t}{2}\left|f'(a)\right|^q+\frac{t}{2}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right] \\
&=\frac{b-a}{2^{2+\frac{1}{q}}\Psi(1)}\left(\int_0^1|\Psi(t)|\,dt\right)^{1-\frac{1}{q}}\left[\left(B_1\left|f'(a)\right|^q+B_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(B_2\left|f'(a)\right|^q+B_1\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

which completes the proof. □

Remark 4.10. Under assumption of Theorem 4.6 with $\varphi(t)=t$, then we have the inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{8}\left[\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right] \\
&\quad\leq\frac{3^{1-\frac{1}{q}}}{8}(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \qquad (19)
\end{aligned}$$

Proof. The proof of the second inequality is obvious from the inequality (18) $a_1 = |f'(a)|^q$, $b_1 = 2|f'(b)|^q$, $a_2 = 2|f'(a)|^q$ and $b_2 = |f'(b)|^q$. □

Remark 4.11. Under assumption of Theorem 4.6 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then Theorem 4.6 reduces to Theorem 5 in [30].

Remark 4.12. Under assumption of Theorem 4.6 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then Theorem 4.6 reduces to Theorem 3.1 in [8].

5. Trapezoid Type Inequalities for Differentiable Functions with Generalized Fractional Integral Operators

In this section, firstly we need to give a lemma for differentiable functions which will help us to prove our main theorems. Then, we present some trapezoid type inequalities which are the generalization of those given in earlier studies.

Lemma 5.1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$. If $f' \in L[a,b]$, then we have the following identity for generalized fractional integral operators:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_\varphi f(b) + {}_{\left(\frac{a+b}{2}\right)-}I_\varphi f(a)\right]$$
$$= \frac{b-a}{4\Psi(1)}\left[\int_0^1 \Phi(t) f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt - \int_0^1 \Phi(t) f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt\right]. \tag{20}$$

where the mapping $\Phi(t)$ is defined by

$$\Phi(t) = \int_t^1 \frac{\varphi\left(\frac{b-a}{2}u\right)}{u}du,$$

with $\Phi(0) = \Psi(1)$.

Proof. Integrating by parts, we have

$$I_3 = \int_0^1 \Phi(t) f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt$$
$$= -\frac{2}{b-a}\Phi(t) f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\Bigg|_0^1 - \frac{2}{b-a}\int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t} f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt$$
$$= \frac{2}{b-a}\Psi(1) f(b) - \frac{2}{b-a}\,{}_{\left(\frac{a+b}{2}\right)+}I_\varphi f(b) \tag{21}$$

and similarly we get

$$I_4 = \int_0^1 \Phi(t) f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt = -\frac{2}{b-a}\Psi(1) f(a) + \frac{2}{b-a}\,{}_{\left(\frac{a+b}{2}\right)-}I_\varphi f(a). \tag{22}$$

Thus, we have

$$\frac{b-a}{4\Psi(1)}\left(I_3-I_4\right)=\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[\ _{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+\ _{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right].$$

This completes the proof. □

Remark 5.1. Under assumption of Lemma 5.1 with $\varphi(t)=t$, then the identity (20) reduces to the identity (2).

Remark 5.2. Under assumption of Lemma 5.1 with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then Lemma 5.1 reduces to Lemma 2 (for $x=\frac{a+b}{2}$) in [25].

Corollary 5.2. *Under assumption of Lemma 5.1 with $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following important identity for k-fractional integrals*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{\left(\frac{a+b}{2}\right)+}f(b)+I^{\alpha,k}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\\&=\frac{b-a}{4}\left[\int_0^1\left(1-t^{\frac{\alpha}{k}}\right)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1\left(1-t^{\frac{\alpha}{k}}\right)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right].\end{aligned}$$

Theorem 5.3. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $|f'|$ is a convex function, then we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\right|\\&\leq\frac{b-a}{4\Psi(1)}\left(\int_0^1|\Phi(t)|\,dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\end{aligned}$$

Proof. From Lemma 5.1, by the using convexity of $|f'|$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[_{\left(\frac{a+b}{2}\right)+}I_{\varphi}f(b)+_{\left(\frac{a+b}{2}\right)-}I_{\varphi}f(a)\right]\right|\\&\leq\frac{b-a}{4\Psi(1)}\left[\int_0^1|\Phi(t)|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1|\Phi(t)|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right]\\&\leq\frac{b-a}{4\Psi(1)}\left[\int_0^1|\Phi(t)|\left[\frac{t}{2}\left|f'(a)\right|+\frac{2-t}{2}\left|f'(b)\right|\right]dt+\int_0^1|\Phi(t)|\left[\frac{2-t}{2}\left|f'(a)\right|+\frac{t}{2}\left|f'(b)\right|\right]dt\right]\\&=\frac{b-a}{4\Psi(1)}\left(\int_0^1|\Phi(t)|\,dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\end{aligned}$$

which completes the proof. □

Remark 5.3. Under assumption of Theorem 5.3 with $\varphi(t)=t$, then Theorem 5.3 reduce to Theorem 2.3 in [7].

