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A study on K- paracontact and (κ, μ)- paracontact manifold admitting vanishing Cotton tensor and Bach tensor

V. Venkatesha, N. Bhanumathi, and C. Shruthi

Abstract. The object of the present paper is to study K-paracontact manifold admitting parallel Cotton tensor, vanishing Cotton tensor and to study Bach flatness on K-paracontact manifold. In that we prove for a K-paracontact metric manifold M^{2n+1} has parallel Cotton tensor if and only if M^{2n+1} is an η-Einstein manifold and $r = -2n(2n+1)$. Further we show that if g is an η-Einstein K-paracontact metric and if g is Bach flat then g is an Einstein. Also we study vanishing Cotton tensor on (κ, μ)-paracontact manifold for both $\kappa > -1$ and $\kappa < -1$. Finally, we prove that if M^{2n+1} is a (κ, μ)-paracontact manifold for $\kappa \neq -1$ and if M^{2n+1} has vanishing Cotton tensor for $\mu \neq \kappa$, then M^{2n+1} is an η-Einstein manifold.

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1. Introduction

In 1921, the notion of Bach tensor was introduced by R. Bach [1] to study conformal relativity. This is a symmetric traceless $(0, 2)$-type tensor B on an n-dimensional Riemannian manifold (M, g), defined by

$$B(X,Y) = \frac{1}{n-1} \sum_{i,j=1}^{n} \big((\nabla_{e_i} \nabla_{e_j} W)(X, e_i, e_j, Y)\big) + \frac{1}{n-2} \sum_{i,j=1}^{n} Ric(e_i, e_j) W(X, e_i, e_j, Y), \tag{1}$$

where $(e_i), i = 1, ..., n$, is a local orthonormal frame on $(M; g)$, Ric is the Ricci tensor of type $(0, 2)$ and C is the $(0, 3)$-type Cotton tensor defined by[9]

$$C(X,Y)Z = (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) - \frac{1}{2(n-1)}[g(Y,Z)(X_r) - g(X,Z)(Y_r)] \tag{2}$$

and W denotes the Weyl tensor of type $(0, 3)$ defined by[9]

$$W(X,Y)Z = R(X,Y)Z - g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y). \tag{3}$$

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After Bach[1], many people worked on Bach tensor; In 1993 Pedersen and Swann[13] studied Einstein-Weyl geometry, the Bach tensor and conformal scalar curvature. In 2013-14 H.D. Cao and others ([6] and [7]) studied Bach tensor on gradient shrinking and steady Ricci soliton. In 2017 Ghosh and Sharma [10] studied Sasakian manifolds with purely transversal Bach tensor. In that article they shows a $(2n+1)$-dimensional Sasakian manifold (M, g) with a purely transversal Bach tensor has constant scalar curvature $\geq 2n(2n+1)$, equality holding if and only if (M, g) is Einstein. For dimension 3, M is locally isometric to the unit sphere S^3. For dimension 5, if in addition (M, g) is complete, then it has positive Ricci curvature and is compact with finite fundamental group $\pi_1(M)$. Recently in 2019 Ghosh and Sharma [9] studied classification of (κ, μ)-contact manifold with divergence free Cotton tensor and vanishing Bach tensor.

The study of paracontact geometry was introduced by Kaneyuki and Williams in [11]. A systematic study of paracontact metric manifolds started with the paper [16], were the Levi-Civita connection, the curvature and a canonical connection (analogue to the Tanaka Webster connection of the contact metric case) of a paracontact metric manifold have been described.

There are differences between a contact metric (κ, μ)- space $(M^{2n+1}, \phi, \xi, \eta, g)$ and a paracontact metric (κ, μ)-space $(M^{2n+1}, \phi, \xi, \eta, g)$. Namely, unlike in the contact Riemannian case, a paracontact (κ, μ)-manifold such that $\kappa = -1$ in general is not para-Sasakian. In fact, there are paracontact (κ, μ)-manifolds such that $h^2 = 0$ (which is equivalent to take $\kappa = -1$) but with $h \neq 0$. For 5-dimensional, Cappelletti Montano and Di Terlizzi gave the first example of paracontact metric $(-1, 2)$-space $(M^{2n+1}, \phi, \xi, \eta, g)$ with $h^2 = 0$ but $h \neq 0$ in [5] and then Cappelletti Montano et. al., gave the first paracontact metric structures defined on the tangent sphere bundle and constructed an example with arbitrary n in [2]. Later, for 3-dimensional, the first numerical example was given in [8]. Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric (κ, μ)-spaces the constant κ can not be greater than 1, paracontact metric (κ, μ)-space has no restriction for the constants κ and μ.

