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The total edge Steiner number of a graph

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Abstract. A total edge Steiner set of G is an edge Steiner set W such that the subgraph $\langle W \rangle$ induced by has no isolated vertex. The minimum cardinality of a total edge Steiner set of G is the total edge Steiner number of G and is denoted by $s_{te}(G)$. Some general properties satisfied by this concept are studied. The total edge Steiner number of certain classes of graphs is determined. Connected graphs of order p with total edge Steiner number 2 or 3 are characterized. Necessary conditions for total edge Steiner number to be p or $p-1$ is given. It is shown that for every pair a and b of integers with $2 \leq a < b$ and $b > a+1$, there exists a connected graph G such that $s_e(G) = a$ and $s_{te}(G) = b$. Also it shown that for every pair a and b of integers with $4 \leq a < b$ and $b > a+1$, there exists a connected graph G such that $s_t(G) = a$ and $s_{te}(G) = b$.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1]. The *open neighborhood of* v is $N(v) = \{u/uv \in E\}$. The *degree of a vertex* v is $deg_G(v) = |N(v)|$. If the degree of a vertex is 0, it is called an *isolated vertex*, while if the degree is 1, it is called an *end-vertex*. The *subgraph induced* by set S of vertices of a graph G is denoted by $\langle S \rangle$ with $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. A u-v path of length $d(u, v)$ is called a *u-v geodesic*. It is known that the distance is a metric on the vertex set of G. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, $radG$ and the maximum eccentricity is its *diameter*, $diamG$ of G. Two vertices u and v are said to be *antipodal* if $d(u, v) = diam(G)$. For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a *Steiner tree* with respect to W or a *Steiner W-tree*. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. If v is an end vertex of a Steiner W-tree, then $v \in W$. Also if $\langle W \rangle$ is connected, then any Steiner W-tree contains the elements of W only. The Steiner distance of a graph was introduced in

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[4].The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $\langle W\rangle$ is connected, then $S(W) = W$. If $S(W) = V$, then W is called a *Steiner set* for G. A Steiner set of minimum cardinality is a *minimum Steiner set* or simply a *s-set* of G and this cardinality is the *Steiner number* $s(G)$ of G. If W is a Steiner set of G and v a cut vertex of G, then v lies in every Steiner W-tree of G and so $W \cup \{v\}$ is also a Steiner set of G. The Steiner number of a graph was introduced in [5] and further studied in [2, 3, 6-9, 11-16, 17-19, 21]. The set of edges of G that lie on some Steiner *W-tree* is denoted by $S_e(W)$. If $S_e(W) = E(G)$, then W is called an *edge Steiner set* for G. An Steiner set of minimum cardinality is a minimum *edge Steiner set* or simply a edge *s_e-set* of G and this cardinality is the *edge Steiner number* $s_e(G)$ of G. The edge Steiner number of a graph was introduced in [16] and further studied in [20]. A *total Steiner* set of G is a Steiner set W such that the subgraph induced by W has no isolated vertex. The minimum cardinality of a total Steiner set of G is the *total Steiner number* of G and is denoted by $s_t(G)$. The total Steiner number of a graph was introduced in [10]. Steiner trees have application in transportation networks. For example, it may be desired to connect a certain set of leading cities with sub-cities that uses the least number of transportation links. In a transportation network, leading cities are a set of cities which cover the entire sub-cities with the property of the shortest transportation link. Automobile repair shops are present in the leading cities. If the leading cities are isolated and also if repair shops in a particular leading city fail, then repair of a vehicle is impossible as help from other leading city will not be available. The problem in the vehicle can be fixed only when there is an edge between each pair of leading cities or no leading city is isolated. Consider a transportation network as a graph model and each city as a vertex. Then the minimum cardinality of a set of leading cities with at least two leading cities are connected or no leading city is isolated is a minimum total edge Steiner set for the graph representing the transportation network. In this paper, we have introduced and studied the concept of the total edge Steiner number of a graph. The following theorems are used in the sequel.

