Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 46(2), 2019, Pages 445–456
ISSN: 1223-6934

Identification of parameters of Richards equation using Grey Wolf Optimizer algorithm

W. O. Sawadogo, O. Selt, P. O. F Ouédraogo, K. Somé, and B. Somé

Abstract. In this paper, it is a question of identification of the parameters in the equation of Richards modelling the flow in unsaturated porous medium. The mixed formulation pressure head-moisture content has been used. The direct problem was solved by the finite difference method. The equation being strongly non-linear, we will use the Picard's method. The function cost used is built by using the infiltration. The optimization method used is a meta-heuristic called Grey Wolf Optimizer Algorithm (GWO). A test on experimental data has been carried. A comparison with genetic algorithm show that the GWO is a powerful algorithm for identifying parameters of the Richards equation.

2010 Mathematics Subject Classification. 65M32,65M06,74F10,90C59.
Key words and phrases. unsaturated porous medium, equation of Richards, finite difference method, inverse problem, Grey Wolf Optimizer algorithm.

1. Introduction

The fluid movement in unsaturated porous is governed by the Richards equation [3, 4] which contains parameters that take into account type of the considered soil. The calculation of the water balance on a soil-scale requires knowledge of infiltration that is obtained by solving the unsaturated flow equation. However the hydrodynamic parameters of the soils involved in the equation are, in most cases, badly known. The values given in the literature are not precise values but intervals, hence the importance of the inverse modeling.

There exist many methods to solve the inverse problems [1, 2]. Most computing softwares in hydrogeology use deterministic methods. However most of these methods require a good knowledge of the solution. Indeed, these algorithms can not detect a global optimum and can stop with a local optimum. Moreover, these algorithms require a certain regularity of the functions to be optimized. However, this regularity is not always checked.

Meta-heuristic optimization techniques are adapted better to the problems of optimization in which the size of the space of research is important, where the parameters interact in a complex way and where very little information on the function to be optimized is available [6, 7]. They do not require a particular assumption on the regularity of the function objective. Meta-heuristic algorithms do not use in particular the successive derivative of the functions to be optimized; no assumption on continuity is necessary. The function to be optimized can thus be the result of a simulation. These

Received November 28, 2018. Accepted November 15, 2019.
This paper has been presented at the Conference MOCASIM, Marrakesh, 26-27 November 2018.

algorithms are often much more robust in their capacity to identify the total optimum with less sensitivity to the initial condition.

Grey Wolf Optimizer (GWO) algorithm is a meta-heuristic proposed by S. Mirjalili et al [7]. This algorithm is inspired by grey wolves. He mimics the leadership hierarchy and hunting mechanism of grey wolves in nature. Tests were successfully performed on test functions [7]. In this work we use it for identifying parameters of the Richards equation. The rest of the paper is organized as follows: the second part is devoted to the equation of Richards in one dimension and his resolution by finite differences method; in the third part, we present the problem inverse to solve; the fourth part is devoted to the identification of the parameters of the equation of Richards GWO algorithm; the fifth section present the results and discussions.

2. Direct problem

2.1. Mathematical model. There exist several formulations of the equation of Richards which models the flow in unsaturated porous medium but in this work, we use the mixed formulation pressure head-moisture content because the numerical solutions obtained with his mixed formulation are more precise [4, 5, 8].
In one dimension, the mixed formulation is given by :

$$\begin{cases} \frac{\partial\theta}{\partial t} - \frac{\partial}{\partial z}[K(h)(\frac{\partial h}{\partial z} - 1) = f & \text{in} \quad [0,Z]\times[0,T] \\ h(z,0) = h_0(z) & \text{in} \quad [0,Z] \\ h(0,t) = h_{sup}(t) & \text{in} \quad [0,T] \text{ (top limit)} \\ h(Z,t) = h_{inf}(t) & \text{in} \quad [0,T] \text{ (bottom limit)} \end{cases} \tag{1}$$

where

- z denotes the vertical dimension;
- $h[L]$ is the pressure head;
- $\theta[L^3/L^3]$ is the moisture content given by:

$$\theta(h) = \frac{\theta_s - \theta_r}{(1 + (\alpha|h|)^n)^m} + \theta_r, \tag{2}$$

