Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 48(1), 2021, Pages 18–31
ISSN: 1223-6934

Boundary value problem for nonlinear fractional differential equations involving Erdélyi–Kober derivative on unbounded domain

Maria Titraoui and Yacine Arioua

Abstract. In this paper, we establish sufficient conditions for the existence of bounded solution for a class of boundary value problem for nonlinear fractional differential equations involving the Erdélyi–Kober differential operator on unbounded domain. Our results are based on a fixed point theorem of Schauder combined with the diagonalization argument method in a special Banach space. To that end, an example is presented to illustrate the usefulness of our main results.

2010 Mathematics Subject Classification. Primary 34A08; Secondary 34A37.
Key words and phrases. fractional differential equations, boundary value problems, Erdélyi–Kober derivative, Fixed point theorems, diagonalization argument.

1. Introduction

Differential equations of fractional-order have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. More details are available, for instance, in the books Das 2008 [9], Diethelm 2010 [10], Kilbas et al. 2006 [17], Mathai and Haubold 2018 [20], Polubny 1999 [22], Sabatier et al. 2007 [23], and Samko et al. 1993 [24]. Among the various definitions of fractional differentiation. the Riemann-Liouville and Caputo fractional derivatives are widely used in the literature. Besides these integrals, there is another kind of integral operator, introduced by Arthur Erdélyi and Hermann Kober [11] in 1940, which is known as Erdélyi–Kober fractional integral operator. For details and applications of the Erdélyi–Kober fractional integrals, we refer the reader to a series of papers and texts [11, 15, 17, 18, 25, 27]. For some recent contributions on fractional boundary value problems on unbounded domain, see([1], [3], [12], [21]) and the references therein. Very recently, in [2], the authors considered the following boundary value problem on the semi-infinite interval:

$$\begin{cases} y'' + \phi(t) f(t, y, y') = 0, \ 0 < t < \infty, \\ y(0) = 0, \ y \text{ bounded on } [0, \infty), \end{cases}$$

where $f : [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous that satisfies an appropriate condition. Then, the technique they used to established the existence of the solution

Received December 05, 2019. March 15, 2020.

is based on (i) establishing new results (see [4] also) on the finite interval $[0, n]$ for each $n \in \mathbb{N}^*$ and (ii) a diagonalisation argument.

In [6], Arara et al. studied the existence solutions of fractional differential problem of the form:

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), \ t \in [0, \infty), \\ y(0) = y_0, \ y \text{ is bounded on } [0, \infty), \end{cases}$$

by using the fixed point theorem of Schauder combined with the diagonalization method. Where, $1 < \alpha \leq 2$ and ${}^cD^\alpha$ denotes the Caputo fractional derivative, $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $y_0 \in \mathbb{R}$.

In [13], Agarwal et al. considered the following fractional boundary value problems:

$$\begin{cases} D^\alpha y(t) = f(t, y(t)), \ t \in [0, \infty), \ 1 < \alpha \leq 2, \\ y(0) = 0, \ y \text{ is bounded on } [0, \infty), \end{cases}$$

where, D^α is the Riemann-Liouville fractional derivative and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a given function. They used the nonlinear alternative of Leary-Schouder type combined with the diagonalisation method.

The aim of this paper is to study the existence of bounded solution for the boundary value problem of nonlinear fractional differential equation involving Erdélyi–Kober differential operator on unbounded domain

$$\mathcal{D}_\beta^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t \in J = (0, \infty) \tag{1}$$

with the boundary conditions

$$\lim_{t \to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta, m-\delta} u(t) = 0, \text{ with } k = \overline{0, \ m-2}, \ u(t) \text{ bounded on } J, \tag{2}$$

where $\mathcal{D}_\beta^{\gamma,\delta}$ denotes the Erdélyi–Kober fractional derivative operator of order δ and $\mathcal{I}^{\delta+\gamma, m-\delta}$ is the Erdélyi–Kober fractional integral of order $m - \delta$, with $m - 1 < \delta \leq m$, $-m < \gamma < 1 - m, m \in \mathbb{N}, m \geq 2$, $\beta > 0$ and f is a given function required to satisfy the following conditions:

(H1) $f : J \times \mathbb{R} \longrightarrow [0, \infty)$ is continuous.

(H2) There exist $\psi(t) : (0, \infty) \to (0, \infty)$ continuous and in $L^1(0, \infty)$ and $\omega(t) \in ((0, \infty), (0, \infty))$ and non-decreasing such that

$$\left| t^{\beta(1+\gamma)-1} f(t, u) \right| \leq \psi(t)\omega(|u|) \text{ on } (0, \infty) \times \mathbb{R}.$$

In the rest of the paper, we describe some preliminary concepts related to the proposed study in Section 2, while the main existence results are established in Section 3 by applying Schauder's fixed point theorem combined with the diagonalization argument method. Finally we present an example for illustration of our main results.

2. Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory that will be used to derive our main results.

Definition 2.1 ([19])**.** The space of functions C_α^n, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, consists of all functions $f(t), t > 0$, that can be represented in the form $f(t) = t^p f_1(t)$ with $p > \alpha$ and $f_1 \in C^n([0, \infty))$.

Definition 2.2 (*Erdélyi–Kober* fractional integral [19]). The right-hand Erdélyi–Kober fractional integral of the order δ of the function $u \in C_\alpha$ is defined by

$$\left(\mathcal{I}_\beta^{\gamma,\delta} u\right)(t) = \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+\delta)} \int_0^t \left(t^\beta - s^\beta\right)^{\delta-1} s^{\beta(\gamma+1)-1} u(s)\, ds, \ \delta, \beta > 0, \ \gamma \in \mathbb{R}, \quad (3)$$

where Γ is the Euler gamma function.

