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Fast growing solutions of linear differential equations with analytic coefficients in the unit disc

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Abstract. In this paper, we investigate the growth of solutions of higher order linear differential equations with analytic coefficients of φ-order in the unit disc. We introduce new definitions of the lower φ-order and the lower φ-type related to the φ-order concepts in order to generalise and extend previous results due to Chyzhykov-Semochko [6], Semochko [14], Belaïdi [1, 2, 3], Hu-Zheng [12].

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1. Introduction and main results

Throughout all of this paper, we assume that the reader is familiar with the fundamental notions of Nevanlinna value distribution theory of meromorphic functions in the whole complex plane $\mathbb{C}$ and in the unit disc Δ, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, (see [8, 13, 16]). For all $r \in [0,1)$, we define $\exp_1 r = \exp r = e^r$ and $\exp_{p+1} r = \exp(\exp_p r)$, $p \in \mathbb{N} = \{1, 2, \dots\}$. Inductively, $\log^+ r = \max\{0, \log r\}$, $\log_1^+ r = \log^+ r$ and $\log_{p+1}^+ r = \log^+(\log_p^+ r)$, $p \in \mathbb{N} \cup \{0\}$. We also denote $\exp_0 r = r = \log_0^+ r$, $\exp_{-1} r = \log_1^+ r$ and $\log_{-1}^+ r = \exp_1 r$. In addition, the sets $F \subset [0,1)$ (resp. $E \subset [0,1)$) are not necessarily the same at each occurrence, but they are always of finite logarithmic measure (resp. infinite logarithmic measure), that is $\int_F \frac{dr}{1-r} < +\infty$ (resp. $\int_E \frac{dr}{1-r} = +\infty$). Furthermore, in this paper the phrasing "r, $r \to 1^-$ "when occurs is simply a shorthand notation of "for all $r \in (0,1)$ sufficiently close to 1 ".

Definition 1.1 ([4])**.** The iterated p-order of an analytic function f in Δ is defined by

$$\tilde{\rho}_p(f) := \limsup_{r \longrightarrow 1^-} \frac{\log_{p+1}^+ M(r,f)}{-\log(1-r)},\ p \in \mathbb{N},$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$ is the maximum modulus of f. If f is meromorphic in Δ, the iterated p-order is defined by

$$\rho_p(f) := \limsup_{r \longrightarrow 1^-} \frac{\log_p^+ T(r,f)}{-\log(1-r)},\ p \in \mathbb{N},$$

where $T(r,f)$ is the Nevanlinna characteristic of f.

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Definition 1.2 ([11]). The iterated p-type of an analytic function f in Δ with $0 < \tilde{\rho}_p(f) < +\infty$ is defined by

$$\tilde{\tau}_p(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\tilde{\rho}_p(f)} \log_p^+ M(r,f).$$

Heittokangas et al. in [11] investigated the iterated p-order of solutions of the complex linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \tag{1}$$

where $k \geq 2$ and the coefficients $A_0 \not\equiv 0, \cdots, A_{k-1}$ are analytic functions in the unit disc Δ. They obtained the following result.

Theorem 1.1 ([11]). *Let $p \in \mathbb{N}$. If the coefficients $A_0, A_1, \ldots, A_{k-1}$ are analytic functions in Δ such that $\tilde{\rho}_p(A_j) < \tilde{\rho}_p(A_0)$ for $j = 1, \cdots, k-1$, then all solutions $f \not\equiv 0$ of (1) satisfy $\tilde{\rho}_{p+1}(f) = \tilde{\rho}_p(A_0)$.*

Observe that $A_0(z)$ is the only one dominant coefficient. In [7], Hamouda gave an improvement of Theorem 1.1 by considering more than one dominant coefficient. He proved the following theorem.

Theorem 1.2 ([7]). *Let $p \in \mathbb{N}$. If the coefficients $A_0, A_1, \ldots, A_{k-1}$ are analytic functions in Δ such that $\tilde{\rho}_p(A_j) \leq \tilde{\rho}_p(A_0)$ for $j = 1, \cdots, k-1$, and*

$$\max\{\tilde{\tau}_p(A_j) : \tilde{\rho}_p(A_j) = \tilde{\rho}_p(A_0)\} < \tilde{\tau}_p(A_0),$$

then all solutions $f \not\equiv 0$ of (1) satisfy $\tilde{\rho}_p(f) = \tilde{\rho}_{p+1}(A_0)$.

There are many extensions of Theorem 1.1 and Theorem 1.2 by considering the so called $[p,q]$-order (see some results in [1, 12]). Unfortunately, both of the iterated p-order and the $[p,q]$-order do not cover an arbitrary growth, i.e., there exist functions of infinite iterated p-order and $[p,q]$-order for an arbitrary $p \in \mathbb{N}$ as it was shown in [6, Example 1.4]. To avoid this disadvantage, Chyzhykov and Semochko [6, 14] used a more general scale to measure the growth of solutions of equation (1) called the φ-order (cf. [15]).

Definition 1.3 ([14]). Let φ be an increasing unbounded function on $(0,\infty)$. The φ-orders of an analytic function f in Δ are defined by

$$\tilde{\rho}_\varphi^0(f) = \limsup_{r \longrightarrow 1^-} \frac{\varphi(M(r,f))}{-\log(1-r)}, \qquad \tilde{\rho}_\varphi^1(f) = \limsup_{r \longrightarrow 1^-} \frac{\varphi(\log M(r,f))}{-\log(1-r)}.$$

If f is meromorphic in Δ, then the φ-orders are defined by

$$\rho_\varphi^0(f) = \limsup_{r \longrightarrow 1^-} \frac{\varphi(e^{T(r,f)})}{-\log(1-r)}, \qquad \rho_\varphi^1(f) = \limsup_{r \longrightarrow 1^-} \frac{\varphi(T(r,f))}{-\log(1-r)}.$$

Note that for $\varphi(r) = \log_p^+ r,\ p \in \mathbb{N}$ and if f is an analytic function in Δ, then

$$\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_p(f).$$

By Φ we define the class of positive unbounded increasing functions on $(0,\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e., $\forall c > 0 : \lim_{t \to +\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1$. One can see that $\varphi(r) = \log_p^+ r$, $(p \geq 2)$ belongs to the class Φ and $\varphi(r) = \log^+ r \notin \Phi$. We recall now some basic properties of functions from the class Φ.

