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Boundary stabilization of an overhead crane with beam model

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Abstract. In this paper, we study the boundary feedback stabilization problem of a hybrid system consisting of a flexible beam attached to a platform moving along a straight rail and carrying at the free end a load which is free to move in a horizontal plane. The model proposed in this paper fits a large class of real-life applications such as an overhead crane with a beam. Using the Riesz basis approach of general second-differential equation systems with non-separated boundary conditions, it is shown that the Riesz basis property holds for the system and as a consequence, the exponential stability is concluded. To verify the theoretical developments, numerical study of the spectrum is performed by Legendre approximation, also the numerical simulations are presented to show the effectiveness of the proposed control.

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1. Introduction

In this paper, we consider a model of an overhead crane system with a beam which consist of a flexible beam attached to a platform moving along a straight rail of mass I_m and attached rigidly at the free end to a tip body of mass M and moment of inertia J. Furthermore, the beam is supposed to be clamped at the platform. The stabilization was achieved using the application of high derivative boundary damping.

The design of the high derivative feedback controllers in literature is mainly based on the principle of passivity that makes the closed loop system dissipative, so the system is at least asymptotically stable by Lyapunov function method. There are many technics of designing controllers that make the system practically uniformly stable, but there is no dissipativity which usually brings the difficulty of theoretical proof of the uniform stability of the system. We use in the present paper the Riesz basis approach, which was recently used to study the basis generation, exponential stability and distribution of eigenvalues.

There are many different models in literature describing the vibration of a flexible beam with a tip rigid body [7, 9]. The transversal displacement $y(x,t)$ at position x and time t is governed by one partial differential equation (the Euler-Bernoulli beam equation for vibrations of a beam) coupled with two ordinary differential equations (the Newton-Euler equations for oscillations of a rigid body), this set of equations forms what is often called in the literature a hybrid system.

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In this paper, we consider the following system

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x,t) + \frac{\partial^4 y}{\partial x^4}(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ \frac{\partial y}{\partial x}(0,t) = 0, & t > 0, \\ \left(\frac{\partial^3 y}{\partial x^3} + I_m \frac{\partial^2 y}{\partial t^2} + \alpha y + \beta \frac{\partial y}{\partial t} + K \frac{\partial^4 y}{\partial x^3 \partial t}\right)(0,t) = 0, & t > 0, \\ M \frac{\partial^2 y}{\partial t^2}(1,t) - \frac{\partial^3 y}{\partial x^3}(1,t) = 0, & t > 0, \\ J \frac{\partial^3 y}{\partial x \partial t^2}(1,t) + \frac{\partial^2 y}{\partial x^2}(1,t) = 0, & t > 0, \end{cases} \tag{1}$$

where $\alpha > 0$, $\beta > 0$, and $K \neq 0$. When $I_m = 0$, the uniform stabilization of the hybrid system is obtained in [1] by means of a feedback law taking into account only the position and the velocity of the platform. i.e. $K = 0$. When $I_m \neq 0$, no result is available about the uniform stabilization of this hybrid system. It should be noted that, in engineering, it is usually difficult to directly measure $\frac{\partial^4 y}{\partial x^3 \partial t}(0,t)$, but measuring the strain signal $\frac{\partial^3 y}{\partial x^3}(0,t)$ by strain gauges can be easily obtained. Combining these measures with the actuator equations, such as those of an electrical motors with drivers of speed reference type, produces indirectly the signal $\frac{\partial^4 y}{\partial x^3 \partial t}(0,t)$. Details can be found in Section V of [8].

The rest of this paper is organized as follows. In section 2, the asymptotic expressions of eigenvalues and eigenfunctions are derived. In Section 3, we show that there is a sequence of generalized eigenfunctions of system (1), which forms a Riesz basis for the state Hilbert space and the exponential stability of the system is proved. Numerical simulation of the distribution of eigenvalues is presented in the first part of section 4 after relating the stability of the system to a finite dimensional eigenvalue problem and the effectiveness of the control is presented in the second part of section 4.

2. Abstract formulation and asymptotic behavior of the eigenpairs

We consider the system (1) on the following complex Hilbert space

$$H := \mathbb{V} \times L^2(0,1) \times \mathbb{C}^3, \text{ where } \mathbb{V} = \{\phi \in H^2(0,1) / \phi'(0) = 0\} \tag{2}$$

equipped with the norm

$$\|(u,v,a,b,c)\|^2 := \int_0^1 [|u''|^2 + |v|^2] dx + \alpha |u(0)|^2 + \frac{|a|^2}{I_m} + \frac{|b|^2}{M} + \frac{|c|^2}{J}, \tag{3}$$

and the state variable

$$Y(t) := (y(.,t), \frac{\partial y}{\partial t}(.,t), I_m \frac{\partial y}{\partial t}(0,t) + K \frac{\partial^3 y}{\partial x^3}(0,t), M \frac{\partial y}{\partial t}(1,t), J \frac{\partial^2 y}{\partial x \partial t}(1,t)). \tag{4}$$

Then the system (1) can be written as

$$\frac{\partial Y}{\partial t}(t) = AY(t), \tag{5}$$

where the associated system operator is

$$A(\phi, \psi, a, b, c) = (\psi, -\phi'''', -\phi'''(0) - \alpha\phi(0) - \beta\psi(0), \phi'''(1), -\phi''(1)) \tag{6}$$

$$D(A) = \left\{ (\phi, \psi, a, b, c) \in (H^4(0,1) \cap \mathbb{V}) \times \mathbb{V} \times \mathbb{C}^3 \;\middle|\; \begin{array}{l} a = I_m \psi(0) + K\phi'''(0), \\ b = M\psi(1), \\ c = J\psi'(1). \end{array} \right\}.$$

Lemma 2.1. *A^{-1} exists and is compact on H. Hence $\sigma(A)$, the spectrum of A, consists of isolated eigenvalues only.*