Corollary 5.4. *Under assumption of Theorem 5.3 with* $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, *then we have the following important fractional inequality related to right-hand side of the inequality 4,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right]\right| \leq \frac{b-a}{2}\frac{\alpha}{\alpha+1}\left[\frac{|f'(a)|+|f'(b)|}{2}\right].$$

Remark 5.4. Under assumption of Theorem 5.3 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$,then we have the following important inequality related to right-hand side of the Hermite-Hadamard inequality for k-fractional integrals,

$$\left|\frac{f(a)+f(b)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{\left(\frac{a+b}{2}\right)+} f(b) + I^{\alpha,k}_{\left(\frac{a+b}{2}\right)-} f(a)\right]\right| \leq \frac{b-a}{2}\left(\frac{\alpha+1-k}{\alpha+1}\right)\left[\frac{|f'(a)|+|f'(b)|}{2}\right].$$

Theorem 5.5. *Let* $f : [a,b] \to \mathbb{R}$ *be differentiable function on* (a,b) *with* $a < b$. *If* $|f'|^q$, $q > 1$, *is a convex function, then we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi} f(b) + {}_{\left(\frac{a+b}{2}\right)-}I_{\varphi} f(a)\right]\right| \\
&\leq \frac{b-a}{4\Psi(1)}\left(\int_0^1 |\Phi(t)|^p\, dt\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{2^{\frac{2}{q}}\Psi(1)}\left(\int_0^1 |\Phi(t)|^p\, dt\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right] \qquad (23)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q} = 1$ *and the mappings* Ψ *and* Φ *are defined as above.*

Proof. Similar to proof of Theorem 4.4, by using the well-known Hölder inequality and convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)+}I_{\varphi} f(b) + {}_{\left(\frac{a+b}{2}\right)-}I_{\varphi} f(a)\right]\right| \\
&\leq \frac{b-a}{4\Psi(1)}\left[\int_0^1 |\Phi(t)|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right| dt + \int_0^1 |\Phi(t)|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right| dt\right] \\
&\leq \frac{b-a}{4\Psi(1)}\left(\int_0^1 |\Phi(t)|^p\, dt\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof of first inequality in (23)

The proof of second inequality in (23) is obvious from the inequality (18). □

Remark 5.5. Under assumption of Theorem 5.5 with $\varphi(t) = t$, then we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right|$$
$$\leq \frac{b-a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right]$$
$$\leq \frac{b-a}{4} \left(\frac{4}{p+1} \right)^{\frac{1}{p}} \left[|f'(a)| + |f'(b)| \right].$$

Corollary 5.6. *Under assumption of Theorem 5.5 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we have the following important fractional inequality related to right-hand side of the inequality 4,*

$$\left| \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \right|$$
$$\leq \frac{b-a}{4} \left(\frac{1}{p\alpha+1} \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right]$$
$$\leq \frac{b-a}{4} \left(\frac{4}{p\alpha+1} \right)^{\frac{1}{p}} \left[|f'(a)| + |f'(b)| \right].$$

Corollary 5.7. *Under assumption of Theorem 5.5 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following important inequality related to right-hand side of the Hermite-Hadamard inequality for k-fractional integrals,*

$$\left| \frac{f(a)+f(b)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} \left[I^{\alpha,k}_{\left(\frac{a+b}{2}\right)^+} f(b) + I^{\alpha,k}_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \right|$$
$$\leq \frac{b-a}{4} \left(\frac{k}{p\alpha+k} \right)^{\frac{1}{p}} \left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right]$$
$$\leq \frac{b-a}{4} \left(\frac{4k}{p\alpha+k} \right)^{\frac{1}{p}} \left[|f'(a)| + |f'(b)| \right].$$

Proof. Using the fact that $|t_1^w - t_2^w| \leq |t_1 - t_2|^w$ for $w \in (0,1]$ and $\forall\, t_1, t_2 \in [0,1]$, we have

$$\int_0^1 |\Phi(t)|^p \, dt = \left[\frac{1}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{2} \right)^{\frac{\alpha}{k}} \right]^p \int_0^1 \left(1 - t^{\frac{\alpha}{k}}\right)^p dt$$
$$\leq \left[\frac{1}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{2} \right)^{\frac{\alpha}{k}} \right]^p \int_0^1 (1-t)^{\frac{\alpha}{k}p} \, dt$$
$$= \left[\frac{1}{\Gamma_k(\alpha+k)} \left(\frac{b-a}{2} \right)^{\frac{\alpha}{k}} \right]^p \frac{k}{p\alpha+k}$$

which completes the proof. □

Theorem 5.8. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $|f'|^q$, $q\geq 1$, is convex function, then we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)^+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)^-}I_{\varphi}f(a)\right]\right|\\&\leq\frac{b-a}{2^{2+\frac{1}{q}}\Psi(1)}\left(\int_0^1|\Phi(t)|\,dt\right)^{1-\frac{1}{q}}\\&\times\left[\left(B_5|f'(a)|^q+B_6|f'(b)|^q\right)^{\frac{1}{q}}+\left(B_6|f'(a)|^q+B_5|f'(b)|\right)^{\frac{1}{q}}\right]\end{aligned}$$

where the mappings Ψ and Φ as above and the constants B_5 and B_6 are defined by

$$B_5=\int_0^1|\Phi(t)|\,tdt\ \text{ and }\ B_6=\int_0^1|\Phi(t)|\,(2-t)\,dt.$$

Proof. The case of the $q=1$ is obvious from the Theorem 5.3.