Proposition 1.3 ([6]). *If $\varphi \in \Phi$, then*

$$\forall m > 0,\ \forall k \geq 0: \qquad \frac{\varphi^{-1}(\log x^m)}{x^k} \longrightarrow +\infty, \qquad x \longrightarrow +\infty, \tag{2}$$

$$\forall \delta > 0: \quad \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} \longrightarrow +\infty, \quad x \longrightarrow +\infty. \tag{3}$$

Remark 1.1. By (3) and the definition of the limit, for all $K > 0$ and sufficiently large x, we obtain

$$\forall \delta > 0: \left[\varphi^{-1}(x)\right]^K \leq \varphi^{-1}\left((1+\delta)x\right).$$

Remark 1.2 ([6]). One can show that (3) implies that

$$\forall c > 0,\ \varphi(cx) \leq \varphi(x^c) \leq (1+o(1))\varphi(x), \quad x \longrightarrow +\infty. \tag{4}$$

Proposition 1.4 ([14]). *Let $\varphi \in \Phi$ and f be an analytic function in Δ. Then*

$$\rho_\varphi^1(f) = \tilde{\rho}_\varphi^1(f).$$

Remark 1.3. For an analytic function f in Δ and $\varphi \in \Phi$, the equality $\rho_\varphi^0(f) = \tilde{\rho}_\varphi^0(f)$ is not verified when $\varphi(.) = \log_2(.) = \log\log(.)$ which belongs to the class Φ and satisfies $\rho_{\log_2}^0(f) = \rho_1(f)$ and $\tilde{\rho}_{\log_2}^0(f) = \tilde{\rho}_1(f)$. Since for an analytic function f at $|z| = r < 1$, we have

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{1+3r}{1-r} T\left(\frac{1+r}{2}, f\right)$$

(see [8, Theorem 1.6] or [13, Proposition 2.2.2]), then $\rho_{\log_2}^0(f) \leq \tilde{\rho}_{\log_2}^0(f) \leq \rho_{\log_2}^0(f) + 1$. However, we have $\rho_\varphi^0(f) = \tilde{\rho}_\varphi^0(f)$ if $\varphi(.) = \log_{p+1}(.) = \log(\log_p)(.),\ p \geq 2$.

The following theorem due to Semochko [14] used the concept of φ-order which improves Theorem 1.1.

Theorem 1.5 ([14]). *Let $\varphi \in \Phi$ and $A_0, A_1, \ldots, A_{k-1}$ be analytic functions in Δ such that $\max\{\tilde{\rho}_\varphi^0(A_j), j = 1, \ldots, k-1\} < \tilde{\rho}_\varphi^0(A_0)$. Then, every solution $f \not\equiv 0$ of (1) satisfies $\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_\varphi^0(A_0)$.*

By analogous manner to Definition 1.3, we introduce the following quantities.

Definition 1.4. Let φ be an increasing unbounded function on $(0, \infty)$. The lower φ-orders of an analytic function f in Δ are defined by

$$\tilde{\mu}_\varphi^0(f) = \liminf_{r \longrightarrow 1^-} \frac{\varphi(M(r,f))}{-\log(1-r)}, \qquad \tilde{\mu}_\varphi^1(f) = \liminf_{r \longrightarrow 1^-} \frac{\varphi(\log M(r,f))}{-\log(1-r)}.$$

If f is meromorphic in Δ, then the lower φ-orders are defined by

$$\mu_\varphi^0(f) = \liminf_{r \longrightarrow 1^-} \frac{\varphi(e^{T(r,f)})}{-\log(1-r)}, \qquad \mu_\varphi^1(f) = \liminf_{r \longrightarrow 1^-} \frac{\varphi(T(r,f))}{-\log(1-r)}.$$

Using a similar proof as in [14, Proposition 1], one can show the following result.

Proposition 1.6. *Let $\varphi \in \Phi$ and f be an analytic function in Δ. Then*

$$\mu_\varphi^1(f) = \tilde{\mu}_\varphi^1(f).$$

Definition 1.5. Let φ be an increasing unbounded function on $(0,\infty)$. We define the φ-types of an analytic function f in Δ with $0<\tilde{\rho}_{\varphi}^{i}(f)<+\infty,\ \ (i=0,1)$ by

$$\tilde{\tau}_{\varphi}^{0}(f)=\limsup_{r\longrightarrow 1^{-}}\,(1-r)^{\tilde{\rho}_{\varphi}^{0}(f)}\exp\{\varphi(M(r,f))\},$$

$$\tilde{\tau}_{\varphi}^{1}(f)=\limsup_{r\longrightarrow 1^{-}}\,(1-r)^{\tilde{\rho}_{\varphi}^{1}(f)}\exp\{\varphi(\log M(r,f))\}.$$

If f is meromorphic in Δ, then the φ-types with $0<\rho_{\varphi}^{i}(f)<+\infty,\ \ (i=0,1)$ are defined by

$$\tau_{\varphi}^{0}(f)=\limsup_{r\longrightarrow 1^{-}}\,(1-r)^{\rho_{\varphi}^{0}(f)}\exp\{\varphi(e^{T(r,f)})\},$$

$$\tau_{\varphi}^{1}(f)=\limsup_{r\longrightarrow 1^{-}}\,(1-r)^{\rho_{\varphi}^{1}(f)}\exp\{\varphi(T(r,f))\}.$$

Definition 1.6. Let φ be an increasing unbounded function on $(0,\infty)$. We define the lower φ-types of an analytic function f in Δ with $0<\tilde{\mu}_{\varphi}^{i}(f)<+\infty,\ \ (i=0,1)$ by

$$\underline{\tilde{\tau}}_{\varphi}^{0}(f)=\liminf_{r\longrightarrow 1^{-}}\,(1-r)^{\tilde{\mu}_{\varphi}^{0}(f)}\exp\{\varphi(M(r,f))\},$$

$$\underline{\tilde{\tau}}_{\varphi}^{1}(f)=\liminf_{r\longrightarrow 1^{-}}\,(1-r)^{\tilde{\mu}_{\varphi}^{1}(f)}\exp\{\varphi(\log M(r,f))\}.$$

If f is meromorphic in Δ, then the lower φ-types with $0<\mu_{\varphi}^{i}(f)<+\infty,\ \ (i=0,1)$ are defined by

$$\underline{\tau}_{\varphi}^{0}(f)=\liminf_{r\longrightarrow 1^{-}}\,(1-r)^{\mu_{\varphi}^{0}(f)}\exp\{\varphi(e^{T(r,f)})\},$$

$$\underline{\tau}_{\varphi}^{1}(f)=\liminf_{r\longrightarrow 1^{-}}\,(1-r)^{\mu_{\varphi}^{1}(f)}\exp\{\varphi(T(r,f))\}.$$

The main purpose of this paper is to generalise Theorems 1.1, 1.2 and 1.5 by considering the concepts of the φ-orders and the φ-types. Our results are counterparts of theorems listed in [2, 3, 6], where the coefficients $A_j(z)$ in equation (1) are entire functions.