Proof. For any $(f, g, a, b, c) \in H$, solving

$$\begin{aligned}&A(\phi, \psi, I_m\psi(0) + K\phi'''(0), \zeta, \delta)\\ &= (\psi, -\phi^{(4)}, -\phi'''(0) - \alpha\phi(0) - \beta\psi(0), \phi'''(1), -\phi''(1)) = (f, g, a, b, c),\end{aligned}$$

produces the unique solution (notice that $\phi'(0) = 0$) $\psi = f \in \mathbb{V}$ and $\phi \in H^4(0,1) \cap \mathbb{V}$

$$\begin{aligned}\phi(x) =& \frac{1}{\alpha}[\frac{b\alpha - 6b}{6} - a - \int_0^1 g(t)dt - \beta f(0) + \alpha \int_0^1 \frac{t^3}{6} g(t)dt]\\ &- \frac{1}{2}x[b + \int_0^1 t^2 g(t)dt] - \frac{c}{2}x^2 + \frac{b}{6}(x-1)^3 - \frac{1}{6}\int_1^x (x-t)^3 g(t)dt.\end{aligned} \tag{7}$$

The result then follows from the Sobolev's embedding theorem [11] and the details are omitted. □

Lemma 2.2. *For any $\lambda = i\tau^2 \in \sigma(A)$, there is a unique eigenfunction (up to a scalar)*

$$(\phi, \lambda\phi, \lambda I_m\phi(0) + K\phi'''(0), M\lambda\phi(1), J\lambda\phi'(1)) \tag{8}$$

where

$$\begin{aligned}\phi(x) =& -(1 + MJ\tau^4)\cosh\,\tau x + [2J\tau^3 \sin\,\tau + (-1 + MJ\tau^4)\cos\,\tau]\cosh\,\tau(1-x)\\ &+ [2J\tau^3 \sinh\,\tau - (1 + MJ\tau^4)\cos\,\tau + (-1 + MJ\tau^4)\cosh\,\tau]\cos\,\tau(1-x)\\ &+ [(-1 + MJ\tau^4)\sin\,\tau - 2M\tau\cos\,\tau]\sinh\,\tau(1-x)\\ &+ [(1 - MJ\tau^4)\sinh\,\tau - (1 + MJ\tau^4)\sin\,\tau + 2M\tau\cosh\,\tau]\sin\,\tau(1-x)\end{aligned} \tag{9}$$

Proof. Solving the eigenvalue problem

$$A(\phi, \psi, a, b, c) = \lambda(\phi, \psi, a, b, c), \text{ where } (\phi, \psi, a, b, c) \in D(A),$$

one has $a = I_m\psi(0) + K\phi'''(0)$, $b = M\psi(1)$, $c = J\psi'(1)$, $\psi = \lambda\phi$, and

$$\begin{aligned}&\phi'''' + \lambda^2\phi = 0,\\ &(1 + \lambda K)\phi'''(0) + (I_m\lambda^2 + \alpha + \beta\lambda)\phi(0) = 0,\\ &\phi'(0) = 0,\\ &\phi'''(1) - M\lambda^2\phi(1) = 0,\\ &\phi''(1) + J\lambda^2\phi'(1) = 0.\end{aligned} \tag{10}$$

Let $f(x) = \phi(1-x)$. Then f satisfies

$$\begin{aligned}&f'''' + \lambda^2 f = 0,\\ &f'(1) = -(1 + \lambda K)f'''(1) + (I_m\lambda^2 + \alpha + \beta\lambda)f(1) = 0,\\ &f'''(0) + M\lambda^2 f(0) = 0,\\ &f''(0) - J\lambda^2 f'(0) = 0.\end{aligned} \tag{11}$$

Let $\lambda = i\tau^2$, it is easily seen that for any $\lambda = i\tau^2$, the general solution of the following equation

$$f'''' + \lambda^2 f = 0,$$
$$f'''(0) + M\lambda^2 f(0) = 0,$$
$$f''(0) - J\lambda^2 f'(0) = 0,$$

is of the form

$$\begin{aligned} f(x) =&[(d_1 - d_2) - MJ\tau^4(d_1 + d_2)] \cosh\ \tau x + [(d_1 - d_2) + MJ\tau^4(d_1 + d_2)] \cos\ \tau x \\ &+ 2M\tau[d_1 \sinh\ \tau x + d_2 \sin\ \tau x], \end{aligned}$$

where d_1 and d_2 are arbitrary constants. By $f'(1) = 0$, one has (up to a scalar)

$$\begin{aligned} d_1 &= (1 + MJ\tau^4) \sinh\ \tau + (-1 + MJ\tau^4) \sin\ \tau - 2M\tau \cos\ \tau, \\ d_2 &= (1 - MJ\tau^4) \sinh\ \tau - (1 + MJ\tau^4) \sin\ \tau + 2M\tau \cosh\ \tau, \\ d_1 - d_2 &= 2MJ\tau^4 \sinh\ \tau + 2MJ\tau^4 \sin\ \tau - 2M\tau \cos\ \tau - 2M\tau \cosh\ \tau, \\ d_1 + d_2 &= 2 \sinh\ \tau - 2 \sin\ \tau - 2M\tau \cos\ \tau + 2M\tau \cosh\ \tau. \end{aligned}$$

Hence (again up to a scalar)

$$\begin{aligned} f(x) =& - (1 + MJ\tau^4) \cosh\ \tau(1 - x) + [2J\tau^3 \sin\ \tau + (-1 + MJ\tau^4) \cos\ \tau] \cosh\ \tau x \\ &+ [2J\tau^3 \sinh\ \tau - (1 + MJ\tau^4) \cos\ \tau + (-1 + MJ\tau^4) \cosh\ \tau] \cos\ \tau x \\ &+ [(-1 + MJ\tau^4) \sin\ \tau - 2M\tau \cos\ \tau] \sinh\ \tau x \\ &+ [(1 - MJ\tau^4) \sinh\ \tau - (1 + MJ\tau^4) \sin\ \tau + 2M\tau \cosh\ \tau] \sin\ \tau x. \end{aligned}$$

□

Lemma 2.3. *The characteristic equation that λ satisfies is*

$$\begin{aligned} &\tau^3(1 + iK\tau^2)[-2M\tau \cosh\tau \cos\tau + (-1 + MJ\tau^4) \cos\tau \sinh\tau \\ &+ (-1 + MJ\tau^4) \cosh\tau \sin\tau + 2J\tau^3 \sinh\tau \sin\tau] \\ &+ (-I_m\tau^4 + i\beta\tau^2 + \alpha)[(1 + MJ\tau^4) - (M\tau + J\tau^3) \sin\tau \cosh\tau \\ &+ (1 - MJ\tau^4) \cos\tau \cosh\tau + (M\tau - J\tau^3) \cos\tau \sinh\tau] = 0. \end{aligned} \tag{12}$$