Definition 2.3 (*Erdélyi–Kober* fractional derivative [19]). Let $n-1 < \delta \leq n, \ n \in \mathbb{N}^*$. The right-hand Erdélyi–Kober fractional derivative of the order δ of the function $u \in C_\alpha^n$ is defined by

$$\left(\mathcal{D}_\beta^{\gamma,\delta} u\right)(t) = \prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_\beta^{\gamma+\delta, n-\delta} u\right)(t), \quad (4)$$

where

$$\prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_\beta^{\gamma+\delta, n-\delta} u\right) = \left(\gamma + 1 + \frac{1}{\beta} t \frac{d}{dt}\right) \dots \left(\gamma + n + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_\beta^{\gamma+\delta, n-\delta} u\right).$$

Lemma 2.1 ([19]). *Let $\delta, \beta > 0, \ \gamma \in \mathbb{R}$, and $u \in C_\alpha$. The Erdélyi–Kober fractional integral defined by (3) has the following properties:*

$$\begin{aligned}
\left(\mathcal{I}_\beta^{\gamma,\delta} x^{\lambda\beta} u\right)(t) &= x^{\lambda\beta} \left(\mathcal{I}_\beta^{\gamma+\lambda,\delta} u\right)(t), \\
\left(\mathcal{I}_\beta^{\gamma,\delta} \mathcal{I}_\beta^{\gamma+\delta,\alpha} u\right)(t) &= \left(\mathcal{I}_\beta^{\gamma,\delta+\alpha} u\right)(t), \\
\left(\mathcal{I}_\beta^{\gamma,\delta} \mathcal{I}_\beta^{\alpha,\eta} u\right)(t) &= \left(\mathcal{I}_\beta^{\alpha,\eta} \mathcal{I}_\beta^{\gamma,\delta} u\right)(t).
\end{aligned}$$

Lemma 2.2 ([19]). *Let $n - 1 < \delta < n, n \in \mathbb{N}^*$, $\alpha \geq -\beta(\gamma + 1)$, and $u \in C_\alpha^n$. Then, the following relationship between the E–K fractional derivative and the E–K fractional integral of order δ is given by*

$$\left(\mathcal{I}_\beta^{\gamma,\delta} \mathcal{D}_\beta^{\gamma,\delta} u\right)(t) = u(t) - \sum_{k=0}^{n-1} c_k t^{-\beta(1+\gamma+k)},$$

where,

$$c_k = \frac{\Gamma(n-k)}{\Gamma(\delta-k)} \lim_{t \to 0} t^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} \left(1 + \gamma + i + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_\beta^{\gamma+\delta, n-\delta} u\right)(t). \quad (5)$$

Definition 2.4 (Equicontinuous). Let E be a Banach space; a subset P in $C(E)$ is called equicontinuous if

$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall u, v \in E, \ \forall \mathcal{A} \in P, \ \|u - v\| < \delta \Rightarrow |\mathcal{A}(u) - \mathcal{A}(v)| < \varepsilon.$$

Theorem 2.3 (Ascoli–Arzela). *Let E be a compact space. If $\mathcal{P}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{P}$ is relatively compact.*

Definition 2.5 (Relatively compact subset). A subset $\mathcal{P}$ of a topological space E is a relatively compact if and only if any sequence in $\mathcal{P}$ has sub-sequence convergent in E.

Definition 2.6 (Completely continuous). Let E be a Banach space; we say that $\mathcal{A} : E \to E$ is completely continuous if for any bounded subset P of E, the set $\mathcal{A}(P)$ is relatively compact.

Theorem 2.4 (Schauder's fixed point [5]). *Let E be a Banach space and let P be a closed, convex and nonempty subset of E. Let $\mathcal{A} : P \to P$ be a continuous mapping such that:*
$\mathcal{A}(P)$ is a relatively compact subset of E. Then $\mathcal{A}$ has at least one fixed point in P.

3. Main results

In this section, we assume that $T_n \in J$, $n \in \mathbb{N}^*$, such that $0 < T_1 < T_2 < ... < T_n < ...$ with $T_n \to \infty$ as $n \to \infty$. In the sequel we set $J_n = (0, T_n]$. In this paper, we use the space $C_l(0,\infty)$ to study the problem (1)-(2), which is denoted by

$$C_l(0,\infty) = \left\{ u \;\middle|\; \begin{array}{l} u \text{ is a continuous function on } (0,+\infty) \text{ such that} \\ \lim_{t\to 0} u(t) \text{ and } \lim_{t\to+\infty} u(t) \text{ exist} \end{array} \right\},$$

from [8],[28], $C_l(0,\infty)$ is a Banach space with the norm

$$\|u\|_{C_l(0,\infty)} = \sup_{t\in(0,\infty)} |u(t)|,$$

furthermore

$$C_l(0,T_n] = \left\{ u \;\middle|\; \begin{array}{l} u \text{ is a continuous function on } (0,T_n] \text{ such that} \\ \lim_{t\to 0} u(t) \text{ exists} \end{array} \right\}.$$

It is easily seen that $C_l(0,T_n]$ is a Banach space with the norm

$$\|u\|_{C_l(0,T_n]} = \sup_{0<t\leq T_n} |u(t)|.$$

3.1. Finite Interval Problem.

To present existence theory for the problem (1) -(2) we begin with the following existence principle for the problem on the finite interval. Fix $n \in \mathbb{N}^*$, for $m = \{2,3,\ldots\}$, with $m-1 < \delta \leq m$, $-m < \gamma < 1-m$, $\beta > 0$, we consider the following boundary value problem

$$\begin{cases} \mathcal{D}_\beta^{\gamma,\delta} u(t) + f(t,u(t)) = 0, \; t \in J_n, \\ \lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta,m-\delta} u(t) = 0, \text{ with } k = \overline{0,m-2}, \\ u(T_n) = 0. \end{cases} \tag{6}$$

Based on the previous lemma, we will define the integral solution of the finite interval problem (6).