  where θ_r is the moisture content to saturation $(L^3.L^{-3})$, θ_s is the residual moisture content $(L^3.L^{-3})$, α is a parameter of form related to the mean size of the pores (L^{-1}), n is a parameter related to the distribution of the sizes of pores $(-)$. According to Mualem [11], we have $m = 1 - 1/n$;
- $K(h)$ is the insaturated hydraulic conductivity $[L/T]$. We use the relation of Van Genuchten [3] given by

$$K(S_e) = K_s S_e^{1/2}(1 - (1 - S_e^{1/m})^m)^2, \tag{3}$$

  where K_S is the effective saturated hydraulic conductivity $[L/T]$, S_e is the effective saturation given by:

$$S_e = \begin{cases} \frac{\theta - \theta_r}{\theta_s - \theta_r} & \text{if } h < 0, \\ 1 & \text{if } h \geq 0, \end{cases} \tag{4}$$

h and θ are related by the moisture capacity function $C(h)[1/L]$ defined by

$$C(h) = \frac{\partial\theta}{\partial h}, \tag{5}$$

which gives

$$C(h) = -\alpha n(\theta_r - \theta_s) sign(h)(\frac{1}{n} - 1)(\alpha|h|)^{n-1}(1 + (\alpha|h|)^n)^{1/n-2}. \tag{6}$$

To solve the problem (1), the five parameters should be known: α, θ_S, θ_r, n and K_S.

2.2. Numerical resolution. In this part, we present the numerical resolution of equation (1). We will use the finite differences method for the space discretization. A implicit scheme of Euler will be used for the temporal discretization. The equation being strongly non-linear, we will use the method of Picard to linearize its.

To discretize $[0, Z] \times [0, T]$, we introduce a step of space $\Delta z = \frac{Z}{N_z+1}$ (N_z an integer stricly positive) and a step of time $\Delta t = \frac{T}{M}$ (M an integer stricly positive), and we define the nodes of a regular meshing:

$$(z_i, t_j) = (i\Delta z, j\Delta t) \text{ for } (i, j) \in \{0, 1, \ldots, N_z + 1\} \times \{0, 1, \ldots, M\}$$

we denote $h_i^j \approx h(x_i, t_j)$, $\theta_i^j \approx \theta(x_i, t_j)$. The discretization of (1) by an implicit scheme is given by

$$\frac{\theta_i^{j+1} - \theta_i^j}{\Delta t} = \frac{1}{\Delta z}\left[K_{i+\frac{1}{2}}^{j+1}\left(\frac{h_{i+1}^{j+1} - h_i^{j+1}}{\Delta z} - 1\right) - K_{i-\frac{1}{2}}^{j+1}\left(\frac{h_i^{j+1} - h_{i-1}^{j+1}}{\Delta z} - 1\right)\right] + f_i^{j+1}$$
$$i = 1, \ldots, N_z,\ j = 0, \ldots, M - 1 \tag{7}$$

$K_{i-\frac{1}{2}} = \sqrt{K(h_i^{j+1}) \times K(h_{i-1}^{j+1})}, K_{i+\frac{1}{2}} = \sqrt{K(h_i^{j+1}) \times K(h_{i+1}^{j+1})}$ and $f_i^{j+1} = f(x_i, t_{j+1})$.

By applying the method of Picard the equation (7) is written:

$$\begin{aligned}\frac{\theta_i^{j+1,k+1} - \theta_i^j}{\Delta t} &= \frac{1}{\Delta z} K_{i+\frac{1}{2}}^{j+1,k}\left(\frac{h_{i+1}^{j+1,k+1} - h_i^{j+1,k+1}}{\Delta z} - 1\right) \\ &\quad - \frac{1}{\Delta z} K_{i-\frac{1}{2}}^{j+1,k}\left(\frac{h_i^{j+1,k+1} - h_{i-1}^{j+1,k+1}}{\Delta z} - 1\right) + f_i^{j+1},\end{aligned} \tag{8}$$

where $i = 1, \ldots, N_z$, $j = 0, \ldots, M - 1$ and k is the indice of iteration of the Picard's method.