Proof. For any $\lambda = i\tau^2$. In order f to be a solution of (11), it is necessary and sufficient that $-(1 + \lambda K)f'''(1) + (I_m\lambda^2 + \alpha + \beta\lambda)f(1) = 0$ which induces (12), proving the lemma. □

Lemma 2.4. *There is a family of eigenvalues $\{\lambda_n = i\tau_n^2, -i\tau_n^2\}$ of A with the following asymptotic expression*

$$\lambda_n = i\tau_n^2 = -\frac{I_m}{K} + i(\frac{2}{M} + (s\pi)^2) + O(\frac{1}{n}), \tag{13}$$

where $s = n - \frac{1}{4}$, n is a sufficiently large positive integer. A corresponding eigenfunction is of the form

$$\begin{aligned} \Phi_n &= (\phi_n, \lambda_n\phi_n, \lambda_n I_m\phi_n(0) + K\phi_n'''(0), M\lambda_n\phi_n(1), J\lambda_n\phi_n'(1)), \\ &= (\phi_n, \lambda_n\phi_n, -\lambda_n^{-1}((\alpha + \lambda_n\beta)\phi_n(0) + \phi_n'''(0)), \lambda_n^{-1}\phi_n'''(1), \lambda_n^{-1}\phi_n''(1)), \end{aligned} \tag{14}$$

where

$$\begin{aligned}\phi_n(x) =& -(1+MJ\tau_n^4)\cosh\ \tau_n x + [2J\tau_n^3 \sin\ \tau_n + (-1+MJ\tau_n^4)\cos\ \tau_n]\cosh\ \tau_n(1-x)\\ &+ [2J\tau_n^3 \sinh\ \tau_n - (1+MJ\tau_n^4)\cos\ \tau_n + (-1+MJ\tau_n^4)\cosh\ \tau_n]\cos\ \tau_n(1-x)\\ &+ [(-1+MJ\tau_n^4)\sin\ \tau_n - 2M\tau_n \cos\ \tau_n]\sinh\ \tau_n(1-x)\\ &+ [(1-MJ\tau_n^4)\sinh\ \tau_n - (1+MJ\tau_n^4)\sin\ \tau_n + 2M\tau_n\cosh\ \tau_n] \times \sin\ \tau_n(1-x).\end{aligned} \tag{15}$$

Proof. Note that for a large positive integer n, in a uniformly bounded small neighborhood of $s\pi = (n - \frac{1}{4})\pi$

$$|\sin\tau| \le C,\ |\cos\tau| \le C,\ |e^{-\tau}\sinh\tau| \le C,\ |e^{-\tau}\cosh\tau| \le C,$$

uniformly for all n with some constant C. By multiplying $-\frac{e^{-\tau}\tau^{-9}}{iKMJ}$ on both sides of (12), we can write in a uniformly bounded small neighborhood of $s\pi = (n-\frac{1}{4})\pi$ for each n to be

$$\left.\begin{aligned}&\sin\tau + \cos\tau = O(\frac{1}{|\tau|})\ \ (\sin\ 2\tau = -1 + O(\frac{1}{|\tau^2|})),\\ \text{or}&\\ &\sin\tau + \cos\tau = \frac{1}{\tau}(\frac{2}{M} + \frac{iI_m}{K})\cos\tau + O(\frac{1}{|\tau|^2}).\end{aligned}\right\} \tag{16}$$

Applying Rouche's Theorem [6] in a small neighborhood of $s\pi = (n-\frac{1}{4})\pi$ where n is a large positive integer, we obtain a solution τ_n which is of the form

$$\tau = \tau_n = s\pi + O(\frac{1}{n}), \tag{17}$$

for sufficiently large n. Substituting (17) into the second equation of (16), yields

$$2O(\frac{1}{n}) = \frac{1}{s\pi}(\frac{2}{M} + \frac{iI_m}{K}) + O(\frac{1}{n^2}),$$

and so

$$\tau_n = s\pi + \frac{1}{s\pi}(\frac{1}{M} + \frac{iI_m}{2K}) + O(\frac{1}{n^2}).$$

□

3. Riesz basis and exponential stability

3.1. Preliminaries. We consider the following second-order differential equation system with one-spatial variable in the general form [5]:

$$\begin{cases}\frac{d^2y}{dt^2} + L(y) = 0, 0 < x < 1,\ t > 0\\ U_j(y) = B_{1,j}(y) = 0,\ j = 1, 2, ..., n_1 - 1,\\ U_j(y) = B_{1,j}(y),\ j = n_1, ...n_2 - 1,\\ U_j(y, y_t) = B_{1,j}(y) + B_{2,j}(y_t) = 0,\ j = n_2, n_2 + 1, ..., n_3 - 1,\\ U_j(y, y_t) = B_{1,j}(y) + B_{2,j}(y_t) = 0,\ j = n_3, n_2 + 1, ..., n_4 - 1,\\ U_j(y, y_t, y_{tt}) = B_{1,j}(y) + B_{2,j}(y_t) + B_{3,j}(y_{tt}) = 0,\ j = n_4, n_4 + 1, ..., n,\end{cases} \tag{18}$$

where

- $L(y)$ is an ordinary differential operator of order $n = 2m \in \mathbb{N}$,

$$L(y)(x,t) = (-1)^m \frac{\partial^n y(x,t)}{\partial x^n} + \sum_{s=2}^{n} f_s(x) \frac{\partial^{(n-s)} y(x,t)}{\partial x^{(n-s)}}, \tag{19}$$

