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Stability of probability of return on locally compact groups under quasi-isometry

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Abstract. The aim of this article is to demonstrate the invariance of the probability of return to the origin on the locally compact groups by quasi isometries. This is a generalization of a theorem of Ch. Pittet and L. Salof-Coste (see [1]) who established this invariance in the case of discrete groups finitely generated.

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1. Prelimiaries

Let G be a locally compact, separable, compactly generated and unimodular group. Let e be the unit element of G. We denote by μ the Haar measure on G. In case we manipulate several groups, we will denote μ_G the Haar measure on G.

The notation ν denotes a probability measure on G associated to a density F with respect to the Haar measure μ.

We will say that F verifies the natural assumptions if it satisfies the following conditions:

(1) F is bounded and in $L^1(G, \mu)$,
(2) F is symmetric,
(3) There exists a relatively compact, symmetric open neighbourhood U containing the neutral e and generating G, such that $F_{|U} > C$ for a constant $C > 0$.
(4) F has a finite second moment with respect to the word distance associated to the compact set $adh(U)$. (Notice that this condition doesn't depend on the choice of the compact generating set).

We consider

- $\Omega = G^{\mathbb{N}}$, equipped with the product Borelian structure,
- $P = \delta_e \otimes \nu^{\otimes \mathbb{N}}$ is the product probability on Ω, where δ_e is the Dirac measure at e,
- The canonical projection $X_n : \Omega \to G$, with X_0 the sure variable equal to e.
- $Z_n(\omega) = \prod_{i=0}^{n} X_i(\omega); \omega \in \Omega$ defines as in [1, 2] the random walk on G associated to ν.

If f and g are two non-negative functions defined on the positive real axis, we use the notation $f \lesssim g$ if there exist constants $a, b > 0$, such that for x large enough, $f(x) \leq ag(bx)$. If the symmetric relation also holds, we write $f \simeq g$.

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When a function is defined only on the integers, we extend it to the positive real axis by linear interpolation. We will use the same name for the original function and its extension.

The asymptotic behaviour of f is the coset with respect to this relation. If $f \lesssim g$ holds without $f \simeq g$, we write $f \lnsim g$.

For a Borelian $A \subset G$, the probability that the walk started at the neutral reaches A at time n is $P(Z_n \in A)$. The asymptotic behavior of the random walk is given by $P(Z_{2n} \in U)$ which is also the asymptotic of $F^{*2n}(e)$, when F is a density verifying the natural assumptions (see [1, 2]).

We denote by $F^{*2n}(e) = p_{2n}(e, e)$ the probability of return to the origin at time $2n$. Given the above notations and asumptions, we have :

Proposition 1.1. ([3])

$$P(Z_{2n} \in U) \simeq \nu^{*2n}(U) \simeq F^{*2n}(e).$$

It is well known that (see [1, 2]) the asymptotic behaviour of $F^{*2n}(e)$ is an invariant of the group and it doesn't depend on the choice of F. In the sequel, we denote by $\Phi_G(n)$ the asymptotic behaviour of the probability of return on the group G. Several results about the asymptotic behavior on locally compact groups are obtained by Varopoulos, Ch. Pittet, Saloff-Coste, S. Mustapha, A. Erschler, D. Gretete and others (see [2, 4, 5, 6, 7, 10, 11]).

Definition 1.1. Let G, H be two locally compact groups. A quasi isometry from G to H is a map ψ from G to H such that:

there exists a constant $C > 0$ satisfying for all $x, y \in G$

$$C^{-1} d_1(x, y) - C \leq d_2(\psi(x), \psi(y)) \leq C d_1(x, y) + C$$

and for all $x \in H$, there exists $z \in G$ such that

$$d_2(x, \psi(z)) \leq C.$$

It is well known the works of Ch. Pittet and L. Saloff-Coste emphasize that the asymptotic behavior of the return probability is invariant under quasi isometry in the case of discrete finitely generated groups. More precisely, we have the following theorem :

Theorem 1.2. (see [1], p. 3) *Let (G, S) and (H, T) be Cayley graphs of two finitely generated groups G, H witch are quasi-isometric. Then,*

$$\Phi_G(n) \simeq \Phi_H(n).$$

As a consequence of this theorem, we can deduce that the probability of return on a finitely generated group is equivalent to that of a subgroup of finite index. This follows from the fact that, in this case, the canonical surjection is a quasi isometry.

2. The main result

Theorem 2.1. *Let G_1, G_2 be two locally compact compactly generated groups unimodular. Let's suppose that there exist a quasi isometry ψ from G_1 to G_2 preserving the Haar measure and symtric preserving, then*

$$\psi_{G_2}(n) \simeq \psi_{G_1}(n).$$

Proof. There is no loss of generality in assuming that G_1, G_2 are amenable.

Indeed, if G_1 is not amenable and G_2 is quasi isometric to G_2. Then by theorem 2 in [12] the group G_2 is also non-amenable. Therefore, if we apply the celebrated Kesten's theorem in [13], we get

$$\Phi_{G_1}(n) \simeq exp(-n) \simeq \Phi_{G_2}(n).$$

In the rest of this proof we can therefore assume that the groups are amenable. Let U be an open set of G_1 relatively compact, neighborhood of the unit element e, generating G. Let K_1 be the closure of U.