We pose $h^j = (h_1^j, \ldots, h_{N_z}^j)$ and $\theta^j = (\theta_1^j, \ldots, \theta_{N_z}^j)$, the developpement in Taylor series of θ respect h at point $h^{j+1,k}$ gives [4]

$$\theta^{j+1,k+1} = \theta^{j+1,k} + \frac{d\theta}{dh}|^{j+1,k}(h^{j+1,k+1} - h^{j+1,k}) + 0(\delta^2),$$

where $h^{j+1,k+1}$ and $\theta^{j+1,k+1}$ represent respectively the vector of pressure and the vector of the moisture content to the step of $j + 1$ and the iteration $k + 1$.
By truncating and by using the relation (5), we have :

$$\theta^{j+1,k+1} \approx \theta^{j+1,k} + C^{j+1,k}(h^{j+1,k+1} - h^{j+1,k}).$$

The equation (7) becomes:

$$C_i^{j+1,k}\left(\frac{h_i^{j+1,k+1}-h_i^{j+1,k}}{\Delta t}\right)-\frac{1}{\Delta z^2}K_{i+\frac{1}{2}}^{j+1,k}.h_{i+1}^{j+1,k+1}$$
$$-\frac{1}{\Delta z^2}\left[\left(K_{i+\frac{1}{2}}^{j+1,k}+K_{i-\frac{1}{2}}^{j+1,k}\right)h_i^{j+1,k+1}+K_{i-\frac{1}{2}}^{j+1,k}.h_{i-1}^{j+1,k+1}\right]$$
$$=-\frac{(\theta_i^{j+1,k}-\theta_i^j)}{\Delta t}-\frac{1}{\Delta z}\left(K_{i+\frac{1}{2}}^{j+1,k}+K_{i-\frac{1}{2}}^{j+1,k}\right)+f_i^{j+1}. \tag{9}$$

Posing $\delta_i^k=h_i^{j+1,k+1}-h_i^{j+1,k}$, the system (9) becomes:

$$C_i^{j+1,k}\frac{\delta_i^k}{\Delta t}-\frac{1}{(\Delta z)^2}\left[K_{i+\frac{1}{2}}^{j+1,k}\left(\delta_{i+1}^k-\delta_i^k\right)-K_{i-\frac{1}{2}}^{j+1,k}\left(\delta_i^k-\delta_{i-1}^k\right)\right]$$
$$=\frac{1}{(\Delta z)^2}\left[K_{i+\frac{1}{2}}^{j+1,k}\left(h_{i+1}^k-h_i^k\right)-K_{i-\frac{1}{2}}^{j+1,k}\left(h_i^k-h_{i-1}^k\right)\right]$$
$$-\frac{(\theta_i^{j+1,k}-\theta_i^j)}{\Delta t}+\frac{1}{\Delta z}\left(K_{i+\frac{1}{2}}^{j+1,k}+K_{i-\frac{1}{2}}^{j+1,k}\right)+f_i^{j+1}. \tag{10}$$

By regrouping we have:

$$\frac{K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}\delta_{i-1}^k+\left[\frac{C_i^{j+1,k}}{\Delta t}-\left(\frac{K_{i+\frac{1}{2}}^{j+1,k}+K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}\right)\right]\delta_i^k-\frac{K_{i+\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}\delta_{i+1}^k$$
$$=\frac{1}{(\Delta z)^2}\left[K_{i+\frac{1}{2}}^{j+1,k}\left(h_{i+1}^k-h_i^k\right)-K_{i-\frac{1}{2}}^{j+1,k}\left(h_i^k-h_{i-1}^k\right)\right]$$
$$-\frac{(\theta_i^{j+1,k}-\theta_i^j)}{\Delta t}+\frac{1}{\Delta z}\left(K_{i+\frac{1}{2}}^{j+1,k}+K_{i-\frac{1}{2}}^{j+1,k}\right)+f_i^{j+1}, \tag{11}$$

which is written in matrix form, with each iteration k of Picard k

$$A^k\delta^k=R^k, \tag{12}$$

where A^k is a tridiagonal matrix of size $N_z\times N_z$, and R^k is a vector of size N_z. We have