- $B_{1,j}(y)$, $B_{2,j}(y_t)$ and $B_{3,j}(y_{tt})$ are linear forms of differentiations of their variables in x at most of order $n-1$ evaluated at the two point $x = 0$ or $x = 1$. That is

$$B_{1,j}(y) = \sum_{s=0}^{k_{1j}} (\alpha_{1js} \frac{\partial^{(k_{1j}-s)} y(x,t)}{\partial x^{(k_{1j}-s)}}|_{x=0} + \beta_{1js} \frac{\partial^{(k_{1j}-s)} y(x,t)}{\partial x^{(k_{1j}-s)}}|_{x=1}), \tag{20}$$

$$B_{2,j}(y_t) = \sum_{s=0}^{k_{2j}} (\alpha_{2js} \frac{\partial^{(k_{2j}-s+1)} y(x,t)}{\partial x^{(k_{2j}-s)} \partial t}|_{x=0} + \beta_{2js} \frac{\partial^{(k_{2j}-s+1)} y(x,t)}{\partial x^{(k_{2j}-s)} \partial t}|_{x=1}), \tag{21}$$

and

$$B_{3,j}(y_{tt}) = \sum_{s=0}^{k_{3j}} (\alpha_{3js} \frac{\partial^{(k_{3j}-s+2)} y(x,t)}{\partial x^{(k_{3j}-s)} \partial t^2}|_{x=0} + \beta_{3js} \frac{\partial^{(k_{3j}-s+2)} y(x,t)}{\partial x^{(k_{3j}-s)} \partial t^2}|_{x=1}), \tag{22}$$

with $k_{ij} \in \{0, 1, ...n-1\}$ and α_{ljs}, $\beta_{ljs} \in \mathbb{C}$. The order of the boundary conditions of (18) is defined as

$$\gamma = \tilde{k_1} + \tilde{k_2} + ... + \tilde{k_n}, \tag{23}$$

with

$$\tilde{k_j} = \begin{cases} k_{1j},\ j = 1, 2, ..., n_2 - 1, \\ \max\{k_{1j}, m + k_{2j}\},\ j = n_2, n_2 + 1, ..., n_4 - 1, \\ \max\{m + k_{2j}, n + k_{3j}\},\ j = n_4, n_4 + 1, ..., n. \end{cases}$$

We assume that the following conditions are satisfied:

- (H_1) the coefficient functions f_s, $(2 \le s \le n)$ in (19) are sufficiently smooth, say $\mathbb{C}^{n-s}$, in x.
- (H_2) $|\alpha_{lj0}| + |\beta_{lj0}| > 0$ for $l = 1,\ 1 \le j \le n,\ l = 2,\ n_2 \le j \le n,\ l = 3,\ n_4 \le j \le n$.
- (H_3) $\max_{1 \le j \le n_1 - 1}\{k_{1j}\} < m$, $\min_{n_1 \le j \le n_2 - 1}\{k_{1j}\} \ge m$.
- (H_4) $\max_{n_2 \le j \le n_3 - 1}\{k_{2j}\} < m$, $m \le \min_{n_3 \le j \le n_4 - 1}\{k_{2j}\}$.
- (H_5) $\max_{n_4 \le j \le n}\{k_{3j}\} < m$.
- (H_6) the boundary conditions are already normalized in the sense that for any equivalent boundary condition $\{\tilde{U}\}_{j=1}^{n}$ of order $\tilde{\gamma}$, it always holds that $\gamma \le \tilde{\gamma}$.

Let $H_E^m(0,1) = \{f(x) \in H^m(0,1)/B_{1j}(f) = 0,\ j = 1, 2, ..., n_1 - 1\}$ and define a Hilbert space $\mathcal{H}$ by $\mathcal{H} = H_E^m(0,1) \times L^2(0,1) \times \mathbb{C}^{n_0}$, where $n_0 = n - n_3 + 1$ with the norm

$$\|(y, z, \eta_1, ..., \eta_{n_0})\|_{\mathcal{H}}^2 := \|y\|_{H_E^m}^2 + \|z\|_{L^2} + \textstyle\sum_{j=1}^{n_0} \gamma_j |\eta_j|^2, \tag{24}$$

where γ_j, $j = 1, 2, ..., n_0$ are positive constants.
Define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}(f, g, \eta_1, ..., \eta_{n_0}) = (g, -L(f), \tilde{\eta}_1, ..., \tilde{\eta}_{n_0}) \tag{25}$$

$$D(\mathcal{A}) = \left\{ (f, g, \eta_1, ..., \eta_{n_0}) \in \mathcal{H} \;\middle|\; \begin{array}{l} f \in H^n(0,1), g \in H_E^m(0,1), \\ B_{1,j}(f) = 0, \text{ for } n_1 \leq j \leq n_2 - 1, \\ B_{1,j}(f) + B_{2,j}(g) = 0, \text{ for } n_2 \leq j \leq n_3 - 1, \\ \eta_j = B_{2,j+n_3-1}(f), \text{ for } n_2 \leq j \leq n_4 - n_3, \\ n_j = B_{2,j+n_3-1}(f) + B_{3,j+n_3-1}(g), \\ \qquad\qquad\qquad \text{for } n_4 - n_3 + 1 \leq j \leq n_0. \end{array} \right\}.$$

where $\tilde{\eta}_j = -\beta_{1,j+n_3-1}(f)$ for $j = 1, 2, ...n_0$.

The following result can be obtained from [5]:

Theorem 3.1. *If the ordinary differential system with parameter* $\lambda = \rho^m$

$$\begin{cases} \mathcal{L}(f, \lambda) = L(f) + \lambda^2 f = 0, \\ U_j(f) = B_{1,j}(f) = 0, \; j = 1, 2, ..., n_2 - 1, \\ U_j(f, \lambda f) = B_{1,j}(f) + \lambda B_{2,j}(f) = 0, \; j = n_2, n_2 + 1, ..., n_4 - 1, \\ U_j(f, \lambda f, \lambda^2 f) = B_{1,j}(f) + \lambda B_{2,j}(f) + \lambda^2 B_{3,j}(f) = 0, \; j = n_4, n_4 + 1, ..., n, \end{cases} \tag{26}$$

has strongly regular boundary conditions, then the system of generalized eigenfunctions of $\mathcal{A}$ *forms a Riesz basis in the Hilbert space* $\mathcal{H}$.