These papers leads interest and gives motivation to us to study Bach and Cotton tensor on K-paracontact and (κ, μ)-paracontact manifold.

After the introduction, we discuss preliminary part, it includes some basic definitions and some important properties of K-paracontact and (κ, μ)-paracontact manifold which are related to our paper and in the third section we study vanishing Cotton tensor on K-paracontact manifold, in the next section we study parallel Cotton tensor on K-paracontact manifold. In section five, we study Bach tensor on η-Einstein K-paracontact manifold $(n > 1)$. Finally in the last two sections, we discuss vanishing Cotton tensor on (κ, μ)-paracontact manifold for both $\kappa > -1$ and $\kappa < -1$.

2. Preliminaries

In this section, we recall some basic definitions, which are helpful for our future studies. For more information we refer [3],[12],[15]. A $(2n+1)$-dimensional smooth manifold M^{2n+1} has a almost paracontact structure (φ, ξ, η) if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η such that

$$\varphi^2 = I - \eta \cdot \xi, \quad \varphi(\xi) = 0, \quad \eta \cdot \varphi = 0, \quad \eta(\xi) = 1, \tag{4}$$

for all $X, Y \in TM^{2n+1}$ and the eigen distributions D^+ and D^- of φ corresponding to the respective eigenvalues 1 and -1 have equal dimension n. If an almost paracontact manifold is endowed with a semi-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{5}$$

where signature of g is necessarily $(n+1, n)$ for all $X, Y \in TM^{2n+1}$, then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called an almost paracontact metric manifold. The curvature tensor R is taken with the sign convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (note that an opposite convention is used in [[3],[4],[14]]. By Q and r, we shall denote the Ricci operator determined by $S(X,Y) = g(QX,Y)$ and the scalar curvature of the metric g, respectively. The fundamental 2-form of an almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is defined by $\Phi(X,Y) = g(X, \varphi Y)$. If $d\eta = \Phi$, then the manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be paracontact metric manifold and g the associated metric. In such case η is a contact form (that is, $\eta \wedge (d\eta)^n \neq 0$), ξ is its Reeb vector field and M^{2n+1} is a contact manifold. If, in addition, ξ is a Killing vector field (equivalently, $h = \frac{1}{2}\pounds_\xi \varphi = 0$, where $\pounds$ is the usual Lie derivative), then M^{2n+1} is said to be a paracontact metric manifold. In a K-paracontact manifold, we can easily get the following formulas

$$\begin{aligned} \nabla_X \xi &= -\varphi X + \varphi h X, & (6)\\ \nabla_\xi h &= -\varphi + \varphi h^2 - \varphi l & (7)\\ Ric(\xi,\xi) &= g(Q\xi,\xi) = Trl = Tr(h^2) - 2n, & (8) \end{aligned}$$

for all vector fields X, Y on M, where ∇ is the operator of covariant differentiation of g and Q denotes the Ricci operator associated with the Ricci tensor given by $Ric(X,Y) = g(QX,Y)$ for all vector fields X, Y on M. If the vector field ξ is Killing (equivalently, $h = 0$) then M is said to be a K-paracontact manifold. On K-paracontact manifold, the following formulas hold:

$$\begin{aligned} \nabla_X \xi &= -\varphi X & (9)\\ R(X,\xi)\xi &= -X + \eta(X)\xi & (10)\\ Q\xi &= -2n\xi & (11) \end{aligned}$$

Proposition 2.1. *On a K-paracontact manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, we have (from* [12]*)*

$$\begin{aligned} (i) \quad & (\nabla_X Q)\xi = Q\varphi X + 2n\varphi X, & (12)\\ (ii) \quad & (\nabla_\xi Q)X = Q\varphi X - \varphi Q X, & (13) \end{aligned}$$

for any vector field X on M^{2n+1}.

