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Approximation for multi-quadratic mappings in non-Archimedean spaces

Abasalt Bodaghi, Somaye Salimi, and Ghasem Abbasi

Abstract. In this article, we introduce the multi-quadratic mappings and then unify the system of functional equations defining a multi-quadratic mapping to a single equation namely, the multi-quadratic functional equation. We also apply a fixed point theorem to establish the Hyers-Ulam stability for this new multi-quadratic functional equation in non-Archimedean normed spaces.

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1. Introduction

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty), n \in \mathbb{N}$. For any $s \in \mathbb{N}_0, m \in \mathbb{N}$, $t = (t_1, \ldots, t_m) \in \{-1, 1\}^m$ and $x = (x_1, \ldots, x_m) \in V^m$ we write $sx := (sx_1, \ldots, sx_m)$ and $tx := (t_1x_1, \ldots, t_mx_m)$, where ra stands, as usual, for the rth power of an element a of the commutative group V.

It is well-known that the quadratic (Jordan-von Neumann) equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1}$$

(which is useful in some characterizations of inner product spaces) play a remarkable role. A lot of information about its solutions, stability and its applications is available for instance in [15] and [22].

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [11] that a mapping $f : V^n \longrightarrow W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation) in each variable. Some basic facts on such mappings can be found in [17] and many other sources, where their application to the representation of polynomial functions is also presented. The mapping f is also said to be *multi-quadratic* if it is quadratic in each variable [10]. In [25], Zhao et al. proved that the mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if the following relation holds.

$$\sum_{t \in \{-1,1\}^n} f(x_1 + tx_2) = 2^n \sum_{j_1, j_2, \ldots, j_n \in \{1,2\}} f(x_{1j_1}, x_{2j_2}, \ldots, x_{nj_n}) \tag{2}$$

where $x_j = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. Various versions of multi-quadratic mappings which are recently studied can be found in [5] and [23].

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The stability of a functional equation originated from a question raised by Ulam: " when is it true that the solution of an equation differing slightly from a given one must of necessity be close to the solution of the given equation?" (see [24]). The first answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question was given by Hyers in [14]. Following his result, a great number of papers on the the stability problems of several functional equations have been extensively published as generalizing Ulam's problem and Hyers's theorem in various directions; see for instance [1, 3, 7, 9, 16, 21], and the references given there. In [11] and [10], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively. For more information about stability of multi-quadratic, multi-cubic and multi-quartic mappings, we refer to [4], [5], [6], [19] and [20].

We recall that a general form of the quadratic functional equation (1) is as follows:

$$Q(x+ay)+Q(x-ay)=2Q(x)+2Q^2f(y) \tag{3}$$

where a is a non-zero integer. It is easily verified that the function $Q(x)=cx^2$ is a solution of functional equation (3).

In this paper, we define the multi-quadratic mappings, i.e., they are quadratic (equation (3)) in each variable and present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-quadratic mappings to obtain a single functional equation and we prove the generalized Hyers-Ulam stability of this equation. Indeed, in the proof of our main result (Theorem 3.2), we apply the fixed point method, which was used for the first time by Brzdęk et al., in [8]; for more applications of this approach for the stability of multi-Cauchy-Jensen, multi-quadratic and multi-cubic mappings in non-Archimedean spaces see [2], [5] and [12], respectively.

2. Characterization of multi-quadratic mappings

Throughout this paper, let V and W be vector spaces over the rationals, $n\in\mathbb{N}$ and $x_i^n=(x_{i1},x_{i2},\dots,x_{in})\in V^n$, where $i\in\{1,2\}$. We shall denote x_i^n by x_i or simply x if there is no risk of ambiguity. Let $l_j\in\{1,2\}$. For the element $(x_{l_11},x_{l_22},\dots,x_{l_nn})\in V^n$, we put $s_i=\mathrm{Card}\{l_j: l_j=2\}$. Clearly, $0\le s_i\le n$.

We say the mapping $f:V^n\longrightarrow W$ is *n-multi-quadratic* or *multi-quadratic* if f is quadratic in each variable (see equation (3)). Here, we consider the functional equation

$$\sum_{q\in\{-1,1\}^n} f(x_1+qax_2)=2^n\sum_{\substack{0\le s_i\le n\\ l_1,\dots,l_n\in\{1,2\}}} a^{2s_i}f(x_{l_11},x_{l_22},\dots,x_{l_nn}). \tag{4}$$

In this section, we reduce the system of n equations defining the n-mixed quadratic mapping to obtain equation (4) and show that if a mapping $f:V^n\longrightarrow W$ satisfies equation (4), then it is multi-quadratic. Put $\mathbf{m}:=\{1,\dots,m\}$, $m\in\mathbb{N}$. For a subset $T=\{j_1,\dots,j_i\}$ of $\mathbf{m}$ with $1\le j_1<\dots<j_i\le m$ and $x=(x_1,\dots,x_m)\in V^m$,

$${}_Tx:=(0,\dots,0,x_{j_1},0,\dots,0,x_{j_i},0,\dots,0)\in V^m$$

denotes the vector which coincides with x in exactly those components, which are indexed by the elements of T and whose other components are set equal zero. Note that ${}_{\phi}x = 0$, ${}_{\mathbf{m}}x = x$. We use these notations in the proof of upcoming lemma.