3.2. Riesz basis property of the eigenfunctions of A**.** To further solve the eigenvalue problem (10), we follow the procedure in Birkhoff [2] and Naimark [10] and divide the complex plane into eight distinct sectors,

$$S_k = \left\{ \rho \in \mathbb{C} / \frac{k\pi}{4} \leq \arg \rho \leq \frac{(k+1)\pi}{2} \right\}, \; k = 0, 1, ..., 7, \tag{27}$$

and let ω_1, ω_2, ω_3, and ω_4 be the roots of equations $\theta^4 + 1 = 0$ that are arranged so that

$$\Re(\rho\omega_1) \leq \Re(\rho\omega_2) \leq \Re(\rho\omega_3) \leq \Re(\rho\omega_4), \; \forall \rho \in S_k. \tag{28}$$

Setting $\lambda = \rho^2$, in each sector S_k, we have the following result about the fundamental solutions of the system (10):

Lemma 3.2. [10] *For* $\rho \in S_k$ *with* $|\rho|$ *large enough, the equation*

$$\phi^{(4)}(x) + \rho^4 \phi(x) = 0, \tag{29}$$

has four linearly independent asymptotic fundamental solutions ϕ_i, $i = 1, 2, 3, 4$ *such that*

$$\phi_i(x, \rho) = e^{\rho\omega_i x} \left(1 + O(\rho^{-1})\right), \tag{30}$$

and hence their derivatives for $i = 1, 2, 3, 4$ *and* $j = 1, 2, 3$ *are given by*

$$\frac{d^j}{dx^j} \phi_i(x, \rho) = (\rho\omega_i)^j e^{\rho\omega_i x} \left(1 + O(\rho^{-1})\right). \tag{31}$$

Substituting (30) and (31) into the boundary conditions of (10), we obtain asymptotic expressions for the boundary conditions for large enough $|\rho|$:

$$\begin{aligned} V_4(\phi_i, \rho) &= \rho\omega_i + O(\rho^{-1}) = \rho\omega_i(1 + O(\rho^{-2})), \\ V_3(\phi_i, \rho) &= (1 + \rho^2 K)(\rho\omega_i)^3 + (I_m \rho^4 + \alpha + \beta\rho^2) + O(\rho^{-1}), \\ &= \rho^5((K\omega_i^3 + \frac{I_m}{\rho}) + O(\rho^{-2})), \end{aligned}$$

$$V_2(\phi_i,\rho) = (\rho\omega_i)^3 e^{\rho\omega_i} - M\rho^4 e^{\rho\omega_i} + O(\rho^{-1}) = \rho^4 (e^{\rho\omega_i}(-M + \frac{\omega_i^3}{\rho}) + O(\rho^{-2})),$$
$$V_1(\phi_i,\rho) = (\rho\omega_i)^2 e^{\rho\omega_i} + J\rho^5 \omega_i e^{\rho\omega_i} + O(\rho^{-1}) = \rho^5 (\omega_i J e^{\rho\omega_i} + O(\rho^{-2})). \tag{32}$$

Theorem 3.3. *Let $\lambda = \rho^2$. For $\rho \in S_1 = \{\rho \in \mathbb{C}/\frac{\pi}{4} \leq \arg\rho \leq \frac{\pi}{2}\}$, the characteristic determinant $\Delta(\rho)$ of system (10) has an asymptotic expansion of the following form*

$$\Delta(\rho) = KMJi\rho^{15}e^{\rho\omega_4} \times \left\{ e^{\rho\omega_2}\left[-2\sqrt{2}i \quad -\frac{2i}{\rho}(\frac{2}{M} + \frac{I_m}{K})\right] + e^{\rho\omega_3}\left[2\sqrt{2} + \frac{2i}{\rho}(\frac{2}{M} - \frac{I_m}{K})\right]\right\} + O(\rho^{-2}).$$

Proof. Set $\lambda = \rho^2$ Now, let us choose ω_i $(i = 1, 2, 3, 4)$ in S_1, to check the regularity of the characteristic determinant $\Delta(\rho)$ as follows

$$\omega_1 = e^{\frac{3\pi i}{4}},\ \omega_2 = e^{\frac{\pi i}{4}},\ \omega_3 = -\omega_2,\ \omega_4 = -\omega_1, \tag{33}$$

consequently, we have for $\rho \in S_1$

$$\begin{cases} \Re(\rho\omega_1) \leq \Re(\rho\omega_2) \leq \Re(\rho\omega_3) \leq \Re(\rho\omega_4), \\ \Re(\rho\omega_1) = -\mid\rho\mid \sin(\arg\rho + \frac{\pi}{4}) \leq -\frac{\sqrt{2}\mid\rho\mid}{2} < 0, \\ \Re(\rho\omega_2) = \mid\rho\mid\cos(\arg\rho + \frac{\pi}{4}) \leq 0. \end{cases} \tag{34}$$

Note that $\lambda \neq 0$ is the eigenvalue of (10) if and only if the characteristic determinant

$$\Delta(\rho) = \begin{vmatrix} V_4(\phi_1,\rho) & V_4(\phi_2,\rho) & V_4(\phi_3,\rho) & V_4(\phi_4,\rho) \\ V_3(\phi_1,\rho) & V_3(\phi_2,\rho) & V_3(\phi_3,\rho) & V_3(\phi_4,\rho) \\ V_2(\phi_1,\rho) & V_2(\phi_2,\rho) & V_2(\phi_3,\rho) & V_2(\phi_4,\rho) \\ V_1(\phi_1,\rho) & V_1(\phi_2,\rho) & V_1(\phi_3,\rho) & V_1(\phi_4,\rho) \end{vmatrix} = 0. \tag{35}$$

So substituting (32) into (35), we get

$$\Delta(\rho) = \begin{vmatrix} \omega_1\rho & \omega_2\rho & \omega_3\rho & \omega_4\rho \\ \rho^5(K\omega_1^3 + \frac{I_m}{\rho}) & \rho^5(K\omega_2^3 + \frac{I_m}{\rho}) & \rho^5(K\omega_3^3 + \frac{I_m}{\rho}) & \rho^5(K\omega_4^3 + \frac{I_m}{\rho}) \\ \rho^4 e^{\rho\omega_1}(-M + \frac{\omega_1^3}{\rho}) & \rho^4 e^{\rho\omega_2}(-M + \frac{\omega_2^3}{\rho}) & \rho^4 e^{\rho\omega_3}(-M + \frac{\omega_3^3}{\rho}) & \rho^4 e^{\rho\omega_4}(-M + \frac{\omega_4^3}{\rho}) \\ \omega_1\rho^5 e^{\rho\omega_1}J & \omega_2\rho^5 e^{\rho\omega_2}J & \omega_3\rho^5 e^{\rho\omega_3}J & \omega_4\rho^5 e^{\rho\omega_4}J \end{vmatrix} + O\left(\rho^{-2}\right). \tag{36}$$