Lemma 3.1. *Let $m-1 < \delta \leq m$, $-m < \gamma < 1-m$, $m \geq 2$, $\beta > 0$ and $y \in C_\alpha^2$, then the fractional differential equation:*

$$\mathcal{D}_\beta^{\gamma,\delta} u(t) + y(t) = 0, \; t \in J_n, \tag{7}$$

with the conditions

$$\lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta,m-\delta} u(t) = 0, \; \textit{with } k = \overline{0,m-2}, \tag{8}$$

$$u(T_n) = 0, \; n \in \mathbb{N}^* \; \textit{is fixed.} \tag{9}$$

has a unique solution given by

$$u(t) = \int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} y(s)\, ds, \tag{10}$$

where

$$G_n(t,s)=\begin{cases}\frac{\beta}{\Gamma(\delta)}\left[t^{-\beta(\gamma+1)}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}-t^{-\beta(\delta+\gamma)}\left(t^{\beta}-s^{\beta}\right)^{\delta-1}\right], & 0<s\le t\le T_n,\\ \frac{\beta}{\Gamma(\delta)}t^{-\beta(\gamma+1)}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}, & 0<t\le s\le T_n,\end{cases} \tag{11}$$

is called the Green function of boundary value problem (7)-(8)-(9).

Proof. Let $m-1<\delta\le m$, with $-m<\gamma<1-m$, and $\beta>0$; it is easy to prove that the operator $\mathcal{I}_{\beta}^{\gamma,\delta}$ has the linearity property for all $\delta>0$. By applying $\mathcal{I}_{\beta}^{\gamma,\delta}$ to equation (7) we obtain

$$\mathcal{I}_{\beta}^{\gamma,\delta}\ \mathcal{D}_{\beta}^{\gamma,\delta}u(t)+\mathcal{I}_{\beta}^{\gamma,\delta}y(t)=0. \tag{12}$$

By using Lemma 2.2, for $m-1<\delta\le m$, we can easily find that

$$\mathcal{I}_{\beta}^{\gamma,\delta}\ \mathcal{D}_{\beta}^{\gamma,\delta}u(t)=u(t)-c_0t^{-\beta(1+\gamma)}+c_1t^{-\beta(2+\gamma)}+\cdots+c_{m-1}t^{-\beta(m+\gamma)},$$

for some constants $c_0,c_1,\cdots c_{m-1}\in\mathbb{R}$. Thus, (12) gives

$$u(t)-c_0t^{-\beta(1+\gamma)}-c_1t^{-\beta(2+\gamma)}+\cdots+c_{m-1}t^{-\beta(m+\gamma)}+\mathcal{I}_{\beta}^{\gamma,\delta}y(t)=0,$$

which means that

$$u(t)=c_0t^{-\beta(1+\gamma)}+c_1t^{-\beta(2+\gamma)}+\cdots+c_{m-1}t^{-\beta(m+\gamma)}-\mathcal{I}_{\beta}^{\gamma,\delta}y(t). \tag{13}$$

From the formula (5) of Lemma 2.2, it follows that

$$\begin{aligned} c_0 &= \lim_{t\to 0}t^{\beta(1+\gamma)}\prod_{i=1}^{m-1}\left(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt}\right)\left(I^{\delta+\gamma,m-\delta}u\right)(t)\\ &= \lim_{t\to 0}t^{\beta(1+\gamma)}\left[\begin{array}{l}(2+\gamma)(3+\gamma)\cdots(m+\gamma)\left(I^{\delta+\gamma,m-\delta}u\right)(t)\\ +\xi_1(n,\gamma)\frac{1}{\beta}t\frac{d}{dt}\left(I^{\delta+\gamma,m-\delta}u\right)(t)+\cdots\\ +\xi_{m-2}(m,\gamma)\frac{1}{\beta^{m-2}}t^{m-2}\frac{d^{(m-2)}}{dt^{(m-2)}}\left(I^{\delta+\gamma,m-\delta}u\right)(t)\\ +\frac{1}{\beta^{m-1}}t^{m-1}\frac{d^{(m-1)}}{dt^{(m-1)}}\left(I^{\delta+\gamma,m-\delta}u\right)(t),\\ \text{with, }\xi_1,\cdots,\xi_{m-2}\in\mathbb{R}\end{array}\right]\end{aligned}$$

$$
\begin{aligned}
c_1 &= \lim_{t\to 0} t^{\beta(2+\gamma)} \prod_{i=2}^{m-1} \left(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt}\right)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
&= \lim_{t\to 0} t^{\beta(2+\gamma)} \left[\begin{array}{l} (3+\gamma)\cdots(m+\gamma)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\ +\xi_1(m,\gamma)\frac{1}{\beta}t\frac{d}{dt}\left(I^{\delta+\gamma,m-\delta}u\right)(t)+\cdots \\ +\xi_{m-3}(m,\gamma)\frac{1}{\beta^{m-3}}t^{m-3}\frac{d^{(m-3)}}{dt^{(m-3)}}\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\ +\frac{1}{\beta^{m-2}}t^{m-2}\frac{d^{(m-2)}}{dt^{(m-2)}}\left(I^{\delta+\gamma,m-\delta}u\right)(t), \\ \text{with, } \xi_1,\cdots,\xi_{m-3}\in\mathbb{R} \end{array}\right]; \\
&\vdots \\
c_{m-2} &= \lim_{t\to 0} t^{\beta(m-1+\gamma)} \prod_{i=m-1}^{m-1} \left(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt}\right)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
&= \lim_{t\to 0} t^{\beta(m-1+\gamma)} \left[(m+\gamma)\left(I^{\delta+\gamma,m-\delta}u\right)(t)+\frac{1}{\beta}t\frac{d}{dt}\left(I^{\delta+\gamma,m-\delta}u\right)(t)\right]; \\
c_{m-1} &= \lim_{t\to 0} t^{\beta(m+\gamma)} \prod_{i=m}^{m-1} \left(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt}\right)\left(I^{\delta+\gamma,m-\delta}u\right)(t) \\
&= \lim_{t\to 0} t^{\beta(m+\gamma)} \left(I^{\delta+\gamma,m-\delta}u\right)(t).
\end{aligned}
$$