Definition 2.1. (See [2]) A paracontact metric (κ,μ)-manifold M^{2n+1} is a paracontact metric manifold for which the curvature tensor field satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{14}$$

for all vector fields X, Y on M^{2n+1} and for some real constants κ and μ .

Further, a paracontact metric manifold M satisfies the following properties

$$\begin{aligned} h^2 &= (1+\kappa)\varphi^2, & (15)\\ Q\xi &= 2n\kappa\xi, & (16)\\ (\nabla_X \varphi)Y &= -g(X - hX, Y)\xi + \eta(Y)(X - hX), \text{ for } \kappa = -1, & (17) \end{aligned}$$

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X =& -(1+\kappa)2g(X,\varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X \\ &+ (1-\mu)\eta(X)\varphi hY - \eta(Y)\varphi hX, \qquad (18)\\ Q =&(2(1-n)+n\mu)I + (2(n-1)+\mu)h \\ &+ (2(n-1)+n(2\kappa-\mu))\eta\otimes\xi, \quad for\ \kappa > -1, \qquad (19)\\ Q =&(2(1-n)+n\mu)I + (2(n+1)+\mu)h \\ &+ (2(n-1)+n(2\kappa-\mu))\eta\otimes\xi, \quad for\ \kappa < -1, \qquad (20)\end{aligned}$$

for any vector fields X, Y on M, where Q denotes the Ricci operator of (M^{2n+1}, g).

Definition 2.2. A Riemannian manifold is called an η-Einstein manifold, if it has *Ricci* tensor Q such that

$$QY = aY + b\eta(Y)\xi \qquad (21)$$

where $a, b \in C^\infty(M^{2n+1})$ and if the function $b = 0$ then it is called Einstein.

3. Vanishing Cotton tensor on K-paracontact manifold

Proposition 3.1. *Let M^{2n+1} be a K-paracontact manifold. Then M^{2n+1} has constant scalar curvature if and only if $C(X,\xi)\xi = 0$*

Proof. Setting $Z = \xi$ in (2) we get.

$$C(X,Y)\xi = g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X) - \frac{1}{4n}[(Xr)\eta(Y) - (Yr)\eta(X)] \qquad (22)$$

Using equation (12) from Proposition [2.1] in the above equation, we get

$$C(X,Y)\xi = -4ng(\varphi X, Y) + g(Q\varphi X, Y) - g(Q\varphi Y, X) - \frac{1}{4n}[(Xr)\eta(Y) - (Yr)\eta(X)]. \quad (23)$$

Replacing X by φX and Y by φY in (23) we obtain,

$$C(\varphi X, \varphi Y)\xi = 4ng(\varphi X, Y) + g(Q\varphi^2 X, \varphi Y) - g(Q\varphi^2 Y, \varphi X) = 0, \qquad (24)$$

which gives

$$-4ng(\varphi X, Y) - g(X, Q\varphi Y) + g(Q\varphi X, Y) = 0. \qquad (25)$$

Admitting (25) in (23), we get,

$$(Xr)\eta(Y) - (Yr)\eta(X) = 0. \qquad (26)$$

Putting $Y = \xi$ and taking X orthogonal to ξ in the above equation gives

$$Xr = 0. \qquad (27)$$

As M is paracontact manifold and $X \in ker\eta$ which implies $Xr = 0$, $\forall X \in TM^{2n+1}$. So r is constant.

Conversely, if r is constant then substituting $Y = \xi$ in the equation (23) gives $C(X,\xi)\xi = 0$.
Hence the proof. ∎

4. Parallel Cotton tensor on K-paracontact manifold M^{2n+1}

Definition 4.1. In a Riemannian manifold M^{2n+1}, if there is a Cotton tensor C such that its covariant differentiation i.e., $(\nabla_W C) = 0$ then the manifold is said to have parallel Cotton tensor.