For $q>1$, using well-known power mean inequality in Lemma 5.1, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)^+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)^-}I_{\varphi}f(a)\right]\right|\\&\leq\frac{b-a}{4\Psi(1)}\left[\int_0^1|\Phi(t)|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1|\Phi(t)|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right]\\&\leq\frac{b-a}{4\Psi(1)}\left(\int_0^1|\Phi(t)|^q\,dt\right)^{1-\frac{1}{q}}\\&\times\left[\left(\int_0^1|\Phi(t)|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1|\Phi(t)|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

By the using convexity of $|f'|^q$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{a+b}{2}\right)^+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)^-}I_{\varphi}f(a)\right]\right|\\&\leq\frac{b-a}{4\Psi(1)}\left(\int_0^1|\Phi(t)|^q\,dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1|\Phi(t)|\left[\frac{t}{2}|f'(a)|^q+\frac{2-t}{2}|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_0^1|\Phi(t)|\left[\frac{2-t}{2}|f'(a)|^q+|f'(b)|^q\frac{t}{2}\right]dt\right)^{\frac{1}{q}}\right]\\&=\frac{b-a}{2^{2+\frac{1}{q}}\Psi(1)}\left(\int_0^1|\Phi(t)|^q\,dt\right)^{1-\frac{1}{q}}\left[\left(B_5|f'(a)|^q+B_6|f'(b)|^q\right)^{\frac{1}{q}}+\left(B_6|f'(a)|^q+B_5|f'(b)|\right)^{\frac{1}{q}}\right].\end{aligned}$$

The proof is completely completed. □

Remark 5.6. Under assumption of Theorem 5.8 with $\varphi(t) = t$, then we have the following inequality

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \int_a^b f(t)dt \right| \\
&\quad \leq \frac{b-a}{8} \left[\left(\frac{|f'(a)|^q + 5|f'(b)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{5|f'(a)|^q + 1|f'(b)|^q}{6} \right)^{\frac{1}{q}} \right] \\
&\leq \frac{6^{1-\frac{1}{q}}}{8} (b-a) \left[|f'(a)| + |f'(b)| \right] \qquad (24)
\end{aligned}$$

Corollary 5.9. *Under assumption of Theorem 5.8 with $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we have the following important fractional inequality related to right-hand side of the inequality 4,*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \right| \\
&\quad \leq \frac{b-a}{2^{2+\frac{1}{q}}} \frac{\alpha}{\alpha+1} \left[\left(\frac{(\alpha+1)}{2(\alpha+2)} |f'(a)|^q + \frac{(3\alpha+7)}{2(\alpha+2)} |f'(b)|^q \right)^{\frac{1}{q}} \right. \\
&\quad\quad \left. + \left(\frac{(3\alpha+7)}{2(\alpha+2)} |f'(a)|^q + \frac{(\alpha+1)}{2(\alpha+2)} |f'(b)| \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Corollary 5.10. *Under assumption of Theorem 5.8 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following important inequality related to right-hand side of the Hermite-Hadamard inequality for k-fractional integrals,*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} \left[I^{\alpha,k}_{\left(\frac{a+b}{2}\right)+} f(b) + I^{\alpha,k}_{\left(\frac{a+b}{2}\right)-} f(a) \right] \right| \\
&\quad \leq \frac{b-a}{2^{2+\frac{1}{q}}} \left(\frac{\alpha}{\alpha+k} \right) \left[\left(\frac{(\alpha+k)}{2(\alpha+2k)} |f'(a)|^q + \frac{(3\alpha+7k)}{2(\alpha+2k)} |f'(b)|^q \right)^{\frac{1}{q}} \right. \\
&\quad\quad \left. + \left(\frac{(3\alpha+7k)}{2(\alpha+2k)} |f'(a)|^q + \frac{(\alpha+k)}{2(\alpha+2k)} |f'(b)| \right)^{\frac{1}{q}} \right]
\end{aligned}$$

Proof. The proof is similar the proof of Corollary 5.7. □

6. Concluding Remarks

In this study, we consider the Hermite-Hadamard for convex function involving generalized fractional integrals defined by Sarikaya and Ertuğral. We also focus on midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are convex via generalized fractional integrals. The results presented in this study would provide generalizations of those given in earlier works.

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