Let K_2 be a compact symmetric neighborhood of e_2 generating G_2. According to Yves Cornulier theorem (see [9]) the change of word distance is invariant under quasi-isometry. In other words, we have :

For all $g \in G_1$, if $l_{K_1}(g) \leq m$ then $l_{K_2}(g) \leq km$ for some positive constant k.

Let F_2 be defined on G_2 by $F_2 = \frac{1}{\lambda_2(K_2^k)} 1_{K_2^k}$. Since K_2^k is a compact set, then by continuity we get

$$(g_1, ..., g_k) \mapsto g_1...g_k,$$

and K_2^k is also neighborhood of e_2. This insures that the density F_2 is lower bounded by a positive constant on an open generating set.

Let F_1 be defined on G_1 by $F_1(x) = F_2(\psi(x))$.

We have $\int_{G_1} F_1(g) d\lambda_1(g) = \int_{G_2} F_2(h) d\lambda_1^{\psi}(h) = 1$ then F_1 is a density of probability. In addition, it satisfies $g \in U, F_1(g) = \frac{1}{\lambda_2(K_2^k)}$, so F_1 is lower bounded by a positive constant on an open symmetric set.

On the other hand, the density F_1 is symmetric. Therefore, using the Theorem 1 in [2], we get:

$$\Phi_{G_i}(n) \simeq F_i^{2n}(e_i), i = 1, 2.$$

Let

$$Z'_{2n} = \prod_{i}^{2n} \psi(X_i)$$

and V be a symmetric neighborhood of e_2 in K_2, for all i we have :

$$P(\psi(X_i) \in V) = P(X_i \in \psi^{-1}(V)) = \int_{G_1} F_1(g) d\lambda_1(g)$$

$$= \int_{G_1} F_2(\psi(g)) d\lambda_1(g) = \int_{G_2} F_2(h) d\lambda_1^{\psi}(h).$$

So F_2 is a density of $\psi(X_i)$ with respect to the measure λ^{ψ}, and then Z'_{2n} has the density F_2^{*2n}.

On the other hand, by quasi-isometry, there exist a positive integer k such that for all $\omega \in \Omega$, $l_{K_2}(Z'_{2n})(\omega) \leq k l_{K_1}(Z_{2n})(\omega)$ and so we get the inclusion

$$[Z_{2n} \in U] \subset [Z'_{2n} \in K_2^k].$$

Consequently,

$$P'[Z_{2n} \in U]) \leq P([Z'_{2n} \in K_2^k]).$$

We also have $P([Z'_{2n} \in K_2^k]) = \int_{K_2^k} F_2^{*2n}(h) d\lambda^{\psi}(h) \leq \int_{K_2^k} F^{*2n}(e_2) d\lambda^{\psi}(h) = F^{*2n}(e_2)\lambda^{\psi}(K_2^k)$.

In conclusion,

$$P'[Z_{2n} \in U]) \leq F_2^{*2n}(e_2)\lambda^{\psi}(K_2^k).$$

So,

$$P'[Z_{2n} \in U]) \lesssim F_2^{*2n}(e_2).$$

Using Proposition 1 in [2] we get, $P(Z_{2n} \in U) \backsimeq F_1^{*2n}(e_1)$ so

$$F_1^{*2n}(e_1) \lesssim F_2^{*2n}(e_2).$$

As the quasi-isometry relationship is symmetric then we get

$$F_1^{*2n}(e_1) \backsimeq F_2^{*2n}(e_2)$$

and the result holds from Proposition 1.1. □

3. Conclusion

The theorem established in this article allows us in particular to switch from the continuous case to the discrete one as shown in the following example:

Let ψ_1 be the odd function on $\mathbb{R}$ that coincides with the integer part on positive reals. Then, $\psi(x_1,..,x_d) = (\psi_1(x_1),...,\psi_1(x_d))$ is a quasi-isometry from $\mathbb{R}^d$ to $\mathbb{Z}^d$ preserving the symmetric and the Haar measure. Therefore, we get the following result :

$$\Phi_{\mathbb{R}^d}(n) \simeq \Phi_{\mathbb{Z}^d}(n).$$

It is obvious that the function $F = \frac{1}{2d}1_{\{-1,1\}^d}$ satisfies the natural assumptions. So, $\Phi_{2n}(\mathbb{Z}) = \left(\frac{C_{2n}^n}{2^{2n}}\right)^d$ and using the Stirling formulae we get: $\Phi_{2n}(\mathbb{Z}) \simeq \frac{1}{n^{d/2}}$ and using the main result we obtain

$$\Phi_{2n}(\mathbb{R}) \simeq \frac{1}{n^{d/2}}.$$

An important question about the principal result remains unanswered though. Is the hypothesis of preservation of the Haar measure unavoidable ? Or is it possible to find an example of two locally compact groups generated unimodular quasi-isometric (non-discrete) on which the probabilities of return do not have the same asymptotic behavior ?

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