From (34), we obtain

$$\Delta(\rho) = KMJ\rho^{15}e^{\rho\omega_4} \times \begin{vmatrix} \omega_1 & \omega_2 & \omega_3 & 0 \\ \omega_1^3 + \frac{I_m}{K\rho} & \omega_2^3 + \frac{I_m}{K\rho} & \omega_3^3 + \frac{I_m}{K\rho} & 0 \\ 0 & e^{\rho\omega_2}(-1 + \frac{\omega_2^3}{M\rho}) & e^{\rho\omega_3}(-1 + \frac{\omega_3^3}{M\rho}) & -1 + \frac{\omega_4^3}{M\rho} \\ 0 & \omega_2 e^{\rho\omega_2} & \omega_3 e^{\rho\omega_3} & \omega_4 \end{vmatrix} + O\left(\rho^{-2}\right). \tag{37}$$

Since

$$\omega_1\omega_2^{-1} = i,\ \omega_2^2 = i,\ \omega_1^3\omega_2^{-1} = 1, \omega_1 + \omega_2 = \sqrt{2}i,$$
$$\omega_1 - \omega_2 = -\sqrt{2}, \omega_1^2 = -i,\ \omega_1\omega_2 = -1,$$

then

$$\Delta(\rho) = KMJ\omega_2^2\rho^{15}e^{\rho\omega_4}$$
$$\times \begin{vmatrix} i & 1 & -1 & 0 \\ 1 + \frac{I_m}{K\rho\omega_2} & i + \frac{I_m}{K\rho\omega_2} & -i + \frac{I_m}{K\rho\omega_2}) & 0 \\ 0 & e^{\rho\omega_2}(-1 + \frac{\omega_2^3}{M\rho}) & -e^{\rho\omega_3}(1 + \frac{\omega_2^3}{M\rho}) & -(1 + \frac{\omega_1^3}{M\rho}) \\ 0 & \omega_2 e^{\rho\omega_2} & -\omega_2 e^{\rho\omega_3} & -\omega_1 \end{vmatrix} + O\left(\rho^{-2}\right).$$

Expanding the above determinant, we obtain

$$\begin{aligned}\Delta(\rho) =& KMJ\omega_2^2\rho^{15}e^{\rho\omega_4}\Big\{2\Big[(\omega_2 - \omega_1)e^{\rho\omega_3} - (\omega_1 + \omega_2)e^{\rho\omega_2}\Big] \\ &+ \frac{1}{\rho}\Big[- \frac{2}{M}(\omega_1\omega_2^3 - \omega_2\omega_1^3) + \frac{I_m}{K\omega_2}(\omega_1 - \omega_2)(i-1)\Big]e^{\rho\omega_3} \\ &+ \frac{1}{\rho}\Big[- \frac{2}{M}(\omega_2\omega_1^3 - \omega_1\omega_2^3) - \frac{I_m}{K\omega_2}(\omega_1 + \omega_2)(i+1)\Big]e^{\rho\omega_2}\Big\} + O(\rho^{-2}) \\ =& KMJi\rho^{15}e^{\rho\omega_4}\Big\{e^{\rho\omega_2}\Big[- 2\sqrt{2}i - \frac{2i}{\rho}(\frac{2}{M} + \frac{I_m}{K})\Big] \\ &+ e^{\rho\omega_3}\Big[2\sqrt{2} + \frac{2i}{\rho}(\frac{2}{M} - \frac{I_m}{K})\Big]\Big\} + O(\rho^{-2}).\end{aligned}$$

□

The characteristic determinant $\Delta(\rho)$ can be written as follows

$$\Delta(\rho) = KMJi\rho^{15}e^{\rho\omega_4}([\theta_{-1}(\rho)]_1 e^{-\rho\omega_2} + [\theta_0(\rho)]_1 + [\theta_1(\rho)]_1 e^{\rho\omega_2}) + O(\rho^{-2}), \tag{38}$$

where $[\theta_j(\rho)]_1 = \theta_{j0} + O(\rho^{-1}) = \begin{cases} 2\sqrt{2} + O(\rho^{-1}) \text{ if } j = -1, \\ -2\sqrt{2}i + O(\rho^{-1}) \text{ if } j = 1, \\ O(\rho^{-1}) \text{ if } j = 0. \end{cases}$

Definition 3.1. [5] The boundary-value problem (26) is said to be regular if

$$|\theta_{-1,0}| \neq 0 \text{ and } |\theta_{1,0}| \neq 0. \tag{39}$$

It is said to be strongly regular if the zeros $\{\rho_i\}$ of $\Delta(\rho) = 0$ are simple and separable in the sense that $\inf_{i\neq j} |\rho_i - \rho_j| > 0$ for all sufficiently large $|\rho_i|$, or equivalently

$$\theta_{00}^2 - 4\theta_{1,0}\theta_{-1,0} \neq 0. \tag{40}$$

In light of definition 3.1, the boundary-value problem (10) is strongly regular. In order to apply theorem 2.9 in [5], it suffices that conditions (H_1) to (H_6) be satisfied. Therefore, theorem 2.9 in [5] can be directly applied to get the principal result of the paper:

Theorem 3.4. *Let A defined as in (6). For any $\alpha > 0$ and real parameters $\beta \neq 0$, and $K \neq 0$, then*

(i) The generalized eigenfunctions of A form a Riesz basis for H.

(ii) Suppose there exists a constant q such that

$$\sum_q = \{\lambda \in \mathbb{C}/\Re(\lambda) > q\} \subset \rho(A). \tag{41}$$

Then A generates a C_0-semigroup e^{At} on H.

(iii) The spectrum-determined growth condition holds true; that is, $S(A) = \omega(A)$, where

$$S(A) = \sup_{\lambda\in\sigma(A)} \Re(\lambda), \tag{42}$$

is the spectral bound, and

$$\omega(A) = \inf\left\{\omega/\exists c > 0 \text{ such that } \|e^{At}\| \le c\, e^{\omega t}\right\}, \tag{43}$$

is the growth order of e^{At}.

3.3. Exponential stability.

Theorem 3.5. *Suppose $\beta \ge \alpha K > 0$. Then there exists an $\omega > 0$ such that $\Re(\lambda) < -\omega$ for all $\lambda \in \sigma(A)$. Therefore the C_0-semigroup e^{At} generated by A is exponentially stable:*

$$\|e^{At}\Phi\| \le c\, e^{-\omega t}\|\Phi\|^2, \tag{44}$$

where $c > 0$ is a constant independent of Φ.