Theorem 1.1. [**10, 16**] *Each extreme vertex of a graph G belongs to every edge Steiner set (total Steiner set) of G. In particular, each end-vertex of G belongs to every edge Steiner set (total Steiner set) of G.*

Theorem 1.2. [**10**] Let G is a connected graph and v an extreme vertex of G. Then every total Steiner set of G contains at least one element of $N(v)$.

2. The total edge Steiner number of a graph

Definition 2.1. Let G be a connected graph with at least 2 vertices. An edge Steiner set W of G is called a *total edge Steiner set* of G if $\langle W\rangle$ has no isolated vertex. The *total edge Steiner number* $s_{te}(G)$ is the minimum cardinality of its total edge Steiner sets and any total edge Steiner set of cardinality $s_{te}(G)$ is a *minimum total edge Steiner set* of G or *s_{te}-set* of G.

Example 2.1. For the graph G given in Figure 2.1, $W_1 = \{v_1, v_3, v_5, v_6, v_8, v_{10}\}$ is a minimum total edge Steiner set of G so that $s_{te}(G) = 6$

Remark 2.1. For the graph G in Figure 2.1, $W_1 = \{v_1, v_3, v_5, v_6, v_8, v_{10}\}$ is a minimum total Steiner set of G so that $s_{te}(G) = 4$. Thus the total Steiner number and total edge Steiner number of a graph are different.

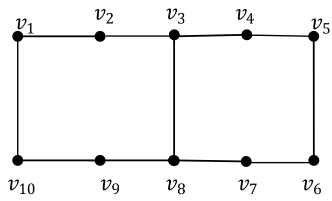


Figure 2.1

Remark 2.2. There can be more than one minimum total edge Steiner set for a graph. For the graph G given in Figure 2.2, $W = \{v_1, v_3, v_4, v_6\}$ and $W_1 = \{v_1, v_2, v_4, v_5\}$ are two different minimum total edge Steiner sets of G.

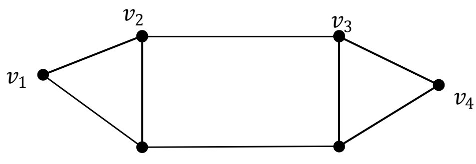


Figure 2.2

Remark 2.3. For any connected graph G, $2 \leq s_e(G) \leq s_{te}(G) \leq p$.

Theorem 2.1. *Each extreme vertex of a graph G belongs to every total edge Steiner set of G. In particular, each end-vertex of G belongs to every total edge Steiner set of G.*

Proof. Since every total edge Steiner set of G is an edge Steiner set of G, the result follows from Theorem 1.1. □

Theorem 2.2. *Let G is a connected graph and v an extreme vertex of G. Then every total edge Steiner set of G contains at least one element of $N(v)$.*

Proof. Assume, to the contrary, that G contains an extreme vertex v and a total edge Steiner set W such that W contains no element of $N(v)$. Then it follows that the vertex v is an isolated vertex of $\langle W \rangle$, which is a contradiction to W is a total edge Steiner set of G. Thus every total edge Steiner set of G contains at least one element of *N(v)*. □

Corollary 2.3. *If G is a connected graph of order p with k extreme vertices, then $max\{2, k\} \leq s_{te}(G) \leq p$.*

Proof. Since, $k \geq 0$ the result follows from Remark 2.3 and Theorem 2.1. □

Corollary 2.4. *For the complete graph $G = K_p$ $(p \geq 2)$, $s_{te}(G) = p$.*

Proof. Since every vertex of the complete graph K_p $(p \geq 2)$ is an extreme vertex, the vertex set of K_p is the unique total edge Steiner set of K_p. Thus $s_{te}(G) = p$. □

Corollary 2.5. *For the star $G = K_{1,p-1}$ $(p \geq 2)$, $s_{te}(G) = p$.*

Proof. This follows from Theorems 2.1 and 2.2. □

Theorem 2.6. *For the complete bipartite graph $G = K_{m,n}$ $2 \leq m \leq n$, $s_{te}(G) = m + n$.*