Theorem 1.7. *Let $A_0,\cdots,A_{k-1}$ be analytic functions in Δ, and let $\varphi\in\Phi$. Assume that* $\max\left\{\tilde{\rho}_{\varphi}^{0}(A_j):j=1,\cdots,k-1\right\}<\tilde{\mu}_{\varphi}^{0}(A_0)\leq\tilde{\rho}_{\varphi}^{0}(A_0)<+\infty$. *Then, every solution $f\not\equiv 0$ of (1) satisfies $\tilde{\mu}_{\varphi}^{0}(A_0)=\tilde{\mu}_{\varphi}^{1}(f)\leq\tilde{\rho}_{\varphi}^{1}(f)=\tilde{\rho}_{\varphi}^{0}(A_0)$.*

Theorem 1.8. *Let $A_0,\cdots,A_{k-1}$ be analytic functions in Δ, and let $\varphi\in\Phi$. Assume that* $\max\{\rho_{\varphi}^{0}(A_j):j=1,\cdots,k-1\}\leq\mu_{\varphi}^{0}(A_0)<+\infty$ *and*

$$\limsup_{r\longrightarrow 1^{-}}\frac{\sum\limits_{j=1}^{k-1}m(r,A_j)}{m(r,A_0)}<1.$$

Then, every solution $f\not\equiv 0$ of (1) satisfies $\rho_{\varphi}^{0}(A_0)\leq\rho_{\varphi}^{1}(f)=\tilde{\rho}_{\varphi}^{1}(f)\leq\tilde{\rho}_{\varphi}^{0}(A_0)$ and $\mu_{\varphi}^{0}(A_0)\leq\mu_{\varphi}^{1}(f)=\tilde{\mu}_{\varphi}^{1}(f)\leq\tilde{\mu}_{\varphi}^{0}(A_0)$.

Theorem 1.9. *Let $A_0, \cdots, A_{k-1}$ be analytic functions in Δ, and let $\varphi \in \Phi$. Assume that*

$$\max\{\tilde{\rho}_\varphi^0(A_j) : j = 1, \cdots, k-1\} \le \tilde{\rho}_\varphi^0(A_0) = \rho_0, \ (0 < \rho_0 < +\infty)$$

and

$$\max\{\tilde{\tau}_\varphi^0(A_j) : \tilde{\rho}_\varphi^0(A_j) = \tilde{\rho}_\varphi^0(A_0), j \neq 0\} < \tilde{\tau}_\varphi^0(A_0) = \tau_0 \ \ (0 < \tau_0 < +\infty).$$

Then, every solution $f \not\equiv 0$ of (1) satisfies $\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_\varphi^0(A_0)$.

Theorem 1.10. *Let $A_0, \cdots, A_{k-1}$ be analytic functions in Δ, and let $\varphi \in \Phi$. Assume that*

$$\max\{\tilde{\rho}_\varphi^0(A_j) : j = 1, \cdots, k-1\} \le \tilde{\mu}_\varphi^0(A_0) = \mu_0, (0 < \mu_0 < +\infty)$$

and

$$\max\{\tilde{\tau}_\varphi^0(A_j) : \tilde{\rho}_\varphi^0(A_j) = \tilde{\mu}_\varphi^0(A_0); j \neq 0\} < \underline{\tilde{\tau}}_\varphi^0(A_0) = \underline{\tau}_0, (0 < \underline{\tau}_0 < +\infty).$$

Then, every solution $f \not\equiv 0$ of (1) satisfies $\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_\varphi^0(A_0) \ge \tilde{\mu}_\varphi^1(f) = \tilde{\mu}_\varphi^0(A_0)$.

2. Preliminary Lemmas

Lemma 2.1 ([5]). *Let f be a meromorphic function in Δ such that $f^{(j)}$ does not vanish identically. Let $\varepsilon > 0$ be a constant, k and j be integers satisfying $0 \le j < k$, F a set with finite logarithmic measure on $[0, 1)$ and $d \in (0, 1)$. Then, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}; T(s(|z|), f)\right\}\right]^{k-j}, \ |z| \notin F,$$

where $s(|z|) = 1 - d(1 - |z|)$. Moreover, if $\rho_1(f) < +\infty$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \left(\frac{1}{1-|z|}\right)^{(k-j)(2+\varepsilon+\rho_1(f))}, \ |z| \notin F.$$

Lemma 2.2 ([9]). *Let f be a meromorphic function in Δ and $k \in \mathbb{N}$. Then, we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log^+ T(r, f) + \log\frac{1}{1-r}\right)$$

possibly outside of an exceptional set $F \subset [0, 1)$ with $\int_F \frac{dr}{1-r} < +\infty$. If $\rho_1(f) < +\infty$, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\frac{1}{1-r}\right).$$

Lemma 2.3. *Let $\varphi \in \Phi$ and $k \in \mathbb{N}$. Let f be a meromorphic function in Δ of order $\rho_\varphi^1(f) =: \rho_1$. Then, for any given $\varepsilon > 0$, we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log^+ \varphi^{-1}\left((\rho_1 + \varepsilon)\log\frac{1}{1-r}\right)\right)$$

holds for all $r \to 1^-$ outside of a set $F \subset [0, 1)$ with $\int_F \frac{dr}{1-r} < +\infty$.

Proof. We proceed by mathematical induction. Firstly, for $k=1$, the definition of $\rho^1_\varphi(f) =: \rho_1$ implies that for any $\varepsilon > 0$ and for all r, $r \to 1^-$ we have

$$T(r,f) \le \varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-r}\right). \tag{5}$$

It follows from Lemma 2.2, (2) and (5) that

$$m\left(r,\frac{f'}{f}\right) = O\left(\log^+\varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-r}\right)\right),\ r\notin F, \tag{6}$$

where $F\subset[0,1)$ with $\int_F \frac{dr}{1-r} < +\infty$. Secondly, for $\varepsilon>0$, we assume that

$$m\left(r,\frac{f^{(k)}}{f}\right) = O\left(\log^+\varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-r}\right)\right),\ r\notin F.$$

Since $N(r,f^{(k)}) \le (k+1)N(r,f)$ and $m(r,f^{(k)}) \le m\left(r,\frac{f^{(k)}}{f}\right) + m(r,f)$, then

$$\begin{aligned}
T(r,f^{(k)}) &= m(r,f^{(k)}) + N(r,f^{(k)})\\
&\le (k+1)T(r,f) + O\left(\log^+\varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-r}\right)\right)\\
&= O\left(\varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-r}\right)\right),\ r\notin F.
\end{aligned}$$

In view of (6), we get

$$m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right) = O\left(\log^+\varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-r}\right)\right),\ r\notin F.$$

Thus, for $r\notin F$ we obtain

$$\begin{aligned}
m\left(r,\frac{f^{(k+1)}}{f}\right) &\le m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right) + m\left(r,\frac{f^{(k)}}{f}\right)\\
&= O\left(\log^+\varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-r}\right)\right).
\end{aligned}$$