$$A_{i,i}^k=\frac{C_i^{j+1,k}}{\Delta t}-\left(\frac{K_{i+\frac{1}{2}}^{j+1,k}+K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}\right),\ i=1,\ldots,N_z$$
$$A_{i-1,i}^k=\frac{K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2},\ i=2,\ldots,N_z$$
$$A_{i,i+1}^k=\frac{K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2},\ i=1,\ldots,N_z-1$$

and

$$R_i^k=\frac{1}{(\Delta z)^2}\left[K_{i+\frac{1}{2}}^{j+1,k}\left(h_{i+1}^k-h_i^k\right)-K_{i-\frac{1}{2}}^{j+1,k}\left(h_i^k-h_{i-1}^k\right)\right]$$
$$-\frac{(\theta_i^{j+1,k}-\theta_i^j)}{\Delta t}+\frac{1}{\Delta z}\left(K_{i+\frac{1}{2}}^{j+1,k}+K_{i-\frac{1}{2}}^{j+1,k}\right)+f_i^{j+1},\ i=1,\ldots,N_z.$$

In summary the resolution by the iterative method of Picard is given by the algorithm 1.

Algorithm 1

At iteration $j+1$ of time, to do
1- Initialize $h^{j+1,0} = h^j$(pressure at j)
2- $k = 0$
3- to do
i- to build the system (12)
ii- To solve the system (12)
iii- To build the new solution: $h^{j+1,k+1} = \delta^{j+1,k} + h^{j+1,k}$
iv- $k = k+1$
v- while $|\delta^{j+1,k}| < \epsilon$, ϵ is the tolerance
vi- if not convergence and $k > I_m$ (maximal number of the iteration of Picard), to change the step of time and to return (1)

3. Inverse problem

3.1. Calculation of infiltration. One of the objectives of the modeling of the flow in unsaturated porous medium is the estimate of the quantity of water which infiltrates to reach the saturated zone. The infiltration describes the process of water penetrating in the ground starting from its surface. In a general way, for a variable initial condition $\theta(0,z)$, the cumulative infiltration I_{cum} is defined by:

$$I_{cum}(t) = \int_0^Z q(t,z)dz,$$

where $q(z,t)$ is the rate of infiltration and Z is the depth of the ground considered. If the initial condition θ_{init} is constant, we have:

$$I_{cum}(t) = \int_0^Z (\theta(t,z) - \theta_{ini})dz, \tag{13}$$

where $\theta(t,z)$ is the moisture content. In discrete form $I_{cum}(t_j)$ is obtained by making an approximation of (13) by the formula of the trapezoids:

$$I_{cum}(t_j) = \Delta z \left[\frac{1}{2} \left(\theta_{sup} - 2\theta_{ini} + \theta_{inf} \right) + \sum_{i=1}^{N_z} \left(\theta_i^j - \theta_{ini} \right) \right], \tag{14}$$

where θ_{inf} is the moisture content at the bottom and θ_{sup} is the moisture content at the top.

3.2. Function cost. Let thus M observations of values of infiltration $I_{obs}(t_j)$ at the moments t_j, $j = 1, \ldots, M$. Let thus J the functional defined by

$$\begin{aligned} J(U) &= \frac{\Delta t}{2} \sum_{j=1}^{M} (I_{cum}(t_j) - I_{obs}(t_j))^2 \\ &= \frac{\Delta t}{2} \sum_{j=1}^{2} (\Delta z \left[\frac{1}{2}(\theta_{sup} - 2\theta_{ini} + \theta_{inf}) + \sum_{i=1}^{N_z} (\theta_i^j - \theta_{ini}) \right] - I_{obs}(t_j))^2, \end{aligned} \tag{15}$$

where U is the vector of parameters to determinate $(\alpha, n, \theta_r, \theta_s, K_s)$.

The inverse problem consists in solving

$$\min_{U \subset \mathcal{D}} J(U), \tag{16}$$

where $\mathcal{D}$ is a bounded subset of $\mathbb{R}^5$.

4. Problem solving by Grey Wolf Optimizer (GWO) algorithm

4.1. Grey Wolf Optimizer algorithm description. In this section, we present GWO algorithm used to solve the problem (16).