The boundary condition (8) implies that $c_{m-1}=c_{m-2}=\cdots=c_1=0$, which means that we can rewrite the integral equation (13) as

$$u(t)=c_0t^{-\beta(1+\gamma)}-\mathcal{I}_\beta^{\gamma,\delta}y(t).$$

In view of the boundary condition (9) we conclude that

$$c_0T_n^{-\beta(1+\gamma)}-\mathcal{I}_\beta^{\gamma,\delta}y(T_n)=0.$$

Consequently, we find that

$$
\begin{aligned}
c_0 &= \frac{\beta T_n^{-\beta(\gamma+\delta)}}{\Gamma(\delta)T_n^{-\beta(1+\gamma)}}\int_0^{T_n}\left(T_n^\beta-s^\beta\right)^{\delta-1}s^{\beta(\gamma+1)-1}y(s)ds \\
&= \frac{\beta}{\Gamma(\delta)}\int_0^{T_n}T_n^{-\beta(\delta-1)}\left(T_n^\beta-s^\beta\right)^{\delta-1}s^{\beta(\gamma+1)-1}y(s)ds \\
&= \frac{\beta}{\Gamma(\delta)}\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^\beta\right)^{\delta-1}s^{\beta(\gamma+1)-1}y(s)ds,
\end{aligned}
$$

and therefore, the unique solution of the problem (7)-(8)-(9) is given by

$$
\begin{aligned}
u(t) &= \frac{\beta}{\Gamma(\delta)}\left[\begin{array}{l} t^{-\beta(1+\gamma)}\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^\beta\right)^{\delta-1}s^{\beta(\gamma+1)-1}y(s)ds \\ -t^{-\beta(\gamma+\delta)}\int_0^t\left(t^\beta-s^\beta\right)^{\delta-1}s^{\beta(\gamma+1)-1}y(s)\,ds \end{array}\right] \\
&= \int_0^{T_n}G_n(t,s)\,s^{\beta(\gamma+1)-1}y(s)ds.
\end{aligned}
$$

□

Now, we present some properties of Green function that form the basis of our main work.

Remark 3.1. For $m-1<\delta\leq m$, $-m<\gamma<1-m$, and $\beta>0$ the following conditions is hold:

1. For all $\forall t,s\in(0,\infty)$, the function $G_n(t,s)\geq 0$.
2. For each $n>0$, the function $t\in J_n\to\int_0^{T_n}|G_n(t,s)|\,ds$ is continuous and bounded on J_n.

We now turn to the question of existence for the problem (6).

Define an integral operator $\mathcal{A}:C_l(0,T_n]\to C_l(0,T_n]$ by

$$\mathcal{A}u(t)=\int_0^{T_n}G_n(t,s)\,s^{\beta(1+\gamma)-1}f(s,u(s))ds,\ t\in J_n, \tag{14}$$

where $G_n(t,s)$ defined by (11).

Clearly, from Lemma 3.1, the fixed points of the operator $\mathcal{A}$ coincide with the solutions of the problem (6).

We put

$$\widetilde{G}_n=\sup\left\{\int_0^{T_n}|G_n(t,s)|\,ds,\ t\in J_n\right\}.\ \psi_n^*=\sup\{\psi(s),\ s\in J_n\}.$$

Lemma 3.2. *If (H1)-(H2) hold, Then $\mathcal{A}:C_l(0,T_n]\to C_l(0,T_n]$ is completely continuous.*

Proof. First, for $u\in C_l(0,T_n]$ we have

$$\begin{aligned}\|\mathcal{A}u(t)\|_{C_l(0,T_n]} &= \sup_{0<t\leq T_n}|\mathcal{A}u(t)|=\sup_{0<t\leq T_n}\left|\int_0^{T_n}G_n(t,s)\,s^{\beta(\gamma+1)-1}f(s,u(s))ds\right| \\ &\leq \sup_{0<t\leq T_n}\int_0^{T_n}|G_n(t,s)|\left|s^{\beta(\gamma+1)-1}f(s,u(s))ds\right|.\end{aligned}$$

Together with conditions (H1) and (H2), it then follows that

$$\begin{aligned}&\sup_{0<t\leq T_n}\int_0^{T_n}|G_n(t,s)|\left|s^{\beta(\gamma+1)-1}f(s,u(s))ds\right| \\ \leq\ &\sup_{0<t\leq T_n}\int_0^{T_n}|G_n(t,s)|\,\psi(s)\omega(|u(s)|)\,ds \\ \leq\ &\psi_n^*\omega\left(\|u\|_{C_l(0,T_n]}\right)\int_0^{T_n}|G_n(t,s)|\,ds \\ \leq\ &\psi_n^*\omega\left(\|u\|_{C_l(0,T_n]}\right)\widetilde{G}_n<\infty.\end{aligned}$$

Hence, $\mathcal{A}:C_l(0,T_n]\to C_l(0,T_n]$ is well-defined.

Choose

$$M\geq\psi_n^*\omega(M)\,\widetilde{G}_n, \tag{15}$$

and let

$$\Omega=\left\{u\in C_l(0,T_n],\|u\|_{C_l(0,T_n]}\leq M,\ M>0\right\}.$$

In what follows we divide the proof into several steps.

Step 1: $\mathcal{A}:\Omega\to C_l\left(0,T_n\right]$ is continuous.