Proof. Let $U = \{u_1, u_2, .., u_m\}$, and $V = \{v_1, v_2, .., v_n\}$ be the bipartition of G. Let $W \subsetneqq U$,then has isolated vertices and so W is not a total edge Steiner set of G. If $W \subsetneqq V$, then it can be proved, as earlier that W is not a total edge Steiner set of G. If W is either U or V, then W is an edge Steiner set of G. However $\langle W \rangle$ has isolated vertices and so W is not a total edge Steiner set of G. If $W' \subsetneqq U \cup V$, such that W contains at least one vertex from each of U and V, then $\langle W \rangle$ is connected and so W is not a total edge Steiner set of G. Thus in any case W is not a total edge Steiner set of G. Hence $W = U \cup V$ is the unique minimum total edge Steiner set of G so that $s_{te}(G) =\mid W \mid= m + n$. □

Theorem 2.7. *For the cycle $G = C_p$ $(p \geq 6)$, $s_{te}(G) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd} \end{cases}$.*

Proof. If n is even, then every pair of antipodal vertices $W = \{u, v\}$ of G is an edge Steiner set of G. However $\langle W \rangle$ contains isolated vertices and so W is not a total edge Steiner set of G. It can be easily verified that no three element subset of G is a total edge Steiner set of G and so $s_{te}(G) \geq 4$. Let ux be an edge of G and y be the antipodal vertex of x. Then $W_1 = \{u, v, x, y\}$ is a total edge Steiner set of G so that $s_{te}(G) = 4$. Let n be odd. Let u and v be any two vertices of G. Then, since any Steiner $\{u, v\}$-tree is the $u-v$ geodesic, it follows that $\{u, v\}$ is not a total edge Steiner set of G. Thus, no 2-element subset of vertices of G is a total edge Steiner set of G. For any vertex u, let v,w be the antipodal vertices of u. Then clearly $W_2 = \{u, v, w\}$ is an edge Steiner set of G. However $\langle W_2 \rangle$ contains isolated vertices and so W_2 is not a total edge Steiner set of G. It can be easily verified that no four element subset of G is a total edge Steiner set of G and so $s_{te}(G) \geq 5$. Let $N(u) = \{x, y\}$. Then $W_2 = \{u, v, w, x, y\}$ is a total edge Steiner set of G so that $s_{te}(G) = 5$. □

Theorem 2.8. *Let G be a connected graph with v a cut-vertex of G and let W be a total edge Steiner set of G. Then every component of $G - v$ contains an element of W.*

Proof. Let v be a cut-vertex of G and W be a total edge Steiner set of G. Suppose there exists a component say G_1 of $G - v$ such that G_1 contains no vertex of W. By Theorem 2.1, W contains all the extreme vertices of G and hence it follows that G_1 does not contain any extreme vertex of G. Thus G_1 contains at least one edge say xy. Since every Steiner W-tree T must have its end-vertex in W and v is a cut-vertex of G, it is clear that no Steiner W-tree would contain the edge xy. This contradicts that W is a total edge Steiner set of G. □

Corollary 2.9. *If v is a cut-vertex of a connected graph G and W is a total edge Steiner set of G, then v lies in every Steiner W-tree of G.*

Proof. Let v be a cut-vertex of G and W a total edge Steiner set of G. Since every component of G-v contains an element of W it is clear that v lies in Steiner W-tree. □

The following two theorems characterize the total edge Steiner of a graph to be 2 or 3.

Theorem 2.10. *For a connected graph G with $p \geq 2$, $s_{te}(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then by Corollary 2.4, $s_{te}(G) = 2$. Let $s_{te}(G) = 2$ and let $W = \{u, v\}$ be a minimum total edge Steiner set of G. Then $S_e(W) = E(G)$. Since $\langle W \rangle$ has no isolated vertices, it follows that W is connected. Therefore $W = V[S_e(W)] = V$. Since $|W| = 2$, it follows that $G = K_2$. □

Lemma 2.11. *For a connected graph G with $p \geq 2$, if $s_{te}(G) = 3$, then $\langle W \rangle$ is connected for any s_{te}- set W of G.*