The GWO algorithm is a meta-heuristic which mimics the leadership hierarchy and hunting mecanism of grey wolwes in nature. This algorithm has been proposed by S. Mirjalili et al [7]. Four types of grey wolves are employed for the simulating the leadership hierarchy (see figure 1):

- the leader of the group called alpha (α) which is mostly responsible for making decisions about hunting, sleeping place, time to wake, and so on;
- the leader alpha is assisted by the beta (β) that help the alpha in decision-making or other pack activities;
- the third level in the hierarchy is delta (δ). Delta wolves have to submit to alphas and betas, but they dominates the omega (ω) wolves that occupy the last level;
- the omega wolves always have to submit to all the other dominant wolves. This assists satisfying the entire pack and maintaining the dominance structure.

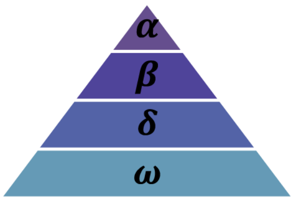

FIGURE 1. Hierarchy of grey wolf [7].

To model mathematically the hunting mechanism of grey wolves, three steps were considered:

- Tracking, chassing and approaching the prey;
- Pursuing, encircling and harassing the prey until it stops moving;
- Attacking the prey.

To respect the hierarchy, the best solution is alpha, the second solution is beta and the third delta. The optimum being the position of the prey.

The grey wolves encircle prey during the hunt. The mathematical model is given:

$$\begin{cases} \vec{D} = |\vec{C}.\vec{X}_p(t) - \vec{X}(t)|, \\ \vec{X}(t+1) = \vec{X}_p(t) - \vec{A}.\vec{D}, \end{cases} \tag{17}$$

where t indicates the current iteration, $\vec{A} = 2a.\vec{r_1}$, $\vec{C} = 2.\vec{r_1}$; $\vec{a}$ are linearly decreased from 2 to 0 over the course of iterations and $\vec{r_1}$, $\vec{r_2}$ are random vectors in $[0, 1]$. $\vec{X}_p$ is the position vector of the prey, and $\vec{X}$ indicates the position vector of a grey wolf.

For better exploration of candidate solutions which tend to diverge when $|\vec{A}| > 1$ and to converge when $|\vec{A}| < 1$.

Grey wolves have the ability to recognize the location of prey and encircle them. Over the course of iterations, the first three fittest solutions we obtain so far are considered as α, β and δ respectively, which guide the optimization processus (the hunting) and are assumed to take the position of the optimum (the prey). To model this procssus, we adapt the positions of population using the following formula:

$$\begin{cases} \vec{D_\alpha} = |\vec{C_1}.\vec{X}_\alpha - \vec{X}|, \\ \vec{D_\beta} = |\vec{C_2}.\vec{X}_\beta - \vec{X}|, \\ \vec{D_\delta} = |\vec{C_3}.\vec{X}_\delta - \vec{X}|, \end{cases} \tag{18}$$

where:
- $\vec{C}_1$,$\vec{C}_2$ and $\vec{C}_3$ are random vectors;
- $\vec{X}_\alpha$, $\vec{X}_\alpha$ and $\vec{X}_\delta$, the positions of alpha, beta and delta respectively;
- $\vec{X}$ the position of prey(current solution).

The next position of the best solution is given by:

$$\vec{X}(t+1) = \frac{\vec{X}_1 + \vec{X}_2 + \vec{X}_3}{3}, \tag{19}$$

where

$$\begin{cases} \vec{X}_1 = \vec{X}_\alpha - \vec{A}_1.\vec{D}_\alpha, \\ \vec{X}_2 = \vec{X}_\beta - \vec{A}_2.\vec{D}_\beta, \\ \vec{X}_3 = \vec{X}_\delta - \vec{A}_3.\vec{D}_\delta, \end{cases} \tag{20}$$

$\vec{A}_1$,$\vec{A}_2$ and $\vec{A}_3$ are random vectors.

To accelerate convergence, we take

$$\vec{X}(t+1) = 0.7 \times \vec{X}_1 + 0.2 \times \vec{X}_2 + 0.1 \times \vec{X}_3. \tag{21}$$

The possible updated positions of a grey wolf in 2D and 3D space are depicted in figure 2.