Let $\left(u_q\right)_{q\in\mathbb{N}}\in\Omega$ be a convergent sequence to u in Ω, from Lemma 3.1 we obtain that

$$\begin{aligned}\left\|\mathcal{A}u_q-\mathcal{A}u\right\|_{C_l(0,T_n]} &= \sup_{0<t\le T_n}\left|\int_0^{T_n} G_n\left(t,s\right)s^{\beta(\gamma+1)-1}\left[f(s,u_q(s))-f(s,u(s))\right]ds\right|\\ &\le \sup_{0<t\le T_n}\left|\frac{\beta}{\Gamma\left(\delta\right)}t^{-\beta(\gamma+1)}\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}\right.\\ &\quad\left.\times\left[f(s,u_q(s))-f(s,u(s))\right]ds\right|\\ &\le \sup_{0<t\le T_n}\frac{\beta}{\Gamma\left(\delta\right)}t^{-\beta(\gamma+1)}\left|\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}\right.\\ &\quad\left.\times f(s,u_q(s))ds-\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}f(s,u(s))ds\right|.\end{aligned}$$

Due to the condition (H2), we get

$$\begin{aligned}\left|\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}f(s,u(s))\right| &\le \left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}\psi(s)\omega\left(\left|u(s)\right|\right)\\ &\le \left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}\psi(s)\omega\left(\left\|u\right\|_{C_l(0,T_n]}\right)\\ &\le \omega\left(M\right)\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}\psi(s).\end{aligned}$$

Since the right hand side of the above inequality is in $L^1\left(0,\infty\right)$ and the function $\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}f(s,u(s))$ is continuous, it follows that the Lebesgue dominated convergence theorem (theorem 12.12, page 199 in [7]) yields $u\to\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}f(s,u(s))ds$ is continuous. Hence, it holds that

$$\begin{aligned}&\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}f(s,u_q(s))ds\\ &\qquad\to\int_0^{T_n}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}s^{\beta(\gamma+1)-1}f(s,u(s))ds\text{ as }q\to\infty.\end{aligned}$$

Therefore, $\left\|\mathcal{A}u_q-\mathcal{A}u\right\|_{C_l(0,T_n]}\to 0$, as $q\to\infty$.

Step 2: $\mathcal{A}\left(\Omega\right)$ is relatively compact.

First, we show that $\mathcal{A}(\Omega)$ is uniformly bounded. Let $u \in \Omega$, by the condition (H2), we obtain

$$\begin{aligned}
\|\mathcal{A}u(t)\|_{C_l(0,T_n]} &= \sup_{0<t\le T_n} |\mathcal{A}u(t)| = \sup_{0<t\le T_n} \left| \int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\
&\le \sup_{0<t\le T_n} \int_0^{T_n} |G_n(t,s)| \left| s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\
&\le \psi_n^* \omega\left(\|u\|_{C_l(0,T_n]}\right) \int_0^{T_n} |G_n(t,s)|\, ds \\
&\le \psi_n^* \omega(M)\, \widetilde{G}_n,
\end{aligned}$$

hence, $\mathcal{A}(\Omega)$ is uniformly bounded.

Next, we show that $\mathcal{A}(\Omega)$ is equicontinuous on J_n.
For all $u \in \Omega$, $t_1, t_2 \in J_n$ and $t_1 \le t_2$, we can find

$$\begin{aligned}
|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| &\le \int_0^{T_n} |G_n(t_2,s) - G_n(t_1,s)| \left| s^{\beta(\gamma+1)-1} f(s,u(s)) \right| ds \\
&\le \psi_n^* \omega(M) \int_0^{T_n} |G_n(t_2,s) - G_n(t_1,s)|\, ds \to 0,
\end{aligned}$$

uniformly as $t_1 \to t_2$ for all $u \in \Omega$.
Hence, $\mathcal{A}(\Omega)$ is locally equicontinuous on J_n. Consequently, $\mathcal{A}(\Omega)$ is relatively compact.

Therefore, $\mathcal{A} : C_l(0,T_n] \to C_l(0,T_n]$ is completely continuous. □

Now, to prove the existence result for the problem (6), we use the fixed point theorem of Schauder.

Theorem 3.3. *Assume that the hypotheses (H1)-(H2) hold, and that there exists $M \in \mathbb{R}$ satisfying (15). Then the fractional boundary value problem (6) has at least one solution $u \in \Omega$.*

Proof. From the proof of Lemma 3.2, we know that $\mathcal{A}$ is a completely continuous operator.

Also we have $\mathcal{A}(\Omega) \subset \Omega$ because of

$$\begin{aligned}
\|\mathcal{A}u(t)\|_{C_l(0,T_n]} &= \sup_{0<t\le T_n} |\mathcal{A}u(t)| = \sup_{0<t\le T_n} \left| \int_0^{T_n} G_n(t,s)\, s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\
&\le \sup_{0<t\le T_n} \int_0^{T_n} |G_n(t,s)| \left| s^{\beta(\gamma+1)-1} f(s,u(s))ds \right| \\
&\le \psi_n^* \omega\left(\|u\|_{C_l(0,T_n]}\right) \int_0^{T_n} |G_n(t,s)|\, ds \\
&\le \psi_n^* \omega(M)\, \widetilde{G}_n \le M.
\end{aligned}$$

Hence, by Theorem 2.4 the boundary value problem (6) has at least one solution u in Ω such that

$$|u(t)| \le M, \text{ for each } t \in J_n.$$

□

3.2. Semi-infinite Interval Problem. The ideas in the previous part together with a diagonalization argument enable us to treat the problem (1)-(2) defined on semi-infinite interval.