Proof. Let $s_{te}(G) = 3$ and let $W = \{u, v, w\}$ be a s_{te}- set of G. Then $S_e(W) = E(G)$. Suppose that $\langle W \rangle$ is not connected. Then $\langle W \rangle$ has isolated vertices, which is a contradiction to W a s_{te}- set of G. Therefore $\langle W \rangle$ is connected. □

Theorem 2.12. *For a connected graph G with $p \geq 3$, $s_{te}(G) = 3$ if and only if $G = K_3$ or P_3 .*

Proof. If G is either K_3 or P_3, then by Corollary 2.4 and Corollary 2.5, $s_{te}(G) = 3$. Conversely let $s_{te}(G) = 3$. Let $W = \{u, v, w\}$ be a s_{te}- set of G. Then by Lemma 2.11, $\langle W \rangle$ is connected. Therefore $W = V[S_e(W)] = V$. Since $\langle W \rangle$ is connected, either u, v, w are adjacent or only one vertex of G is adjacent to other two. Hence it follows that G is either K_3 or P_3. □

Theorem 2.13. *If G is a connected graph of order $p \geq 3$ containing a vertex v of degree $p - 1$, then all the neighbors of v belong to every total edge Steiner set of G.*

Proof. Let v be a vertex of degree $p-1$ and $v_1, v_2, ..., v_{p-1}$ be the neighbors of v in G. Let W be a total edge Steiner set of G. Suppose $v_1 \notin W$. Then the edge vv_1 lies on a Steiner W-tree of G, say T. Since $v_1 \notin W$, v_1 is not an end-vertex of T. Let T' be a tree obtained from T by removing the vertex v_1 in T and joining all the neighbors of v_1 other than v in T to v. Then T' is a Steiner W-tree such that $|V(T')| = |V(T)|-1$, which is a contradiction to T a Steiner W-tree. Therefore every total edge Steiner set of G contains all the neighbors of v. □

Theorem 2.14. *If G is a connected graph of $p \geq 3$ with v a cut vertex of degree $p-1$ such that degree of each vertex of $N(v)$ is greater than one, then $s_{te}(G) = p - 1$.*

Proof. Let v be a cut-vertex of degree $p - 1$ such that degree of each vertex of $N(v)$ is greater than one. By Theorem 2.13, $s_{te}(G) \geq p - 1$. Since degree of each vertex of $N(v)$ is greater than one, we have $W = N(v)$ is a total edge Steiner set of G so that $s_{te}(G) = p - 1$. □

Corollary 2.15. *Let G be a connected graph of order $p \geq 3$ such that $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$ and $| V(K_j) | \geq 2$ for all j. Then $s_{te}(G) = p - 1$.*

Theorem 2.16. *If G is a connected graph of order $p \geq 3$ with a vertex of degree $p-1$ such that degree of at least one vertex of $N(v)$ is one, then $s_{te}(G) = p$.*

Proof. Let v be a cut-vertex of degree $p - 1$ and W be a total edge Steiner set of G. Then by Theorem 2.13, $| W | \geq p - 1$ so that $s_{te}(G) \geq p - 1$. Since degree of at least one vertex of $N(v)$ is one, it follows that $N(v)$ contains an end vertex. Then by Theorem 2.2, v belongs to every total edge Steiner set of G so that $s_{te}(G) \geq p$. Hence $s_{te}(G) = p$. □

Corollary 2.17. *Let G be a connected graph of order $p \geq 3$ such that $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$ and $| V(K_j) | = 1$ for some j. Then $s_{te}(G) = p$.*

Theorem 2.18. *If G is a connected graph of order $p \geq 3$ with v a vertex of degree $p - 1$ such that v is not a cut-vertex of G, then $s_{te}(G) = p$.*

Proof. Let v be a vertex of degree $p - 1$ such that v is not a cut-vertex of G. Let $v_1, v_2, ..., v_{p-1}$ be the neighbors of v in G. Then by Theorem 2.13, $v_1, v_2, ..., v_{p-1}$ belong to every total edge Steiner set of G. Let $W = \{v_1, v_2, ..., v_{p-1}\}$. Since v is not a cut-vertex of G, $\langle W \rangle$ is connected. Hence the edge $vv_i \notin S_e(W)$ for $(1 \leq i \leq p - 1)$ and so W is not a total edge Steiner set of G. Now, it follows that $V(G)$ is the unique total edge Steiner set of G so that $s_{te}(G) = p$. □