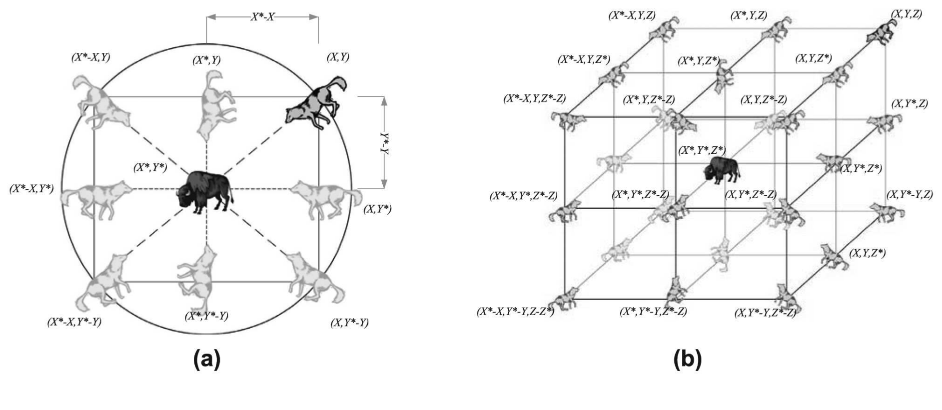

FIGURE 2. 2D and 3D position vectors and their possible next locations [7].

4.2. Problem solving algorithm. In summary the resolution of our problem by GWO algorithm is given by the algorithm below:

Algorithm 2

```
Initialize the input parameters for GWO (N, d, lb, ub, Maxiter)
Initialize Alpha, Beta and Delta Position and Score.
Initialize the random position of search agents.
k ← 0
while k < Maxiter do
  for i=1 to N do
    solving direct problem (1)
    evaluate the score of each search agent using objective function (15)
    if fitness < AlphaScore then
      Update alpha
    end if
    if fitness > AlphaScore and fitness < BetaScore then
      Update beta
    end if
    if fitness > AlphaScore and fitness > BetaScore and fitness <
    DeltaScore then
      Update delta
    end if
  end for
  for i=1 to N do
    Update the Position of search agents including omegas using equation (18-20)
    Update the position of prey using equation(21)
  end for
  l ← l + 1
end while
Return the position of α as the fittest optimum
```

5. Results and discussions

5.1. Application 1. Either an unsaturated medium represented by a domain $\Omega = [0, 20]$ and a simulation time interval $[0, 600]$. Dirichlet conditions were imposed. According to [9], an analytical solution of the problem (1) is given by:

$$h(z,t) = 20.4 \tanh(0.5(z + t/12 - 15)) - 41.5. \tag{22}$$

The source term f is chosen using the analytical solution.

We used the analytical solution to generate data that we will use as observations. The use of these analytical data allows us to see the ability of our algorithm to reconstruct the parameters used.

The simulation conditions and the results are given below:

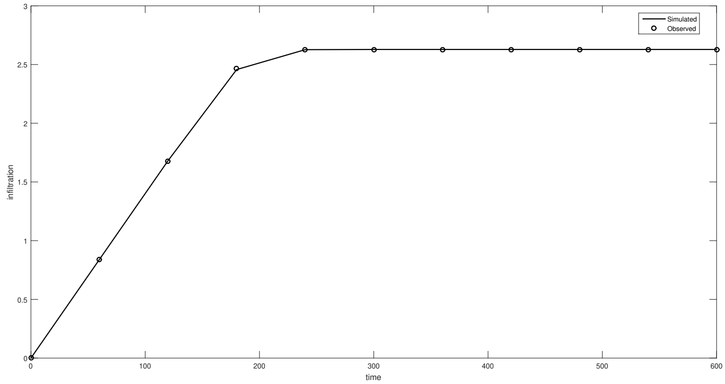

FIGURE 3. Application 1: Curves of infiltration observed and simulated.

Parameters	Range	used values	identified values
θ_s	$[0;4]$	0.368	0.376
θ_r	$[0;4]$	0.102	0.105
α	$[0;1]$	0.0335	0.0302
n	$[0;10]$	2	2.15
K_s	$[0;15]\times 10^{-3}$	9.22×10^{-3}	8.95×10^{-3}

Value of objective function: 1.25×10^{-4}.