Theorem 4.1. *Let M^{2n+1} be a K-paracontact metric manifold. Then M has parallel Cotton tensor if and only if M^{2n+1} is an η-Einstein manifold and $r = -2n(2n+1)$.*

Proof. For a K-paracontact manifold M^{2n+1}, the equation (2) for $Y = \xi$ and $Z = Y$ is gives

$$C(X,\xi)Y = 2ng(\varphi X, Y) + g(Q\varphi X, Y) - \frac{1}{4n}\{(Xr)\eta(Y)\}. \tag{28}$$

Taking $Y = \xi$ in the above equation, we get

$$C(X,\xi)\xi = -\frac{1}{4n}\{(Xr)\}. \tag{29}$$

Using (29) in (22) we calculate the following relations

$$\nabla_W C(X,\xi)\xi = -\tfrac{1}{4n}\{g(\nabla_W X, Dr) + g(X, \nabla_W Dr)\}, \tag{30}$$

$$C(\nabla_W X,\xi)\xi = -\tfrac{1}{4n}\{g(\nabla_W X, Dr)\}, \tag{31}$$

$$C(X,\varphi W)\xi = 4ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W) - g(Q\varphi^2 W, X) - \tfrac{1}{4n}\{-(\varphi Wr)\eta(X)\}, \tag{32}$$

$$C(X,\xi)\varphi W = 2ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W). \tag{33}$$

Making use of above group of equations we obtain

$$\begin{aligned}(\nabla_W C)(X,\xi)\xi = &-\frac{1}{4n}\{g(X, \nabla_W Dr)\} + 4ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W)\\ &-g(Q\varphi^2 W, X) - \frac{1}{4n}\{(\varphi Wr)\eta(X)\} + 2ng(\varphi X, \varphi W) + g(\varphi QX, \varphi W).\end{aligned} \tag{34}$$

Putting $W = \xi$ in the above equation, the parallel Cotton tensor becomes

$$(\nabla_\xi C)(X,\xi)\xi = -\frac{1}{4n}\{g(X, \nabla_\xi Dr)\} = 0. \tag{35}$$

As $\pounds_\xi r = 0, \nabla_\xi Dr = \nabla_{Dr}\xi = -\varphi Dr$, which implies $g(X, \varphi Dr) = 0$, which gives $Dr = 0$ and so r is constant. Then the relation (34) becomes

$$\begin{aligned}6ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W) - g(X, QW) - 2n\eta(X)\eta(W)\\ -g(X, QW) - 2n\eta(X)\eta(W) = 0.\end{aligned} \tag{36}$$

Replacing X by φX and W by φW in (36) and simplifying we get

$$g(Q\varphi X, \varphi W) = -3ng(\varphi X, \varphi W) + \frac{1}{2}g(QX, W) + n\eta(X)\eta(W). \tag{37}$$

Feeding (37) in (36) we obtain

$$\begin{aligned}6ng(X, W) + 6n\eta(X)\eta(W) - 3ng(X, W) + 3n\eta(X)\eta(W) + \frac{1}{2}g(X, \varphi W)\\ +n\eta(X)\eta(W) - 4n\eta(X)\eta(W) - 2g(X, QW) = 0.\end{aligned} \tag{38}$$

Contracting the equation (38) over X and W we have $r = -2n(2n+1)$ and M is an η-Einstein manifold.
Conversely, suppose M is an η-Einstein manifold and $r = -2n(2n+1)$, which implies

$QY = -2nY$. And so this gives $C(X,Y)Z = 0$.
Hence the proof. □

Lemma 4.2. *Let $M^{2n+1}(n > 1)$ be a K-paracontact manifold. If M^{2n+1} satisfies (21), then a and b are constant functions*

Proof. From the condition (21) we have,

$$(\nabla_X Q)Y = (Xa)Y + (Xb)\eta(Y)\xi + b\left\{g(X,\varphi Y)\xi + \eta(Y)\nabla_X\xi\right\}. \tag{39}$$