Theorem 3.4. *Assume that the hypotheses (H1)-(H2) hold, and that there exists $M \in \mathbb{R}$ satisfying (15). Then the fractional boundary value problem (1)-(2) has at least one solution u on $(0,\infty)$.*

Proof. The proof will be given in tow parts.
Part 1: From Theorem 3.3, for all $n \in \mathbb{N}^*$ we show that the following boundary value problems

$$\begin{cases} \mathcal{D}_{\beta}^{\gamma,\delta} u\left(t\right) + f(t, u\left(t\right)) = 0, \ t \in J_n, \\ \lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta,m-\delta} u(t) = 0, \ \text{with } k = \overline{0, m-2}, \\ u(T_n) = 0. \end{cases}$$

have a solution $u_n \in C_l\left(J_n, \mathbb{R}\right)$ verifies that for each $t \in J_n$, $\left|u_n\left(t\right)\right| \leq M$, with

$$u_n\left(t\right) = \int_0^{T_n} G_n\left(t,s\right) s^{\beta(\gamma+1)-1} f(s, u(s)) ds,$$

where, $G_n(t,s)$ defined by (11).
Part 2: Diagonalization argument
Define

$$v_n(t) = \begin{cases} u_n\left(t\right), & t \in (0, T_n], \\ 0, & t \in [T_n, \infty). \end{cases}$$

Then v_n is in $C_l\left(0,\infty\right)$ with $\|v_n(t)\|_{C_l(0,\infty)} \leq M$, $t \in (0,\infty)$.
Let $S = \left\{(v_n)_{n\in\mathbb{N}^*}\right\}$. For $t \in (0, T_1]$, we have

$$|v_n(t)| \leq \left|u_n\left(t\right)\right| \leq M, \ \forall n \in \mathbb{N}^*,$$

which means that, for all $t \in (0, T_1]$; $(v_n)_{n\in\mathbb{N}^*}$ is bounded. Furthermore for all $t_1, t_2 \in (0, T_1]$, $\forall n \in \mathbb{N}^*$, we have

$$\begin{aligned} |v_n(t_2) - v_n(t_1)| &\leq |u_n(t_2) - u_n(t_1)| \\ &\leq \int_0^{T_1} \left|G_1\left(t_2,s\right) - G_1\left(t_1,s\right)\right| \left|s^{\beta(\gamma+1)-1} f(s,u(s))\right| ds \\ &\leq \psi_1^* \omega\left(M\right) \int_0^{T_1} \left|G_1\left(t_2,s\right) - G_1\left(t_1,s\right)\right| ds \to 0 \text{ for } t_1 \to t_2. \end{aligned}$$

It holds that, for all $t \in (0, T_1]$, $(v_n)_{n\in\mathbb{N}^*}$ is equicontinous. Thus, $S = \left\{(v_n)_{n\in\mathbb{N}^*}\right\}$ is relatively compact on $(0, T_1]$. Let $\mathbb{N}^1 = \mathbb{N}^* - \{1\}$. For all $(v_n)_{n\in\mathbb{N}^*} \in S$, the Arzela-Ascoli Theorem 2.3, guarantees that there is a sub-sequence $(v_n)_{n\in\mathbb{N}^1}$ and a function z_1 in $C_l\left(0, T_1\right]$ such that $(v_n)_{n\in\mathbb{N}^1} \to z_1$ uniformly on $(0, T_1]$ as $n \to \infty$.
Let $S_1 = \left\{(v_n)_{n\in\mathbb{N}^1}\right\}$. For $t \in (0, T_2]$, we have

$$|v_n(t)| \leq \left|u_n\left(t\right)\right| \leq M, \ \forall n \in \mathbb{N}^1,$$

which means that for all $t \in (0, T_2]$, $(v_n)_{n\in\mathbb{N}^1}$ is bounded. Furthermore for all $t_1, t_2 \in (0, T_2]$, $n \in \mathbb{N}^*$, we have

$$\begin{aligned} |v_n(t_2) - v_n(t_1)| &\leq |u_n(t_2) - u_n(t_1)| \\ &\leq \int_0^{T_2} |G_2(t_2, s) - G_2(t_1, s)| \left| s^{\beta(\gamma+1)-1} f(s, u(s)) \right| ds \\ &\leq \psi_2^* \omega(M) \int_0^{T_2} |G_2(t_2, s) - G_2(t_1, s)| \, ds \to 0 \text{ for } t_1 \to t_2. \end{aligned}$$

It holds that, for all $t \in (0, T_2]$, $(v_n)_{n\in\mathbb{N}^1}$ is equicontinous. Thus, $S_1 = \left\{ (v_n)_{n\in\mathbb{N}^1} \right\}$ is relatively compact on $(0, T_2]$.
Let $\mathbb{N}^2 = \mathbb{N}^1 - \{2\}$. For all $(v_n)_{n\in\mathbb{N}^1} \in S$, the Arzela-Ascoli Theorem 2.3, guarantees that there is a sub-sequence $(v_n)_{n\in\mathbb{N}^2}$ and a function z_2 in $C_l(0, T_2]$ such that $(v_n)_{n\in\mathbb{N}^2} \to z_2$ uniformly on $(0, T_2]$ as $n \to \infty$. Note that $z_2 = z_1$ on $(0, T_1]$ since $\mathbb{N}^2 \subset \mathbb{N}^1$. Proceed inductively to obtain for $q \in \{3, 4, \cdots\}$ there is a sub-sequence $(v_n)_{n\in\mathbb{N}^q}$ with $\mathbb{N}^q \subset \mathbb{N}^*$ and $\mathbb{N}^q \subset \mathbb{N}^{q-1}$ and a function z_q in $C_l(0, T_q]$ such that $(v_n)_{n\in\mathbb{N}^q} \to z_q$ as $n \to \infty$. Also $z_q = z_{q-1}$ on $(0, T_{q-1}]$.