Figure 3 shows that the infiltration curve identified well coincides with the curve of analytical data.

5.2. Application 2. In this second application, we use data used in [10]. These were measured on a clay soil on a column of $1m$ long. The values of the infiltration were recorded all the 5 mn during 2 hours.

Parameters	Interval	Genitic algorithm	GWO algorithm
θ_s	$[0;1]$	0.0238	0.0205
θ_r	$[0;2]$	0.379	0.392
α	$[0;1]$	0.0879	0.0858
n	$[0;3]$	1.1359	1.141
K_s	$[0;3]\times 10^{-5}$	1.75×10^{-5}	1.62×10^{-5}

The values of the function cost are:
Genetic algorithm: 0.0027.
GWO algorithm: 0.00018.
Figure 4 shows the curves of infiltration obverved and simulated. These curves show good concordance between observed infiltration and simulated infiltration. This translates the quality of parameter estimation.
Figures 5 and 6 represent the curve of pressure and the curve of moisture content respectively.

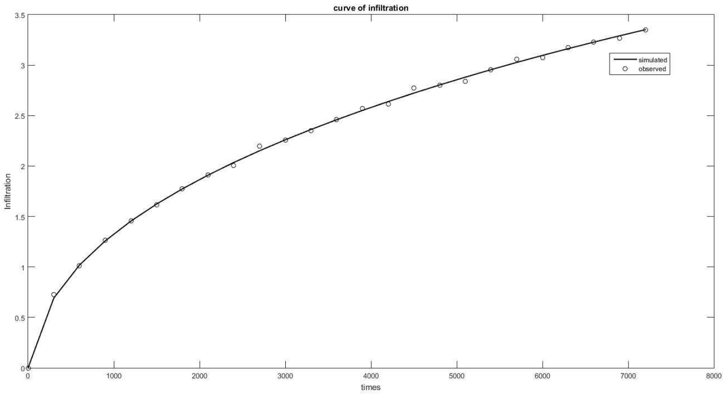

FIGURE 4. Application 2: Curves of infiltration observed and simulated.

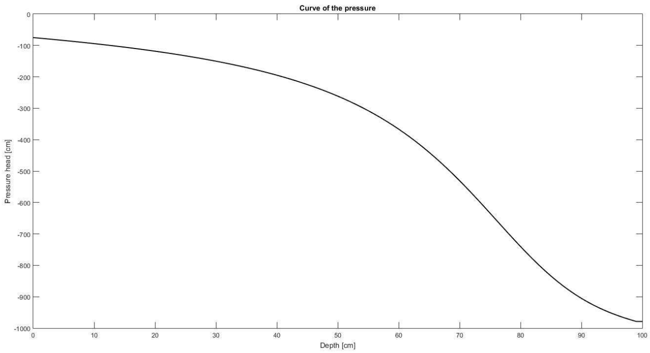

FIGURE 5. Curve of pressure head.

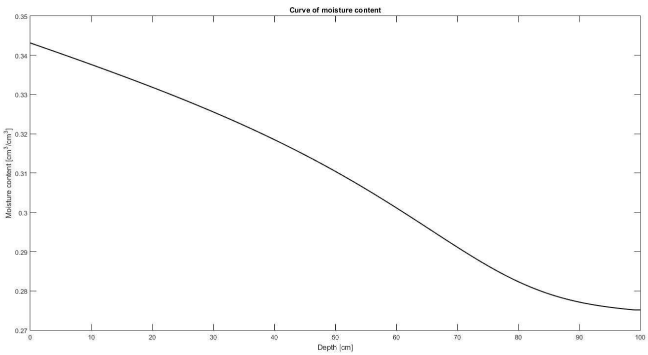

FIGURE 6. Curve of moisture content.

As in [10], the values of the infiltrations observed at the moments $t = 30mn$, $t = 1h$ and $t = 1h30mn$ were not used in the process of identifications. They were used like values test. The Table 1 presents the results. This table shows that the GWO algorithm is better than the genetic algorithm which is confirmed by the convergence curve in Figure 7.