From η-Einstein condition, $-2n = a + b$, so $(Xa) = -(Xb)$.
Therefore

$$(\nabla_X Q)Y = (Xa)Y - (Xa)\eta(Y)\xi + \{-2n - a\}\left\{g(X,\varphi Y)\xi - \eta(Y)\varphi X\right\}. \tag{40}$$

Contracting the above equation over X with respect to the orthonormal frame field we get

$$\sum_{i=1}^{2n+1} \epsilon_i \left\langle(\nabla_{e_i} Q)Y, e_i\right\rangle = \sum_{i=1}^{2n+1} \epsilon_i (e_i a) g(Y, e_i) + (\xi a) \tag{41}$$

where $\xi = g(e_i, e_i)$, as $\xi r = 0$ gives $\xi a = 0$. But we know that $\sum_{i=1}^{2n+1} \left\langle(\nabla_{e_i} Q)Y, e_i\right\rangle = \frac{1}{2}(Yr)$ which gives

$$\frac{1}{2}(Yr) \quad = \quad g(Y, Da) \tag{42}$$

as $Yr = 2$, so $(n-1)Ya = 0$ for $n > 1$ becomes $Ya = 0$, therefore a is constant.
This completes the proof. □

5. Bach tensor on η-Einstein K-paracontact manifolds for $(n > 1)$

Bach tensor for $2n+1$-dimensional manifold is given by

$$B(X,Y) = \frac{1}{2n-1}\left\{\sum_{i=1}^{2n+1} \epsilon_i (\nabla_{e_i} C)(e_i, X, Y) + \sum_{i,j=1}^{2n+1} \epsilon_i g(Qe_i, e_j) W(X, e_i, e_j, Y)\right\} \tag{43}$$

By lemma (4.2) we know that a and b are constants then equation (39) becomes

$$(\nabla_X Q)Y = b\left\{g(X,\varphi Y)\xi - \eta(Y)\varphi X\right\}. \tag{44}$$

We know that from the lemma (4.2) r is constant and simplifying the cotton tensor using (44)

$$C(X,Y)Z \quad = \quad bg(X,\varphi Y)\eta(Z) - b\eta(Y)g(\varphi X, Z) - bg(Y,\varphi X)\eta(Z) + bg(\varphi Y, Z)\eta(X).$$

Applying ∇_W on both side of the above equation gives

$$\begin{aligned}(\nabla_W C)(X,Y)Z =& b\nabla_W \left\{2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y, Z) + \eta(Y)g(X,\varphi Z)\right\} \\ =& b2g(X,(\nabla_W\varphi)Y)\eta(Z) + bg(X,\varphi Y)g(W,\varphi Z) + bg((\nabla_W\varphi)Y,Z)\eta(X) \\ &+ bg(\varphi Y, Z)g(W,\varphi X) + bg(X,(\nabla_W\varphi)Z)\eta(Y) + bg(X,\varphi Z)g(W,\varphi Y).\end{aligned} \tag{45}$$

On contracting above equation over X and W gives

$$\begin{aligned}\sum_{i=1}^{2n+1}\epsilon_i(\nabla_{e_i}C)(e_i,Y)Z &= b\left\{\sum_{i=1}^{2n+1}\epsilon_i g(e_i,(\nabla_{e_i}\varphi)Y)\eta(Z+g(e_i,(\nabla_{e_i}\varphi)Z)\eta(Y)\right\}+2bg(\varphi Y,\varphi Z)\\ &= b\left\{\sum_{i=1}^{2n+1}\epsilon_i\left\langle R(\xi,e_i)Y,e_i\right\rangle\eta(Z)+g(R(\xi,e_i)Z,e_i)\eta(Y)\right\}+2bg(\varphi Y,\varphi Z)\\ &= b\left\{-S(Y,\xi)\eta(Z)-S(Z,\xi)\eta(Y)\right\}+2bg(\varphi Y,\varphi Z)\\ &= b\left\{4n\eta(Y)\eta(Z)+2g(\varphi Y,\varphi Z)\right\}.\end{aligned}$$