Proof. It is seen from (13) that if $K < 0$, then system (1) is never exponentially stable. Since the spectrum-determined growth condition holds, it follows from (ii) of Theorem 3.4 that e^{At} is exponentially stable under condition $\beta \ge \alpha K > 0$ if and only if $\Re(\lambda) < -\omega$ for all $\lambda \in \sigma(A)$.

Clearly, if λ is a real number, it must have $\lambda < 0$. Notice that $\lambda = 0$ is always not in the spectrum of A. Suppose that $\lambda = \lambda_1 + i\lambda_2\,(\lambda_2 \ne 0)$. Multiplying ϕ on both sides of the first equation in (10), integrating by parts from 0 to 1 with respect to x and taking the imaginary parts, yields

$$2\lambda_1[\textstyle\int_0^1 |\phi(x)|^2dx + M|\phi(1)|^2 + J|\phi'(1)|^2] + \frac{2I_m\lambda_1 + I_mK|\lambda|^2+\beta-\alpha K}{|1+\lambda K|^2}|\phi(0)|^2 = 0.$$

There are two cases. When $\lambda_1 \ne 0$ it is obvious that $\lambda_1 < 0$ as $\beta \ge \alpha K > 0$. While as $\lambda_1 = 0$, it must be $\phi(0) = 0$ and so $\phi'''(0) = 0$ from the boundary condition of (10). In this case, the solution of (10) shall be (we may assume that $\lambda_2 > 0$) $\phi(x) = \cosh\sqrt{\lambda_2}x - \cos\sqrt{\lambda_2}x$. But from the boundary condition $\phi'''(1) = M\lambda_2\phi(1)$, we arrive the contradiction that

$$\sinh\sqrt{\lambda_2} - \sin\sqrt{\lambda_2} = -\lambda_2^2\, M(\cosh\sqrt{\lambda_2} - \cos\sqrt{\lambda_2}). \tag{45}$$

$\square$

4. Numerical simulations

4.1. Simulation part I: Spectrum of the closed-loop system. Our approximation process starts from the eigenvalue problem (10). Set

$$\Phi(x) = f\left(\frac{x+1}{2}\right). \tag{46}$$

Then Φ satisfies

$$\begin{aligned} \Phi^{(4)}(x) + \frac{\lambda^2}{16}\Phi(x) = 0,\ -1 < x < 1, \\ \Phi'(-1) = 0, \\ \Phi'''(-1)(1+K\lambda) + \frac{1}{8}(I_m\lambda^2 + \lambda\beta + \alpha)\Phi(-1) = 0, \\ \Phi'''(1) - \frac{1}{8}\lambda^2 M\Phi(1) = 0, \\ \Phi''(1) + \frac{1}{2}\lambda^2 J\Phi'(1) = 0. \end{aligned} \tag{47}$$

Let $P_n(x)$ be the Legendre polynomial of degree n, satisfying

$$\frac{d}{dx}\left((1-x^2)\frac{dP_n(x)}{dx}\right) + n(n+1)P_n(x) = 0,\quad P_n(1) = 1. \tag{48}$$

We approximate $\Phi(x)$ by

$$\Phi_N(x) = \sum_{n=1}^{N} a_n P_n(x). \tag{49}$$

For more details on the procedure we refer to [3]. Here we take $N = 100$, $I_m = 3$, $M = J = \alpha = 1$. Using this method, the total of 101 eigenvalues on the up half complex plane is easily calculated by MATLAB. As it is indicated in [3], computing eigenvalues of boundary value problems with any discretization method, only those numerical values of small magnitude have significant accuracy, with that we can be sure of the accuracy of the first 50 eigenvalues on the up half complex plane, although for our system, the same can be said for large magnitude eigenvalues. We denote $L_{asp} = -\frac{I_m}{K}$, the asymptote of the eigenvalues claimed in (13).

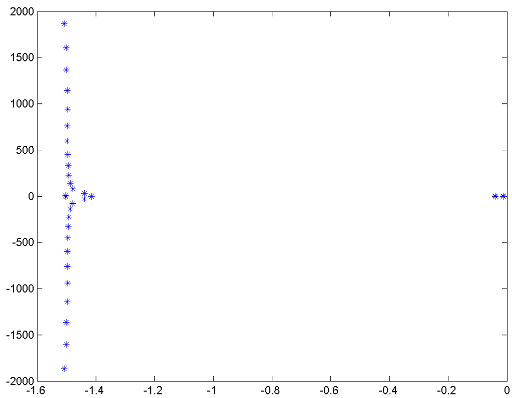

FIGURE 1. Distribution of eigenvalues $I_m = 3,\ \alpha = \beta = m = J = 1,\ K = 2$.

The figures 1 and 2 plot the distribution of the eigenvalues obtained on the complex plane for different values of K. In figure 1, $L_{asp} = -1.5$ and in figure 2, $L_{asp} = -3$.

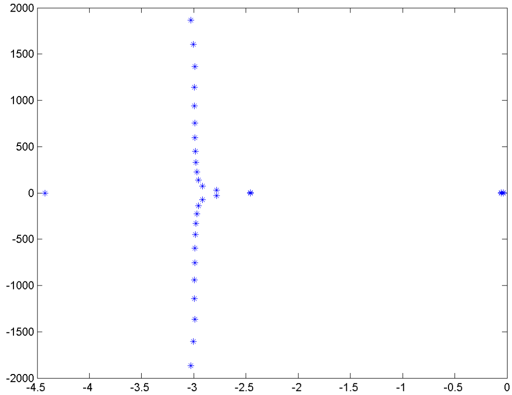

FIGURE 2. Distribution of eigenvalues $I_m = 3,\ \alpha = \beta = m = K = J = 1$.

It is apparent that the numerical results and the theoretical estimate (13) relatively coincident.

Figure 3 plots the functional of $S(L)$ with respect to β, where we have $0.01 \leq \beta \leq 10$, while Figure 4 demonstrates the same functional of $S(L)$ with respect to K, $0.01 \leq K \leq 10$. Both cases suggest that the optimal value β^* for $K = 1$ and K^* for $\beta = 1$ do exist, moreover an interesting fact which can be observed from these two figures is that the assumption $\beta \geq \alpha K > 0$ may be necessary since from figures 3 and 4, $S(L)$ can be positive outside region $\beta \geq \alpha K > 0$.