□

Lemma 2.4 ([10]). *Let $A_0, A_1, \cdots, A_{k-1}$ be analytic functions in $\Delta_R = \{z\in\mathbb{C} : |z| < R\}$, where $0 < R \le +\infty$ and f be a solution of (1) in Δ_R, $1\le p<+\infty$. Then, for all $0\le r<R$ we have*

$$[m_p(r,f)]^p \le C\left(1+\sum_{j=0}^{k-1}\int_0^{2\pi}\int_0^r \left|A_j(se^{i\theta})\right|^{\frac{1}{k-j}}\,ds\,d\theta\right),$$

where $C>0$ is a constant depending on p and on the initial values of $f(z)$ at the point z_θ, where $A_j(z_\theta)\ne 0$ for some $j=0,1,\cdots,k-1$ and where

$$[m_p(r,f)]^p = \frac{1}{2\pi}\int_0^{2\pi}\left[\log^+\left|f(re^{i\theta})\right|\right]^p d\theta.$$

Lemma 2.5 ([10]). *Let f be a solution of (1) in $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$, where $0 < R \leq +\infty$. Let $n_c \in \{1, 2, \cdots, k\}$ be the number of non-zero coefficients $A_j (j = 0, 1, \cdots, k-1)$, and let $\theta \in [0, 2\pi]$ and $\varepsilon > 0$. If $z_0 = \nu e^{i\theta} \in \Delta_R$ is such that $A_j \neq 0$ for some $j = 0, 1, \cdots, k-1$, then for all $\nu \leq r < R$, we have*

$$|f(re^{i\theta})| \leq C \exp\left(n_c \int_{\nu}^{r} \max_{j=0,1,\cdots,k-1} |A_j(te^{i\theta})|^{\frac{1}{k-j}} dt\right),$$

where C is a constant satisfying

$$C \leq (1+\varepsilon) \max_{j=0,1,\cdots,k-1} \left(\frac{|f^{(j)}(z_0)|}{n_c^j \max\limits_{n=0,1,\cdots,k-1} |A_n(z_0)|^{\frac{j}{k-n}}} \right).$$

Lemma 2.6. *Let $\varphi \in \Phi$ and f be a meromorphic function in Δ with $\mu_\varphi^1(f) < +\infty$. Then, there exists a set $E \subset [0, 1)$ with infinite logarithmic measure such that for all $r \in E$, $r \to 1^-$, we have for any $\varepsilon > 0$*

$$T(r, f) < \varphi^{-1}\left((\mu_\varphi^1(f) + \varepsilon) \log \frac{1}{1-r}\right).$$

Proof. The definition of $\mu_\varphi^1(f)$ implies that there exists a sequence $\{r_n, n \geq 1\}$ tending to 1^- satisfying $1 - d(1 - r_n) < r_{n+1}$, where $d \in (0, 1)$ and

$$\lim_{r_n \to 1^-} \frac{\varphi(T(r_n, f))}{-\log(1 - r_n)} = \mu_\varphi^1(f).$$

Then, for any $\varepsilon > 0$ there exists an integer number n_1 such that for all $n \geq n_1$, we have

$$T(r_n, f) < \varphi^{-1}\left((\mu_\varphi^1(f) + \varepsilon) \log \frac{1}{1-r_n}\right).$$

Set $E = \bigcup\limits_{n=n_1}^{+\infty} \left[1 - \frac{1-r_n}{d}, r_n\right]$, then for any $r \in E \subset [0, 1)$, we get

$$\begin{aligned} T(r, f) \leq T(r_n, f) &< \varphi^{-1}\left((\mu_\varphi^1(f) + \frac{\varepsilon}{2}) \log \frac{1}{1-r_n}\right) \\ &\leq \varphi^{-1}\left((\mu_\varphi^1(f) + \frac{\varepsilon}{2}) \log \frac{1}{d(1-r)}\right) < \varphi^{-1}\left((\mu_\varphi^1(f) + \varepsilon) \log \frac{1}{1-r}\right), \end{aligned}$$

where

$$\int_E \frac{dr}{1-r} = \sum_{n=n_1}^{+\infty} \int_{1-\frac{1-r_n}{d}}^{r_n} \frac{dt}{1-t} = \sum_{n=n_1}^{+\infty} \log \frac{1}{d} = +\infty.$$

$\square$

By a similar proof, one can easily prove the following lemma.

Lemma 2.7. *Let $\varphi \in \Phi$ and f be an analytic function in Δ with $\tilde{\mu}_\varphi^0(f) < +\infty$. Then, there exists a set $E \subset [0, 1)$ with infinite logarithmic measure such that for all $r \in E$, $r \to 1^-$, we have for any $\varepsilon > 0$*

$$M(r, f) < \varphi^{-1}\left((\tilde{\mu}_\varphi^0(f) + \varepsilon) \log \frac{1}{1-r}\right).$$

Lemma 2.8. *Let $A_0, \cdots, A_{k-1}$ be analytic functions in Δ, and let $\varphi \in \Phi$. Assume that $\max\{\tilde{\rho}^0_\varphi(A_j) : j \neq s\} \leq \tilde{\mu}^0_\varphi(A_s) < +\infty$. If $f \not\equiv 0$ is a solution of (1), then*

$$\tilde{\mu}^1_\varphi(f) \leq \tilde{\mu}^0_\varphi(A_s).$$

Proof. By Lemma 2.4, we have

$$T(r,f) = m(r,f) \leq C\left(1 + \sum_{j=0}^{k-1} \int_0^{2\pi} \int_0^r |A_j(se^{i\theta})|^{\frac{1}{k-j}} ds d\theta\right)$$
$$\leq 2\pi C\left(1 + \sum_{j=0}^{k-1} rM(r,A_j)\right). \tag{7}$$

Set $\alpha = \max\left\{\tilde{\rho}^0_\varphi(A_j) : j \neq s\right\}$. By the definition of $\tilde{\rho}^0_\varphi(A_j)$, we have for any $\varepsilon > 0$ and $r \to 1^-$

$$M(r,A_j) \leq \varphi^{-1}\left((\alpha + \frac{\varepsilon}{2}) \log \frac{1}{1-r}\right), \quad (j \neq s). \tag{8}$$

By Lemma 2.7, there exists a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} = +\infty$ such that for $r \in E$, $r \to 1^-$

$$M(r,A_s) < \varphi^{-1}\left((\tilde{\mu}^0_\varphi(A_s) + \frac{\varepsilon}{2}) \log \frac{1}{1-r}\right). \tag{9}$$

By (7)-(9), we have for $r \in E$, $r \to 1^-$

$$T(r,f) \leq O\left(\varphi^{-1}\left((\tilde{\mu}^0_\varphi(A_s) + \varepsilon) \log \frac{1}{1-r}\right)\right). \tag{10}$$