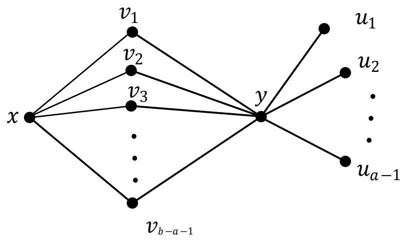
Corollary 2.19. *For the wheel $W_{1,p-1}$, $s_{te}(W_{1,p-1}) = p$.*

Theorem 2.20. *If G is a connected graph of order $p \geq 3$ with at least two vertices of degree $p - 1$, then $s_{te}(G) = p$.*

Proof. Let G contains at least two vertices of degree $p-1$. Then G has no cut vertex. Hence by Theorem 2.18, $s_{te}(G) = p$. □

Theorem 2.21. *For every pair a and b of integers with $2 \leq a < b$ and $b > a + 1$, there exists a connected graph G such that $s_e(G) = a$ and $s_{te}(G) = b$.*

Proof. First we prove that $s_e(G) = a$. Let $V(K_2) = \{x, y\}$ and $V(K_{b-a-1}) = \{v_1, v_2, ..., v_{b-a-1}\}$. Let $H = \overline{K_{b-a-1}} + \overline{K_2}$. Let G be the graph in Figure 2.3 obtained from H by adding $a-1$ new vertices $u_1, u_2, ..., u_{a-1}$ and joining each vertex u_i $(1 \leq i \leq a-1)$ with y. Let $W = \{u_1, u_2, ..., u_{a-1}\}$ be the set of end vertices of G. Then by Theorem 1.1, W is a subset of every edge Steiner set of G. Since $S_e(W) \neq E(G)$, W is not an edge Steiner set of G and so $s_e(G) \geq a$. Let $W' = W \cup \{x\}$. Then $S_e(W') = E(G)$ and so W' is an edge Steiner set of G. Therefore $s_e(G) = a$. Next we prove that $s_{te}(G) = b$. By Theorems 2.1 and 2.2, $W_1 = W \cup \{y\}$ is a subset of every total Steiner set of G. Since $S_e(W_1) \neq E(G)$, W_1 is not a total edge Steiner set of G. Let $W_2 = W_1 \cup \{x\}$. Then W_2 is an edge Steiner set of G. Since $\langle W_2 \rangle$ has isolated vertex, W_2 is not a total edge Steiner set of G. It is easily observed that every total edge Steiner set of G contains each v_i $(1 \leq i \leq b - a - 1)$ and so $s_{te}(G) \geq b$. Hence it follows that $S = V(G)$ is the unique total edge Steiner set of G so that $s_{te}(G) = b$. □



G
Figure 2.3

3. The total edge Steiner number and the total Steiner number of a graph

Theorem 3.1. *Every total edge Steiner set of a connected graph G is a total Steiner set of G.*

Proof. Let G be a connected graph and W be a total edge Steiner set of G. Let $v \in V(G)$. Let uv be an edge of G. Then uv lies on a Steiner W-tree of G. Thus v lies on a Steiner W-tree of G so that W is a Steiner set of G. Since $\langle W \rangle$ has no isolated vertices, G is a total Steiner set of G. □

Corollary 3.2. *For any connected graph G, $s_t(G) \leq s_{te}(G)$.*

Proof. Let W be any total edge Steiner set of G with minimum cardinality. Then $|W| = s_{te}(G)$. By Theorem 3.1, W is a total Steiner set of G so that $s_t(G) \leq |W| = s_{te}(G)$. □

Remark 3.1. The bounds in Corollary 3.2 is strict. For the graph G given in Figure 2.1, $s_t(G) = 4$ and $s_{te}(G) = 6$ so that $s(G) < s_e(G)$. Also the bound in Corollary 3.2 is sharp. For the cycle C_6, $s_t(C_6) = s_{te}(C_6) = 4$.