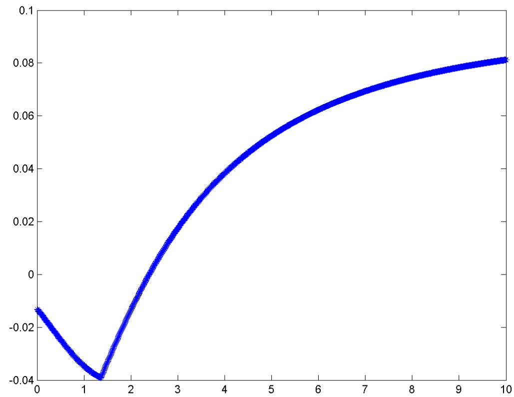

FIGURE 3. Functional relation between $S(L)$ and K.

4.2. Simulation part II: Dynamical behavior of the closed-loop system. Simulations for the hybrid system (1) with the parameters listed in Table 1 are used

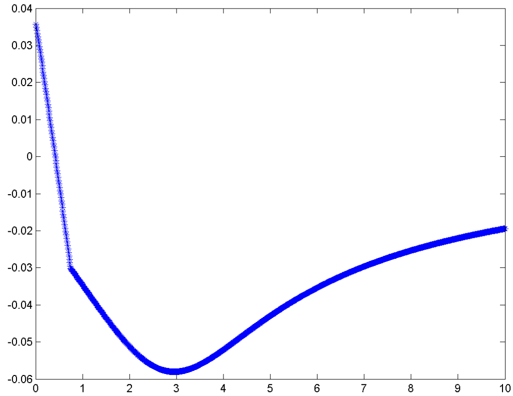

FIGURE 4. Functional relation between $S(L)$ and β.

TABLE 1. Parameters of the hybrid system

Parameter	Description	Value
M	Mass of the tip body	$0.1kg$
J	Moment of inertia of the tip body	$130kg/m$
I_m	Mass of the platform	$20.0kg/m$

to demonstrate the effectiveness of the boundary feedback control. We solve the system (1) using the finite difference method in time and space for the space-time domain $[0,1]\times[0,15]$. In order to ensure the numerical stability of the finite difference numerical scheme, we subdivided the spatial interval into 30 subintervals and the temporal interval into 200000 subintervals.

The initial conditions is selected as

$$y(x,0) = \frac{\sin(\frac{\pi x}{2})}{50}, \text{ and } y_t(x,0) = 0, \quad \forall x \in [0,1]. \tag{50}$$

Figure 5 shows the displacement of the Euler-Bernoulli beam without control input (i.e. $\alpha = 0$, $\beta = 0$, and $K = 0$). It is clear that the system is unstable and the vibration of the beam is quite large. Displacement of the Euler-Bernoulli beam with the boundary feedback control is shown in Figure 6. It can be seen that the vibrations of the Euler-Bernoulli beam can be suppressed greatly within $12s$, by choosing $\alpha = 14.4$, $\beta = 12$, and $K = 2$, which illustrate that the proposed boundary feedback control is able to stabilize the Euler-Bernoulli beam at small neighborhood of its equilibrium position.

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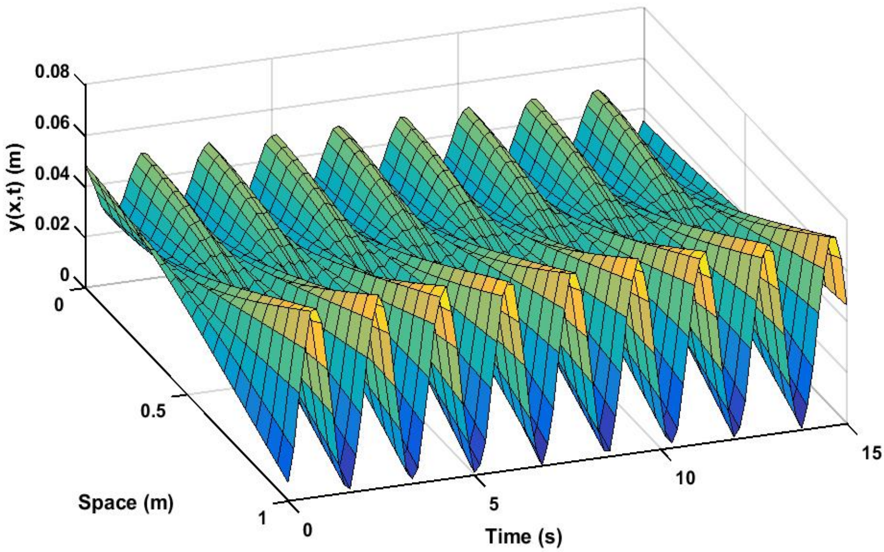

FIGURE 5. Displacement of the beam without control.

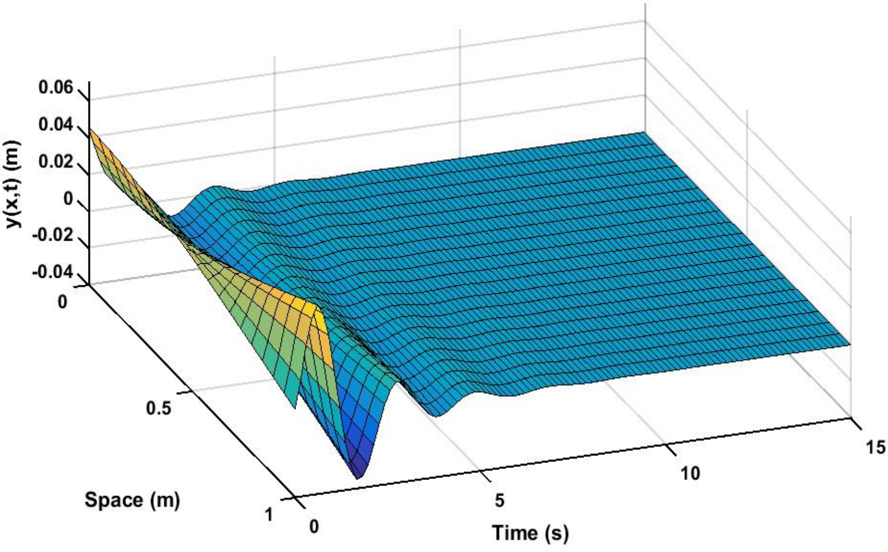

FIGURE 6. Displacement of the beam with boundary control.

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