Then, it follows from (4), (10), by the arbitrariness of $\varepsilon > 0$ and the monotonicity of φ^{-1} that $\tilde{\mu}^1_\varphi(f) \leq \tilde{\mu}^0_\varphi(A_s)$. □

Lemma 2.9. *Let $\varphi \in \Phi$ and $A_0, \cdots, A_{k-1}$ be analytic functions in Δ. Then, every non-zero solution f of (1) satisfies*

$$\tilde{\rho}^1_\varphi(f) \leq \max\{\tilde{\rho}^0_\varphi(A_j) : j = 0, 1, \cdots, k-1\}.$$

Proof. Set $\beta = \max\{\tilde{\rho}^0_\varphi(A_j) : j = 0, 1, \cdots, k-1\}$. By the definition of $\tilde{\rho}^0_\varphi(A_j)$, we have for any $\varepsilon > 0$ and $r \to 1^-$

$$M(r,A_j) \leq \varphi^{-1}\left((\beta + \frac{\varepsilon}{2}) \log \frac{1}{1-r}\right), \; j = 0, 1 \cdots, k-1. \tag{11}$$

By (7) and (11), for any $\varepsilon > 0$ and $r \to 1^-$, we have

$$T(r,f) = m(r,f) \leq O\left(\varphi^{-1}\left((\beta + \varepsilon) \log \frac{1}{1-r}\right)\right). \tag{12}$$

It follows from (4), (12), by the arbitrariness of $\varepsilon > 0$ and the monotonicity of φ^{-1} that

$$\tilde{\rho}^1_\varphi(f) \leq \beta = \max\{\tilde{\rho}^0_\varphi(A_j) : j = 0, 1, \cdots, k-1\}.$$

□

Lemma 2.10. *Let $\varphi \in \Phi$ and f be an analytic function in Δ satisfying $0 < \tilde{\rho}_{\varphi}^{0}(f) = \rho_0 < +\infty$ and $0 < \tilde{\tau}_{\varphi}^{0}(f) = \tau_0 < +\infty$. Then, for any given $0 < \beta < \tilde{\tau}_{\varphi}^{0}(f)$, there exists a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} = \infty$ such that for all $r \in E$, we have*

$$\varphi(M(r,f)) > \log \frac{\beta}{(1-r)^{\rho_0}}.$$

Proof. The definition of $\tilde{\tau}_{\varphi}^{0}(f)$ implies that there exists a sequence $\{r_n, n \geq 1\}$ tending to 1^- satisfying $1 - (1 - \frac{1}{n})(1 - r_n) < r_{n+1}$ and

$$\lim_{n \to +\infty} (1 - r_n)^{\rho_0} \exp\{\varphi(M(r_n, f))\} = \tau_0.$$

Then, for any given $\varepsilon > 0$, there exists an integer n_1 such that for all $n \geq n_1$, we have

$$\exp\{\varphi(M(r_n, f))\} > \frac{\tau_0 - \varepsilon}{(1 - r_n)^{\rho_0}}. \tag{13}$$

For any given β such that $0 < \beta < \tau_0 - \varepsilon$, there exists an integer n_2 such that for all $n \geq n_2$, we have

$$\left(1 - \frac{1}{n}\right)^{\rho_0} > \frac{\beta}{\tau_0 - \varepsilon}. \tag{14}$$

For all $n \geq n_3 := \max\{n_1, n_2\}$ and for any $r \in \left[r_n, 1 - (1 - \frac{1}{n})(1 - r_n)\right]$, it follows from (13) and (14) that

$$\begin{aligned}\exp\{\varphi(M(r,f))\} &\geq \exp\{\varphi(M(r_n,f))\} > \frac{\tau_0 - \varepsilon}{(1 - r_n)^{\rho_0}} \\ &\geq \frac{\tau_0 - \varepsilon}{(1 - r)^{\rho_0}} \left(1 - \frac{1}{n}\right)^{\rho_0} > \frac{\beta}{(1 - r)^{\rho_0}}.\end{aligned}$$

Thus

$$\varphi(M(r,f)) > \log \frac{\beta}{(1-r)^{\rho_0}}.$$

Set $E = \bigcup_{n=n_3}^{+\infty} \left[r_n, 1 - (1 - \frac{1}{n})(1 - r_n)\right]$, then E satisfies

$$\int_E \frac{dr}{1-r} = \sum_{n=n_3}^{+\infty} \int_{r_n}^{1-(1-\frac{1}{n})(1-r_n)} \frac{dr}{1-r} = \sum_{n=n_3}^{+\infty} \log \frac{n}{n-1} = +\infty.$$

$\square$

Lemma 2.11 ([14]). *Let $\varphi \in \Phi$ and f be an analytic function in Δ satisfying $0 < \rho_{\varphi}^{0}(f) = \rho_0 < +\infty$. Then, for any given $0 < \beta < \rho_0$, there exists a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} = \infty$ such that for all $r \in E$, we have*

$$\varphi\left(e^{T(r,f)}\right) > \beta \log \frac{1}{1-r}.$$

3. Proofs of main results

Proof of Theorem 1.7. Suppose that $f \not\equiv 0$ is a solution of equation (1). From Theorem 1.5 we have $\tilde{\rho}_{\varphi}^{1}(f) = \tilde{\rho}_{\varphi}^{0}(A_0)$. So, we only need to prove that $\tilde{\mu}_{\varphi}^{1}(f) = \tilde{\mu}_{\varphi}^{0}(A_0)$. Firstly, we prove the inequality $\mu_1 = \tilde{\mu}_{\varphi}^{1}(f) \geq \tilde{\mu}_{\varphi}^{0}(A_0) = \mu_0$. Suppose that to the contrary, $\mu_1 = \tilde{\mu}_{\varphi}^{1}(f) < \tilde{\mu}_{\varphi}^{0}(A_0) = \mu_0$. Set $\max\left\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \cdots, k-1\right\} = \beta$. We can suppose without reducing the generality that $\mu_1 \leq \beta < \mu_0$. By equation (1), we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right|. \tag{15}$$

For any given ε $(0 < 3\varepsilon < \mu_0 - \beta)$, we have from the definitions of $\tilde{\mu}_{\varphi}^{0}(A_0)$ and $\tilde{\rho}_{\varphi}^{0}(A_j)$ that for $r \to 1^-$

$$|A_0(z)| \geq \varphi^{-1}\left((\mu_0 - \varepsilon)\log\frac{1}{1-r}\right) \tag{16}$$

and

$$|A_j(z)| \leq \varphi^{-1}\left((\beta + \varepsilon)\log\frac{1}{1-r}\right), \quad j = 1, \ldots, k-1. \tag{17}$$