The following theorem gives a realization for the total Steiner number and the total edge Steiner number of a graph.

Theorem 3.3. *For every pair a and b of integers with $4 \leq a < b$ and $b > a+1$, there exists a connected graph G such that $s_t(G) = a$ and $s_{te}(G) = b$.*

Proof. If $a = 4$, $b \geq 5$, let G be the graph in Figure 3.1, obtained from the path on five vertices $P : u_1, u_2, u_3, u_4, u_5$ by adding $b-5$ new vertices $v_1, v_2, ..., v_{b-5}$ and joining each v_i $(1 \leq i \leq b-5)$ with u_2, u_3, u_4. By Theorems 1.1 and 1.2, $\{u_1, u_2, u_4, u_5\}$ is a subset of every total Steiner set of G and so $s_t(G) \geq 4$. It is clear that $\{u_1, u_2, u_4, u_5\}$ is a total Steiner set of G so that $s_t(G) = 4 = a$.

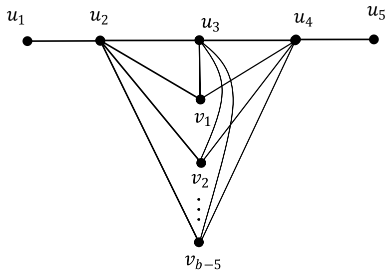


Figure 3.1

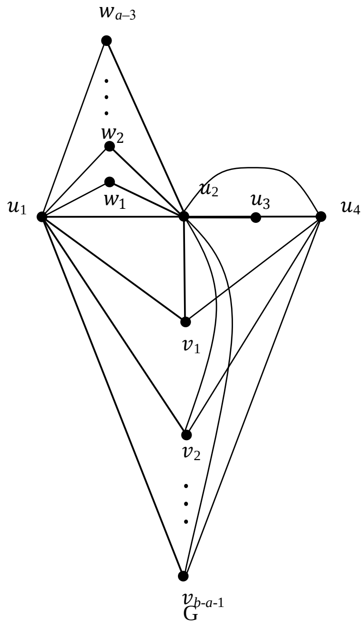


Figure 3.2

By Theorems 2.1 and 2.2, $S = \{u_1, u_2, u_4, u_5\}$ is a subset of every total edge Steiner set of G. It is clear that S is not a total edge Steiner set of G. It is easily observed that every total edge Steiner set of G contains each v_i $(1 \leq i \leq b-5)$ and so $s_{te}(G) \geq 4 + b - 5 = b - 1$. It is easily verified that $S_1 = S \cup \{v_1, v_2, ..., v_{b-5}\}$ is not a total edge Steiner set of G and so $s_{te}(G) \geq b$. However $S_1 = S \cup \{u_3\}$ is a total edge Steiner set of G and so that $s_{te}(G) = b$. If $a \geq 5$ and $b \geq 6$, let G be the graph obtained in Figure 3.2 from the path on four vertices $P : u_1, u_2, u_3, u_4$ by adding the new vertices $v_1, v_2, ..., v_{b-a-1}$ and $w_1, w_2, ..., w_{a-3}$ and joining each v_i $(1 \leq i \leq b-a-1)$ with u_1, u_2, u_4 and also joining each w_i $(1 \leq i \leq a-3)$ with u_1 and u_2. Since each w_i $(1 \leq i \leq a-3)$ is an extreme vertex of G, by Theorem 1.1, each w_i $(1 \leq i \leq a-3)$ belongs to every total Steiner set of G. Also since each w_i $(1 \leq i \leq a-3)$ and u_3 or extreme vertices of G, by Theorem 1.2, every total Steiner set contains either u_1 or u_2 and either u_2 or u_4 and so $s_t(G) \geq a - 3 + 3 = a$.

Let $W = \{w_1, w_2, ..., w_{a-3}, u_1, u_3, u_4\}$. Then W is a total Steiner set of G and so $s_t(G) = a$. Since u_2 is a full degree vertex of G which is not a cut vertex of G, by Theorem 2.18, $s_{te}(G) = a - 3 + 4 + b - a - 1 = b$. □

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