Now we calculate the right hand side of the Bach tensor that is

$$\sum_{i,j=1}^{2n+1}\epsilon_i g(Qe_i,e_j)g(W(X,e_i)e_j,Y) = -\sum_{i,j=1}^{2n+1}\epsilon_i g(Qe_i,W(X,e_i)Y).$$

By η-Einstein condition $Qe_i = ae_i + b\eta(e_i)\xi$, which gives

$$\begin{aligned}\sum_{i,j=1}^{2n+1}\epsilon_i g(Qe_i,e_j)g(W(X,e_i)e_j,Y) &= -\sum_{i,j=1}^{2n+1}\epsilon_i g(e_i+b\eta(e_i)\xi,W(X,e_i)Y)\\ &= \sum_{i=1}^{2n+1}\epsilon_i g(W(X,e_i)e_i,Y)+bg(W(X,\xi)\xi,Y).\end{aligned} \tag{46}$$

From the expression of Weyl tensor W we deduce the following relation

$$\begin{aligned}\sum_{i=1}^{2n+1}\epsilon_i\left\langle W(X,e_i)e_i,Y\right\rangle &= \sum_{i=1}^{2n+1}\epsilon_i(\left\langle R(X,e_i)e_i,Y\right\rangle-\frac{1}{2n-1}[g(Qe_i,e_i)g(X,Y)\\ &\quad -g(QX,e_i)g(e_i,Y)+g(e_i,e_i)g(QX,Y)-g(X,e_i)g(Qe_i,Y)]\\ &\quad +\frac{r}{2n(2n-1)}[g(e_i,e_i)g(X,Y)-g(X,e_i)g(e_i,Y)])\\ &= S(X,Y)-\frac{1}{2n-1}[rg(X,Y)-S(X,Y)+(2n+1)S(X,Y)\\ &\quad -S(X,Y)]+\frac{r}{2n(2n-1)}[(2n+1)g(X,Y)-g(X,Y)]\\ &= 0.\end{aligned} \tag{47}$$

Taking inner product of $W(X,\xi)\xi$ with Y we get,

$$\begin{aligned}\left\langle W(X,\xi)\xi,Y\right\rangle &= \left\langle R(X,\xi)\xi,Y\right\rangle-\frac{1}{2n-1}(-2n\left\langle X,Y\right\rangle+2n\eta(X)\eta(Y)+\left\langle QX,Y\right\rangle\\ &\quad +2n\eta(X)\eta(Y))+\frac{r}{2n(2n-1)}(\left\langle X,Y\right\rangle-\eta(X)\eta(Y))\\ &= \left\langle\varphi\nabla_X\xi,Y\right\rangle+\frac{2n}{2n-1}(X,Y)-\frac{4n}{2n-1}\eta(X)\eta(Y)+\frac{r}{2n(2n-1)}(X,Y)\\ &\quad -\frac{r}{2n(2n-1)}\eta(X)\eta(Y)-\frac{1}{2n-1}S(X,Y)\end{aligned}$$

But $\langle \varphi X, \varphi Y \rangle = -\langle X, Y \rangle + \eta(X)\eta(Y)$, so we get

$$\langle W(X,\xi)\xi, Y \rangle = \frac{1}{2n-1}\left\{\left(1+\frac{r}{2n}\right)(X,Y) - \left(1+2n+\frac{r}{2n}\right)\eta(X)\eta(Y)\right\} - \frac{1}{2n-1}S(X,Y) \quad (48)$$

Using the value of $S(X,Y) = \left(1+\frac{r}{2n}\right)(X,Y) - \left(1+2n+\frac{r}{2n}\right)\eta(X)\eta(Y)$ in (48) gives

$$\langle W(X,\xi)\xi, Y \rangle = 0. \quad (49)$$

Therefore if g is Bach flat,

$$B(Y,Z) = 0 = \frac{b}{2n-1}\left\{4n\eta(Y)\eta(Z) + 2g(\varphi Y, \varphi Z)\right\}. \quad (50)$$

For $Y = Z = \xi$ we obtain $b = 0$.
Hence we can state this result

Theorem 5.1. *Let M^{2n+1} be an η-Einstein K-paracontact manifold. If it has Bach flat then M^{2n+1} is an Einstein manifold.*