By Lemma 2.1, for $j = 1, \cdots, k$ and $|z| \notin F$, where F is a set of finite logarithmic measure on $[0, 1)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T(s(|z|), f)\right]^{j}.$$

It follows from Lemma 2.6 and Proposition 1.6 that there exists a set E of infinite logarithmic measure on $[0, 1)$ such that for $|z| \in E \backslash F$, we have

$$\begin{aligned}
\left|\frac{f^{(j)}(z)}{f(z)}\right| &\leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T(s(|z|), f)\right]^{j} \\
&\leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} \varphi^{-1}\left((\mu_1 + \varepsilon)\log\frac{1}{1 - s(|z|)}\right)\right]^{j}.
\end{aligned} \tag{18}$$

Since $E \backslash F$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in E \backslash F$ tending to 1. Set $R_n = s(|z_n|) = 1 - d(1 - |z_n|)$, $d \in (0, 1)$. We have $1 - |z_n| = \frac{1-R_n}{d}$, $d \in (0, 1)$. By substituting (16)-(18) into (15) for the above ε $(0 < 3\varepsilon < \mu_0 - \beta)$ and using Remark 1.1, we obtain for $R_n \longrightarrow 1^-$ and $R_n \in E \backslash F$ that

$$\begin{aligned}
&\varphi^{-1}\left((\mu_0 - \varepsilon)\log\frac{d}{1-R_n}\right) \\
&\quad\leq k\left[\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} \varphi^{-1}\left((\mu_1 + \varepsilon)\log\frac{1}{1-R_n}\right)\right]^{k} \varphi^{-1}\left((\beta + \varepsilon)\log\frac{d}{1-R_n}\right) \\
&\quad\leq \left[\varphi^{-1}\left((\beta + \varepsilon)\log\frac{d}{1-R_n}\right)\right]^{k+2} \leq \varphi^{-1}\left((\beta + 2\varepsilon)\log\frac{d}{1-R_n}\right).
\end{aligned}$$

By arbitrariness of ε $(0 < 3\varepsilon < \mu_0 - \beta)$ and the monotonicity of φ^{-1} we obtain the contradiction $\mu_0 \leq \beta$. Thus, $\mu_1 = \tilde{\mu}_{\varphi}^{1}(f) \geq \tilde{\mu}_{\varphi}^{0}(A_0) = \mu_0$.

Now, we prove the converse inequality $\tilde{\mu}^1_\varphi(f) \leq \tilde{\mu}^0_\varphi(A_0)$. Let $\theta_0 \in [0, 2\pi]$ be such that $|f(re^{i\theta_0})| = M(r, f)$. By Lemma 2.5, we have

$$\begin{aligned} M(r,f) &\leq C\exp\left(n_c\int_\nu^r \max_{j=0,1,\cdots,k-1}|A_j(te^{i\theta_0})|^{\frac{1}{k-j}}dt\right) \\ &\leq C\exp\left(n_c\int_\nu^r \max_{j=0,1,\cdots,k-1}|M(r,A_j)|^{\frac{1}{k-j}}dt\right) \\ &\leq C\exp\left(n_c(r-\nu)\max_{j=0,1,\cdots,k-1}M(r,A_j)\right). \end{aligned} \tag{19}$$

The definition of $\tilde{\rho}^0_\varphi(A_j)$ implies that for any $\varepsilon > 0$ and $r \to 1^-$, we have

$$M(r,A_j) \leq \varphi^{-1}\left((\beta+\frac{\varepsilon}{2})\log\frac{1}{1-r}\right), \quad (j=1,\cdots,k-1), \tag{20}$$

where $\beta = \max\left\{\tilde{\rho}^0_\varphi(A_j) : j = 1, \cdots, k-1\right\}$. By Lemma 2.7, there exists a set $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} = +\infty$ such that for $r \in E$, $r \to 1^-$

$$M(r,A_0) \leq \varphi^{-1}\left((\tilde{\mu}^0_\varphi(A_0)+\frac{\varepsilon}{2})\log\frac{1}{1-r}\right). \tag{21}$$

We deduce from (19)-(21) that for any $\varepsilon > 0$ and all $r \in E$, $r \to 1^-$

$$\log M(r,f) \leq \varphi^{-1}\left((\tilde{\mu}^0_\varphi(A_0)+\varepsilon)\log\frac{1}{1-r}\right). \tag{22}$$

By arbitrariness of $\varepsilon > 0$ and the monotonicity of φ^{-1}, we obtain $\tilde{\mu}^1_\varphi(f) \leq \tilde{\mu}^0_\varphi(A_0)$. Hence, Theorem 1.7 is proved.

Proof of Theorem 1.8. Let f be a non-zero solution of equation (1). First, we prove $\rho^0_\varphi(A_0) \leq \rho^1_\varphi(f)$. Suppose that to the contrary, $\rho^1_\varphi(f) < \rho^0_\varphi(A_0)$. Let β_1 and β_2 be two real constants satisfying $\rho^1_\varphi(f) < \beta_1 < \beta_2 < \rho^0_\varphi(A_0)$. Equation (1) can be written as

$$A_0 = -\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}\right). \tag{23}$$

By Lemma 2.2 and the above equation, we obtain that

$$\begin{aligned} m(r,A_0) &\leq \sum_{j=1}^{k-1} m(r,A_j) + \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + O(1) \\ &\leq \sum_{j=1}^{k-1} m(r,A_j) + O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right) \end{aligned} \tag{24}$$

holds for all r, $r \to 1^-$ outside of an exceptional set $F \subset [0, 1)$ with $\int_F \frac{dr}{1-r} < +\infty$. Assume that

$$\limsup_{r\to 1^-}\frac{\sum_{j=1}^{k-1} m(r,A_j)}{m(r,A_0)} = \mu < \lambda < 1.$$

Then for $r \to 1^-$, we have

$$\sum_{j=1}^{k-1} m(r, A_j) < \lambda m(r, A_0)$$

and thus for $r \to 1^-$, $r \notin F$ we have

$$(1-\lambda) m(r, A_0) < O\left(\log^+ T(r, f) + \log \frac{1}{1-r}\right). \tag{25}$$

Hence, by Lemma 2.3, we obtain that

$$\begin{aligned} T(r, A_0) &= m(r, A_0) \le O\left(\log^+ T(r, f) + \log \frac{1}{1-r}\right) \\ &= O\left(\log^+ \varphi^{-1}\left(\beta_1 \log \frac{1}{1-r}\right)\right) \end{aligned} \tag{26}$$

holds for $r \to 1^-$ and $r \notin F$. By Lemma 2.11, there exists a set E of infinite logarithmic measure on $[0, 1)$ such that for $r \to 1^-$

$$T(r, A_0) > \log^+ \varphi^{-1}\left(\beta_2 \log \frac{1}{1-r}\right). \tag{27}$$