Define a function u as follows

$$u(t) = \begin{cases} z_q(t), & t \in (0, T_q], \\ 0, & t \in [T_q, \infty). \end{cases}$$

Then $u \in C_l(0, \infty)$, $\lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta, n-\delta} u(t) = 0$, with $k = \overline{0, m-2}$, $m \geq 2$, and $|u(t)| \leq M$, for $t \in (0, \infty)$. Again fix $t \in (0, \infty)$ and let $q \in \mathbb{N}^*$ with $t \leq T_q$. Then for $n \in \mathbb{N}^q$, we have

$$v_n(t) = \int_0^{T_q} G_q(t, s) \, s^{\beta(\gamma+1)-1} f(s, v_n(s)) ds.$$

Passing to limit, as $n \to \infty$ (using [14], p. 38 or [16], p. 35), we obtain

$$z_q(t) = \int_0^{T_q} G_q(t, s) \, s^{\beta(\gamma+1)-1} f(s, z_q(s)) ds,$$

Thus

$$u(t) = \int_0^{T_q} G_q(t, s) \, s^{\beta(\gamma+1)-1} f(s, u(s)) ds,$$

which implies $\lim_{t\to 0} t^{\beta(1+\gamma)} \frac{d^k}{dt^{(k)}} I^{\gamma+\delta, n-\delta} u(t) = 0$, with $k = \overline{0, m-2}$, $u \in C_l(0, \infty)$ and

$$\mathcal{D}_\beta^{\gamma,\delta} u(t) + f(t, u(t)) = 0, \ t \in J = (0, \infty).$$

□

4. An example

Consider the following boundary value problem:

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2}, \frac{3}{2}} u(t) + t^{\frac{3}{2}} \sqrt{\left| \frac{u}{1+t^{\frac{1}{2}}} \right|} e^{-t} = 0, \ t > 0, \\ \lim_{t\to 0} t^{-\frac{1}{2}} \frac{d^k}{dt^{(k)}} I_1^{0, m-\frac{1}{2}} u(t) = 0, \ \text{with } k = \overline{0, m-2}, \ u(t) \text{ bounded on } J. \end{cases} \tag{16}$$

Here, $f(t,u) = t^{\frac{3}{2}}\sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|}e^{-t}$, $\delta = \frac{3}{2}$, $\gamma = -\frac{3}{2}$ and $\beta = 1$.

(H1) Clearly, the function f is continuous for any $(t,u) \in (0,\infty) \times \mathbb{R}$.

(H2) From the expression of the function f, it follows that

$$t^{\beta(1+\gamma)-1} f\left(t,u\right) = \sqrt{|u|}e^{-t}.$$

If we choose $\omega\left(u\right) = \sqrt{u}$, $\psi\left(t\right) = e^{-t}$, then we obtain

$$|F(t,u)| \leq \psi(t)\omega(|u|), \text{ on } (0,\infty)\times\mathbb{R},$$

with $\omega \in C\left(\left(0,\infty\right),\left(0,\infty\right)\right)$ non-decreasing and $\psi(t) : (0,\infty) \to (0,\infty)$ continuous and in $L^1\left(0,\infty\right).$ Then, the condition (H2) holds.
On the other hand, we show that
i) $\psi_n^* = \sup_{t\in(0,T_n]}\psi(t) = 1.$
ii) $\widetilde{G}_n = \sup_{t\in(0,T_n]}\int_0^{T_n}\left|G_n\left(t,s\right)\right|ds$, we have to consider two cases.

Case 1: for $s \leq t$, we have

$$\begin{aligned}
\widetilde{G}_n &= \sup_{t\in(0,T_n]}\frac{\beta}{\Gamma\left(\delta\right)}\int_0^{T_n}\left|t^{-\beta(\gamma+1)}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1} - t^{-\beta(\delta+\gamma)}\left(t^{\beta}-s^{\beta}\right)^{\delta-1}\right|ds \\
&= \sup_{t\in(0,T_n]}\int_0^{T_n}\left|t^{\frac{1}{2}}\left(1-\frac{s}{T_n}\right)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}\right|ds \\
&\leq \sup_{t\in(0,T_n]} t^{\frac{1}{2}}\int_0^{T_n}\left|\left(1-\frac{s}{T_n}\right)^{\frac{1}{2}}\right|ds + \sup_{t\in(0,T_n]}\int_0^{T_n}\left|(t-s)^{\frac{1}{2}}\right|ds \\
&\leq \sup_{t\in(0,T_n]} t^{\frac{1}{2}}\int_0^{T_n}\left(1-\frac{s}{T_n}\right)^{\frac{1}{2}}ds + \sup_{t\in(0,T_n]}\int_0^{T_n}(t-s)^{\frac{1}{2}}\,ds \\
&\leq \sup_{t\in(0,T_n]}\left[\frac{2}{3}t^{\frac{1}{2}}T_n - \frac{2}{3}\left(t-T_n\right)^{\frac{3}{2}} + \frac{2}{3}t^{\frac{3}{2}}\right] \\
&\leq \sup_{t\in(0,T_n]}\frac{2}{3}t^{\frac{1}{2}}T_n - \inf_{t\in(0,T_n]}\frac{2}{3}\left(t-T_n\right)^{\frac{3}{2}} + \sup_{t\in(0,T_n]}\frac{2}{3}t^{\frac{3}{2}} \\
&\leq 2T_n^{\frac{3}{2}}.
\end{aligned}$$

Case 2: for $t \leq s$, we have

$$\begin{aligned}
\widetilde{G}_n &= \sup_{t\in(0,T_n]}\frac{\beta}{\Gamma\left(\delta\right)}\int_0^{T_n}\left|t^{-\beta(\gamma+1)}\left(1-\left(\frac{s}{T_n}\right)^{\beta}\right)^{\delta-1}\right|ds \\
&= \sup_{t\in(0,T_n]} t^{\frac{1}{2}}\int_0^{T_n}\left(1-\frac{s}{T_n}\right)^{\frac{1}{2}}ds \\
&= \sup_{t\in(0,T_n]}\frac{2}{3}t^{\frac{1}{2}}T_n \\
&= \frac{2}{3}T_n^{\frac{3}{2}} \\
&\leq 2T_n^{\frac{3}{2}}.
\end{aligned}$$

Now, If we choose $M \geq 4T_n^3$, then we get

$$\psi_n^* \omega\left(M\right) \widetilde{G}_n = 2T_n^{\frac{3}{2}} \sqrt{M} \leq M.$$

therefore, (15) is satisfied. Hence, all the conditions of Theorem 3.4 hold, which means that the boundary value problem (16) has at least one solution.