Times	Observed	Calculated (Genitic alg.)	Error	Calculated (GWO alg.)	Error
$30mn$	1.5943	1.5732	0.0211	1.5917	0.0026
$1h$	2.322	2.2901	0.0319	2.3285	0.0065
$1h30mn$	2.8143	2.7939	0.0204	2.8224	0.0081

TABLE 1. Comparison to test points

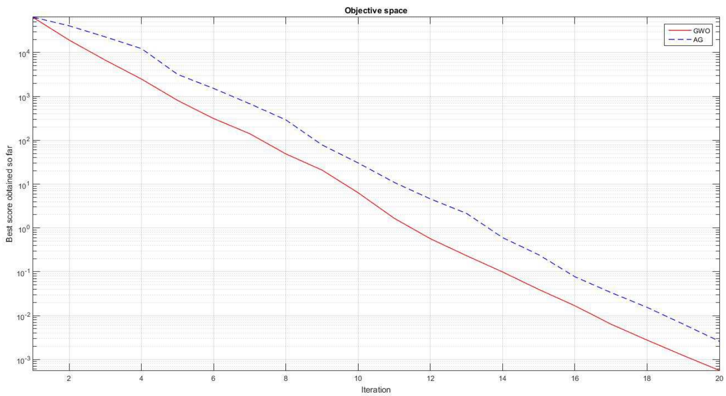

FIGURE 7. Curves of convergence.

6. Conclusion

In this article, the equation simulating the pressure and water content in the unsaturated medium has been reversed. We used Grey Wolves Optimizer (GWO) algorithm to determine parameters of Richards equation using synthetic data and real data. Comparison with the genetic algorithm showed that the GWO algorithm was more effective in identifying parameters involved in the Richards equation.
However, the GWO algorithm despite its effectiveness remains slow. In the future, we intend to propose a parallel version of this algorithm.

References

[1] W. Yeh, Review of parameter identification procedures in groundwater hydrology: the inverse problem, *Water Resources Research* **22** (1986), no. 2, 95–108.

[2] D. McLaughlin, L. R. Townley, A reassessment of the groundwater inverse problem, *Water Resources Research* **32** (1996), no. 5, 1131–1161.

[3] M.T. van Genuchten, A closed-form equation for predicting the hydraulic conductivity of unsaturated soils, *Soil Science Society of American Journal* **44** (1980), 892–898.
[4] M.A. Celia, E.T. Bouloutas, R.L. Zarba, A general mass-conservative numerical solution for unsaturated flow equation, *Water Resources Research* **26** (1990), no. 7, 1483–1496.
[5] P.C.D. Milly, A mass-conservative procedure for time-stepping in models of unsaturated flow, *Advance in Water Ressources* **8** (1985), 32–36.
[6] D.E. Goldberg, *Genetic Algorithms in Search, Optimization and Machine Learning*, Addison Wesley, Reading MA, 1989.
[7] S. Mirjalili, S.M. Mirjalili, A. Lewis, Grey Wolf Optimizer, *Advances in Engineering Software* **69** (2014), 46–61.
[8] K. Rathefelder, L. Abriola, Mass-conservative numerical solutions of the head-based Richards equation, *Water Resources Rechearch* **30** (1994), no. 9, 2579–2586.
[9] P. Sochala, *Méthodes numériques pour les écoulements souterrains et couplage avec le ruissellement*, Thèse de mathémtiques, Paris Est, 2008.
[10] W.O. Sawadogo, N.E. Alaa, Y. Paré, B. Somé, Identification of the parameters of the equation of Richards by the genetic algorithms, *Advances in Theoretical and Applied Mathematics* **11** (2016), no. 1, 17–28.
[11] Y. Mualem, A new model for predicting the hydraulic conductivity of unsaturated porous media, *Water Resources Research* **12** (1976), 513–522.

(W. O. Sawadogo) University of Ouahigouya, Burkina Faso
E-mail address: wenddabo81@gmail.com

(P. O. F Ouédraogo) University of Dédougou, Burkina Faso

(O. Selt) Msila University, Algeria

(K. Somé) Norbert Zongo University, Burkina Faso

(B. Somé) Pr Joseph Ki Zerbo University Ouaga I, Burkina Faso