6. (κ,μ)-paracontact manifold, for $\kappa \neq -1$

In this section we deal with paracontact (κ,μ)-manifolds such that $\kappa > -1$ and $\kappa < -1$.
First for $\kappa > -1$, using (19) we calculate,

$$\begin{aligned}(\nabla_X Q)Y =& g(2(n-1)+\mu)(\nabla_X h)Y \\ &+ (2(n-1)+n(2\kappa-\mu))\{(\nabla_X \eta)Y\xi + \eta(Y)\nabla_X \xi\}\end{aligned} \quad (51)$$

Now considering the Cotton tensor on (κ,μ)-paracontact manifold as from (19), r is constant, which implies

$$C(X,Y)Z = g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)X, Z). \quad (52)$$

Using equation (51) we obtain

$$\begin{aligned}C(X,Y)Z =& (2(n-1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y, X) \\ &- \eta(Y)g(\varphi X, Z)) + (1+\mu)(\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z))\} \\ &+ 2(2(n-1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) + (2(n-1)+n(2\kappa-\mu)) \\ &\times \{\eta(Y)g(-\varphi X + \varphi hX, Z) - \eta(X)g(-\varphi Y + \varphi hY, Z)\}.\end{aligned} \quad (53)$$

Replacing X, Y, Z by $\varphi X, \varphi Y, \varphi Z$ respectively in the above equation then we get $C(\varphi X, \varphi Y)\varphi Z = 0$.
Similarly for $\kappa < -1$ we have from (20)

$$(\nabla_X Q)Y = g(2(n-1)+\mu)(\nabla_X h)Y + (2(n+1)+n(2\kappa-\mu))\{(\nabla_X \eta)Y\xi + \eta(Y)\nabla_X \xi\}$$

Now consider the Cotton tensor with r is constant and substitute $(\nabla_X Q)Y$ and $(\nabla_Y Q)X$ values in Cotton tensor then we get

$$\begin{aligned}C(X,Y)Z =& g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)X, Z) \\ =& (2(n+1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y, X) \\ &- \eta(Y)g(\varphi X, Z)) + (1+\mu)(\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z))\} \\ &+ 2(2(n+1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) + (2(n+1)+n(2\kappa-\mu)) \\ &\times \{\eta(Y)g(-\varphi X + \varphi hX, Z) - \eta(X)g(-\varphi Y + \varphi hY, Z)\}\end{aligned} \quad (54)$$

Replacing X, Y and Z by $\varphi X, \varphi Y$ and φZ respectively in the above equation, then $C(\varphi X, \varphi Y)\varphi Z = 0$.
Form the above two cases, when $\kappa \neq -1$ we obtain the following result;

Proposition 6.1. *On a (κ,μ)-paracontact metric manifold for $\kappa \neq -1$ the projection of the image of Cotton tensor $C/_{\varphi T_P(M^{2n+1})X\varphi T_P(M^{2n+1})}$ in $\varphi T_p(M^{2n+1})$ is zero, i.e., $C(\varphi X, \varphi Y)\varphi Z = 0$, $\forall X, Y, Z \in T_P(M^{2n+1})$*

7. Vanishing Cotton tensor on (κ,μ)-paracontact manifold, for $\kappa \neq -1$

In this section we deal with paracontact (κ,μ)-manifolds such that $\kappa < -1$ and $\kappa > -1$ then we have the Cotton tensor $C(X,Y)Z = 0$.
For $\kappa > -1$, replacing Z by ξ in equation (54) then we get

$$C(X,Y)\xi = 0 = (2(n-1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y))\} + 2(2(n-1)+n(n(2\kappa-\mu))g(X,\varphi Y)$$
$$(2(n-1)+\mu)(1+\kappa) + (2(n-1)+n(2n-\mu)) = 0 \tag{55}$$

Similarly, admitting ξ in the place of X in equation (54) gives,

$$\begin{aligned} C(\xi,Y)Z = 0 \quad &= \quad (2(n-1)+\mu)\{-(1+\kappa)g(\varphi Y,Z) + (1+\mu)g(\varphi hY,Z)\} \\ &\quad +(2(n-1)n(2\kappa-\mu))\{g(\varphi Y,Z) - g(\varphi hY,Z)\} \end{aligned} \tag{56}$$