Since $E \backslash F$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in E \backslash F$ tending to 1. By substituting (27) into (26), we obtain for all $r_n = |z_n| \in E \backslash F$, $r_n \to 1^-$ and any given ε, $0 < \varepsilon < \beta_2 - \beta_1$

$$\begin{aligned} \log^+ \varphi^{-1}\left(\beta_2 \log \frac{1}{1-r_n}\right) &\le O\left(\log^+ \varphi^{-1}\left(\beta_1 \log \frac{1}{1-r_n}\right)\right) \\ &\le \log^+ \varphi^{-1}\left((\beta_1 + \varepsilon) \log \frac{1}{1-r_n}\right). \end{aligned} \tag{28}$$

By arbitrariness of ε, $0 < \varepsilon < \beta_2 - \beta_1$ and the monotonicity of φ^{-1}, from (28) we obtain $\beta_2 \le \beta_1$. This contradiction proves the inequality $\rho_\varphi^0(A_0) \le \rho_\varphi^1(f)$. On the other hand, it follows from Lemma 2.9 that $\rho_\varphi^1(f) \le \tilde{\rho}_\varphi^0(A_0)$. Therefore, $\rho_\varphi^0(A_0) \le \rho_\varphi^1(f) = \tilde{\rho}_\varphi^1(f) \le \tilde{\rho}_\varphi^0(A_0)$.

Now, we prove $\mu_\varphi^0(A_0) \le \mu_\varphi^1(f)$. Suppose that to the contrary, $\mu_\varphi^1(f) < \mu_\varphi^0(A_0)$. Let α_1 and α_2 be two real constants satisfying $\mu_\varphi^1(f) < \alpha_1 < \alpha_2 < \mu_\varphi^0(A_0)$. It follows from Lemma 2.6 that there exists a set E of infinite logarithmic measure on $[0, 1)$ such that for $r \in E$, $r \to 1^-$

$$T(r, f) < \varphi^{-1}\left(\alpha_1 \log \frac{1}{1-r}\right) \tag{29}$$

and for $r \to 1^-$

$$T(r, A_0) > \log^+ \varphi^{-1}\left(\alpha_2 \log \frac{1}{1-r}\right). \tag{30}$$

Since $E \backslash F$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in E \backslash F$ tending to 1. By substituting (29) and (30) into (25), we obtain for all $r_n = |z_n| \in E \backslash F$, $r_n \to 1^-$ and any given ε, $0 < \varepsilon < \alpha_2 - \alpha_1$

$$\log^+ \varphi^{-1}\left(\alpha_2 \log \frac{1}{1-r_n}\right) \le O\left(\log^+ \varphi^{-1}\left(\alpha_1 \log \frac{1}{1-r_n}\right)\right)$$

$$\leq \log^+ \varphi^{-1}\left((\alpha_1+\varepsilon)\log\frac{1}{1-r_n}\right). \tag{31}$$

By arbitrariness of ε, $0<\varepsilon<\alpha_2-\alpha_1$ and the monotonicity of φ^{-1}, from (31) we obtain $\alpha_2 \leq \alpha_1$. This contradiction proves the inequality $\mu_\varphi^0(A_0) \leq \mu_\varphi^1(f)$. By Lemma 2.8, we obtain $\mu_\varphi^1(f) \leq \tilde{\mu}_\varphi^0(A_0)$ and therefore $\mu_\varphi^0(A_0) \leq \mu_\varphi^1(f) = \tilde{\mu}_\varphi^1(f) \leq \tilde{\mu}_\varphi^0(A_0)$.

Proof of Theorem 1.9. If $\max\{\tilde{\rho}_\varphi^0(A_j) : j=1,\cdots,k-1\} < \tilde{\rho}_\varphi^0(A_0) < +\infty$, then by Theorem 1.5 we get $\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_\varphi^0(A_0)$. Suppose now $\max\{\tilde{\rho}_\varphi^0(A_j) : j=1,\cdots,k-1\} = \tilde{\rho}_\varphi^0(A_0) = \rho_0$, $(0<\rho_0<+\infty)$ and $\max\{\tilde{\tau}_\varphi^0(A_j) : \tilde{\rho}_\varphi^0(A_j) = \tilde{\rho}_\varphi^0(A_0),\ j\neq 0\} < \tilde{\tau}_\varphi^0(A_0) = \tau_0$, $(0<\tau_0<+\infty)$. Then we can choose a set $J \subset \{1,\cdots,k-1\}$ such that $\tilde{\rho}_\varphi^0(A_j) = \tilde{\rho}_\varphi^0(A_0) = \rho_0$ and $\tilde{\tau}_\varphi^0(A_j) < \tilde{\tau}_\varphi^0(A_0) = \tau_0$ for all $j \in J$ as well as $\tilde{\rho}_\varphi^0(A_j) < \tilde{\rho}_\varphi^0(A_0)$ for all $j \notin J$.

First, we prove $\tilde{\rho}_\varphi^1(f) \geq \tilde{\rho}_\varphi^0(A_0)$. Suppose that to the contrary, $\rho_1 = \tilde{\rho}_\varphi^1(f) < \tilde{\rho}_\varphi^0(A_0) = \rho_0$. Let β_1 and β_2 be two real constants satisfying $\max\{\tilde{\tau}_\varphi^0(A_j) : j \in J\} < \beta_1 < \beta_2 < \tau_0$. Then, by the definition of $\tilde{\rho}_\varphi^0(A_j)$, for all r, $r \longrightarrow 1^-$, we have

$$|A_j(z)| \leq \varphi^{-1}\left(\log\frac{\beta_1}{(1-r)^{\rho_0}}\right),\ j \in J \tag{32}$$

and

$$|A_j(z)| \leq \varphi^{-1}\left(\gamma\log\frac{1}{1-r}\right) \leq \varphi^{-1}\left(\log\frac{\beta_1}{(1-r)^{\rho_0}}\right),\ j \in \{1,\ldots,k-1\}\setminus J, \tag{33}$$

where $0<\gamma<\rho_0$. By Lemma 2.10, there exists a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} = \infty$ such that for all $r \in E$, we have

$$|A_0(z)| = M(r,A_0) > \varphi^{-1}\left(\log\frac{\beta_2}{(1-r)^{\rho_0}}\right). \tag{34}$$

By Lemma 2.1, for $j=1,\cdots,k$ and $|z| \notin F$, where F is a set of finite logarithmic measure on $[0,1)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T(s(|z|),f)\right]^j.$$

It follows from the definition of $\rho_\varphi^1(f)$, Proposition 1.4 that for any given ε $(0<\varepsilon<\rho_0-\rho_1)$ and $|z| \notin F$, we obtain

$$\begin{aligned}\left|\frac{f^{(j)}(z)}{f(z)}\right| &\leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T(s(|z|),f)\right]^j \\ &\leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} \varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-s(|z|)}\right)\right]^j. \end{aligned} \tag{35}$$