5. Conclusion

In this work, the existence of bounded solution for the nonlinear fractional differential equations with initial conditions comprising the Erdélyi–Kober fractional derivative on unbounded domain have been discussed in a special Banach space $C_l\left(0, \infty\right)$. For our discussion, we have used the fixed point theorem of Schauder combined with diagonalization argument.
The fractional differential operator used in this paper is the Erdélyi–Kober fractional derivative which is generalization of the Rieman-Lioville fractional derivative. Future work will be directed toward fractional coupled systems of differential equations involving Erdélyi–Kober derivatives.

6. Acknowledgements

The authors are deeply grateful to the anonymous referees for their acceptance to read this paper.

References

[1] R.P. Agarwal, M. Bohner, and D. O'Regan, Time scale boundary value problems on infinite intervals, *J. Comput. Appl. Math.* **141** (2002), no. 1-2, 27–34.
[2] R.P Agarwal and D. O'Regan, Boundary value problems of nonsingular type on the semi-infinite interval, *Tohoku. Math. J.* **51** (1999), 391–397.
[3] R.P. Agarwal and D. O'Regan, *Infinite interval problems for differential, difference and integral equations*, Springer Science and Business Media, 2012.
[4] R.P. Agarwal and D. O'Regan, Second and higher order boundary value problems of nonsingular type, *Bull. Belg. Math. Soc. Simon Stevin* **7** (2000), no. 1, 43–52.
[5] B. Ahmad, A. Alsaedi, S.K. Ntouyas, and J. Tariboon, *Hadamard-type fractional differential equations, inclusions and inequalities*,Springer International Publishing, 2017.
[6] A. Arara, M. Benchohra, N. Hamidi, and J. J. Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Anal.* **72** (2010), no. 2, 580–586.
[7] R.G. Bartle, A modern theory of integration, *Grad. Stud. Math.* **32** (2001).
[8] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.
[9] S. Das, *Functional Fractional Calculus for System Identification and Controls*, Springer-Verlag Berlin Heidelberg, 2008.
[10] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer Berlin, 2010.
[11] A. Erdélyi and H. Kober, Some remarks on Hankel transforms, *Quart. J.Math., Oxford, Second Ser.* **11** (1940), 212–221.
[12] A. Granas, R.B. Guenther, J.W. Lee, and D. O'Regan, Boundary value problems on infinite intervals and semiconductor devices, *J. Math. Anal. Appl.* **116** (1986), no. 2, 335–348.
[13] R.P. Agarwal, M. Benchohra, S.Hamani, and S. Pinelas, Boundary value problems for differential equations involving Riemann-Liouville fractional derivative on the half line, *Dynamics*

of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis **18** (2011), 235–244.

[14] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), 18–56.

[15] S.L. Kalla and V.S. Kiryakova, An H-function generalized fractional calculus based upon compositions of Erdélyi-Kober operators in Lp, *Math. Japonica* **35** (1990), 1–21.

[16] B. Kaymakcalan, V. Lakshmikantham, and S. Sivasundaram, *Dynamical Systems on Measure Chains*, Springer US, 1996.

[17] A.A. Kilbas, H.H. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.

[18] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman and John Wiley, New York, 1994.

[19] Y. Luchko and J. Trujillo, Caputo-type modification of the Erdélyi-Kober fractional derivative, *Fract. Calc. Appl. Anal.* **10** (2007), no. 3, 249–267.

[20] A.M. Mathai and H.J. Haubold, *Erdélyi-Kober Fractional Calculus*, Springer Nature Singapore Pte Ltd, 2018.

[21] D. O'Regan, Positive solutions for a class of boundary value problems on infinite intervals, *NoDEA Nonlinear Differential Equations Appl.* **1** (1994), no. 3, 203–228.

[22] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York, 1999.

[23] J. Sabatier, O.P. Agrawal, and J.A. Tenreiro Machado, *Advances in Fractional Calculus Theoretical Developments and Applicationsin Physics and Engineering*, Springer, 2007.

[24] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integral and Derivatives Theory and Applications*, Gordon and Breach, Switzerland, 1993.

[25] I.N. Sneddon, The use in mathematical analysis of the Erdélyi-Kober operators and some of their applications, In: *Lect Notes Math.* 457, Springer–Verlag, New York, 1975, 37–79.

[26] M. Titraoui and Y. Arioua, Unbounded solution of boundary value problem for nonlinear Caputo-type Erdély Kober fractional differential equation on the half-line, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* **28** (2021), 51–76.

[27] S.B. Yakubovich and Yu.F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions. Mathematics and Its Applications*, Springer Netherlands, 1994.

[28] Z. Zhao, Positive solutions of nonlinear second order ordinary differential equations, *Proc. Amer. Math. Soc.* **121** (1994), no. 2, 465–469.

(Maria Titraoui) Laboratory for Pure and Applied Mathematics, University of M'sila, Bp 166 M'sila, 28000, Algeria
E-mail address: maria.titraoui@univ-msila.dz

(Yacine Arioua) Laboratory for Pure and Applied Mathematics, University of M'sila, Bp 166 M'sila, 28000, Algeria
E-mail address: yacine.arioua@univ-msila.dz