Symmetrizing the above equation and replacing Y by hY we obtain

$$(1+\kappa)\{(2(n-1)+\mu)(1+\mu) - (2(n-1)+n(2\kappa-\mu)\} = 0$$

From equation (55) it gives,

$$(1+\kappa)\{(2(n-1)+\mu)(1+\mu) - (2(n-1)+\mu)(1+\kappa)\} = 0$$
$$\Longrightarrow \ (1+\kappa)(\mu-\kappa)(2(n-1)+\mu) = 0$$

The above calculations leads this result.
Case(i) If $\mu \neq \kappa$ then $(2(n-1)+\mu) = 0$. Therefore M^{2n+1} is η-Einstein.
Case(ii) If $\mu = \kappa$ then from equation (55) $\mu = \kappa = 0$ or $\mu = \kappa = 0$. Therefore the we have the following result.

Lemma 7.1. *Let M^{2n+1} be a (κ,μ)-paracontact manifold, admitting vanishing Cotton tensor for $\kappa > -1$ then we have*
i). If $\mu \neq \kappa$ then M^{2n+1} is an η-Einstein manifold,
ii). If $(2(n-1)+\mu) \neq 0$ then $\mu = \kappa = 0$.

Next for $\kappa < -1$, Cotton tensor is

$$\begin{aligned} C(X,Y)Z =& (2(n+1)+\mu)\{(\nabla_X\eta)Y - (\nabla_Y\eta)X\} + (2(n+1)+n(\kappa-\mu))\{(\nabla_X\eta)Y\eta(Z) \\ & - (\nabla_X\eta)X\eta(Z)\} + (2(n-1)+n(2\kappa-\mu))\{\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi\} \\ =& (2(n+1)+\mu)\{-(1+\kappa)2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y,Z) - \eta(Y)g(\varphi X,Z)\} \\ & + (1+\mu)(\eta(X)g(\varphi hX,Z) - \eta(Y)g(\varphi hX,Z)) \\ & + 2(2(n-1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) + (2(n-1)+n(2\kappa-\mu) \\ & \times \{\eta(Y)g(-\varphi X+\varphi hX,Z) - \eta(X)g(-\varphi Y+\varphi hY,Z)\} \end{aligned} \tag{57}$$

Substitute Z by ξ in the above equation become

$$C(X,Y)\xi = 0 = \{(2(n+1)+\mu)(1+\kappa) - (2(n-1)+n(2\kappa-\mu)\} \tag{58}$$

Replace X by ξ in the equation (57) gives

$$C(\xi, Y)Z = 0 =(-2(n-1)+\mu)(1+\kappa)g(\varphi Y, Z) + (2(n-1)+\mu)(1+\mu)g(\varphi hY, Z) + (2(n-1)+n(2\kappa+\mu))\{g(\varphi Y, Z) - g(\varphi hY, Z)\}. \tag{59}$$

On symmetrizing the above equation we have

$$(1+\kappa)(2(n+1)+\mu)(\mu-\kappa) = 0. \tag{60}$$

Therefore we can state the following lemma

Lemma 7.2. *Let M^{2n+1} be a (κ, μ) paracontact metric manifold for $\kappa < -1$, if M^{2n+1} has vanishing Cotton tensor for $\mu \neq \kappa$ then M^{2n+1} is an η- Einstein manifold.*

From case (i) of lemma (7.1) and lemma (7.2) we get the following result.

Theorem 7.3. *Let M^{2n+1} be a (κ, μ)-paracontact manifold for $\kappa \neq -1$. If M^{2n+1} has vanishing Cotton tensor for $\mu \neq \kappa$, then M^{2n+1} is an η-Einstein manifold.*

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(V. Venkatesha, N. Bhanumathi, C. Shruthi) Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, INDIA

E-mail address: vensmath@gmail.com, bhanubhanumathin@gmail.com, c.shruthi28@gmail.com