Since $E\backslash F$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in E\backslash F$ tending to 1. Set $R_n = s(|z_n|) = 1-d(1-|z_n|)$, $d \in (0,1)$. Substituting (32)-(35) into (15) for any given ε $(0<\varepsilon<\min\{\frac{\beta_2-\beta_1}{2},\rho_0-\rho_1\}$, we obtain for $R_n \longrightarrow 1^-$ and $R_n \in E\backslash F$ that

$$\varphi^{-1}\left(\log\frac{\beta_2 d^{\rho_0}}{(1-R_n)^{\rho_0}}\right)$$
$$\leq k\left[\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon}\varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{1-R_n}\right)\right]^k\varphi^{-1}\left(\log\frac{\beta_1 d^{\rho_0}}{(1-R_n)^{\rho_0}}\right)$$
$$\leq\left[\varphi^{-1}\left(\log\frac{(\beta_1+\varepsilon)\,d^{\rho_0}}{(1-R_n)^{\rho_0}}\right)\right]^{k+2}\leq\varphi^{-1}\left(\log\frac{(\beta_1+2\varepsilon)\,d^{\rho_0}}{(1-R_n)^{\rho_0}}\right).$$

The last estimate is verified in view of Remark 1.1. By arbitrariness of ε $(0<\varepsilon<\min\{\frac{\beta_2-\beta_1}{2},\rho_0-\rho_1\}$ and the monotonicity of φ^{-1} we obtain the contradiction $\beta_2\leq\beta_1$. Thus $\tilde{\rho}^1_\varphi(f)\geq\tilde{\rho}^0_\varphi(A_0)$. It follows from Lemma 2.9 that

$$\tilde{\rho}^1_\varphi(f)\leq\max\{\tilde{\rho}^0_\varphi(A_j):j=0,1,\cdots,k-1\}=\tilde{\rho}^0_\varphi(A_0).$$

Therefore, $\tilde{\rho}^1_\varphi(f)=\tilde{\rho}^0_\varphi(A_0)$ and Theorem 1.9 is proved.

Proof of Theorem 1.10. Since $\tilde{\mu}^0_\varphi(A_0)\leq\tilde{\rho}^0_\varphi(A_0)$, then by Theorem 1.9 we obtain $\tilde{\rho}^1_\varphi(f)=\tilde{\rho}^0_\varphi(A_0)$. If $\max\left\{\tilde{\rho}^0_\varphi(A_j):j\neq 0\right\}<\tilde{\mu}^0_\varphi(A_0)$, then by Theorem 1.7 we have $\tilde{\mu}^1_\varphi(f)=\tilde{\mu}^0_\varphi(A_0)$. If $\tilde{\rho}^0_\varphi(A_j)=\tilde{\mu}^0_\varphi(A_0)=\mu_0$ and $\tilde{\tau}^0_\varphi(A_j)\leq\tau<\underline{\tilde{\tau}}^0_\varphi(A_0)=\underline{\tau}_0$.
We first prove the inequality $\mu_1=\tilde{\mu}^1_\varphi(f)\geq\tilde{\mu}^0_\varphi(A_0)=\mu_0$. Suppose that to the contrary, $\mu_1<\mu_0$. It follows from the definitions of $\tilde{\tau}^0_\varphi(A_j)$ and $\underline{\tilde{\tau}}^0_\varphi(A_0)$ that for all r, $r\longrightarrow 1^-$ and for any given $\varepsilon(0<3\varepsilon<\underline{\tau}_0-\tau)$, we have

$$|A_j(z)|\leq\varphi^{-1}\left(\log\frac{\tau+\varepsilon}{(1-r)^{\mu_0}}\right)\tag{36}$$

and

$$|A_0(z)|\geq\varphi^{-1}\left(\log\frac{\underline{\tau}_0-\varepsilon}{(1-r)^{\mu_0}}\right).\tag{37}$$

By Lemma 2.1, for $j=1,\cdots,k$ and $|z|\notin F$, where F is a set of finite logarithmic measure on $[0,1)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon}T(s(|z|),f)\right]^j.$$

By Proposition 1.4 and Lemma 2.6, there exists a set $E\subset[0,1)$ with $\int_E\frac{dr}{1-r}=\infty$ such that for all $r\in E\backslash F$, and any given ε $(0<\varepsilon<\mu_0-\mu_1)$, we obtain

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon}T(s(|z|),f)\right]^j$$
$$\leq\left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon}\varphi^{-1}\left((\mu_1+\varepsilon)\log\frac{1}{1-s(|z|)}\right)\right]^j.\tag{38}$$

Since $E\backslash F$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n=|z_n|\in E\backslash F$ tending to 1. Set $R_n=s(|z_n|)=1-d(1-|z_n|)$, $d\in(0,1)$.

Substituting (36)-(38) into (15) for any given ε $(0 < \varepsilon < \min\{\mu_0 - \mu_1, \frac{\tau_0 - \tau}{3}\}$, we obtain for $R_n \longrightarrow 1^-$ and $R_n \in E\backslash F$ that

$$\begin{aligned}
&\varphi^{-1}\left(\log \frac{(\underline{\tau}_0 - \varepsilon)d^{\mu_0}}{(1-R_n)^{\mu_0}}\right) \\
&\quad \le k\left[\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} \varphi^{-1}\left((\mu_1+\varepsilon)\log\frac{1}{1-R_n}\right)\right]^k \varphi^{-1}\left(\log\frac{(\tau+\varepsilon)d^{\mu_0}}{(1-R_n)^{\mu_0}}\right) \\
&\quad \le \left[\varphi^{-1}\left(\log\frac{(\tau+\varepsilon)d^{\mu_0}}{(1-R_n)^{\mu_0}}\right)\right]^{k+2} \le \varphi^{-1}\left(\log\frac{(\tau+2\varepsilon)d^{\mu_0}}{(1-R_n)^{\mu_0}}\right).
\end{aligned}$$

The last estimate is verified in view of Remark 1.1. By arbitrariness of ε $(0 < \varepsilon < \min\{\mu_0 - \mu_1, \frac{\tau_0 - \tau}{3}\}$ and the monotonicity of φ^{-1} we obtain the contradiction $\underline{\tau}_0 \le \tau$. Thus $\tilde{\mu}^1_\varphi(f) \ge \tilde{\mu}^0_\varphi(A_0)$. It follows from Lemma 2.8 that

$$\tilde{\mu}^1_\varphi(f) \le \tilde{\mu}^0_\varphi(A_0).$$

Therefore, $\tilde{\mu}^1_\varphi(f) = \tilde{\mu}^0_\varphi(A_0)$.

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