We say the mapping $f : V^n \longrightarrow W$ is even in the jth variable if

$$f(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n).$$

To achieve our aim in this section, we need the next lemma.

Lemma 2.1. *If the mapping $f : V^n \longrightarrow W$ satisfies equation (4), then $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.*

Proof. Putting $x_1 = x_2 = (0, \dots, 0)$ in (4), we get

$$2^n f(0, \dots, 0) = 2^n \sum_{s_i=0}^{n} a^{2s_i} f(0, \dots, 0). \tag{5}$$

Thus, $f(0, \dots, 0) = 0$. Letting $x_{1k} = 0$ for all $k \in \{1, \dots, n\} \backslash \{j\}$ and $x_{2k} = 0$ for $1 \leq k \leq n$ in (4), we obtain

$$2^n f(0, \dots, 0, x_{1j}, 0, \dots, 0) = 2^n \sum_{s_i=0}^{n-1} a^{2s_i} f(0, \dots, 0, x_{1j}, 0, \dots, 0) \tag{6}$$

and so $f(0, \dots, 0, x_{1j}, 0, \dots, 0) = 0$. When k components of x_1 are not zero, we denote it by ${}_k x_1$. Therefore, the above process can be repeated to obtain

$$2^n f({}_k x_1) = 2^n \sum_{s_i=0}^{n-k} a^{2s_i} f({}_k x_1). \tag{7}$$

where $1 \leq k \leq n-1$. Hence, $f({}_k x_1) = 0$. This shows that $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero. □

Proposition 2.2. *A mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if it satisfies equation (4).*

Proof. Assume that f is multi-quadratic. We prove f satisfies equation (4) by induction on n. For $n = 1$, it is trivial that f satisfies (3). Assume that (4) is valid for some positive integer $n > 1$. Then,

$$\begin{aligned}
&\sum_{q \in \{-1,1\}^{n+1}} f\left(x_1^{n+1} + qax_2^{n+1}\right) \\
&= 2 \sum_{q \in \{-1,1\}^n} f\left(x_1^n + qax_2^n, x_{1n+1}\right) + 2a^2 \sum_{q \in \{-1,1\}^n} f\left(x_1^n + qax_2^n, x_{2n+1}\right) \\
&= 2^{n+1} \sum_{s_i=0}^{n} a^{2s_i} f(x_{l_1 1}, x_{l_2 2}, \dots, x_{l n_n}, x_{1n+1}) \\
&\quad + 2^{n+1} a^2 \sum_{s_i=0}^{n} a^{2s_i} f(x_{l_1 1}, x_{l_2 2}, \dots, x_{l n_n}, x_{2n+1}) \\
&= 2^{n+1} \sum_{\substack{s_i=0 \\ l_1, \dots, l_n \in \{1,2\}}}^{n+1} a^{2s_i} f(x_{l_1 1}, x_{l_2 2}, \dots, x_{l_n n}, x_{l_{n+1} n+1}).
\end{aligned} \tag{8}$$

This means that (4) holds for $n+1$.

Conversely, suppose that f satisfies equation (4). Now, fix $j\in\{1,\dots,n\}$, put $x_{1j}=0$ and $x_{2k}=0$ for all $k\in\{1,\dots,n\}\backslash\{j\}$. Using Lemma 2.1, we obtain

$$2^{n-1}[f(x_{11},\dots,x_{1j-1},ax_{2j},x_{1j+1},\dots,x_{1n})+f(x_{11},\dots,x_{1j-1},-ax_{2j},x_{1j+1},\dots,x_{1n})] \\ =2^n a^2 f(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n}). \tag{9}$$

Replacing x_{2j} by $-x_{2j}$ in (9), we get

$$2^{n-1}[f(x_{11},\dots,x_{1j-1},-ax_{2j},x_{1j+1},\dots,x_{1n})+f(x_{11},\dots,x_{1j-1},ax_{2j},x_{1j+1},\dots,x_{1n})] \\ =2^n a^2 f(x_{11},\dots,x_{1j-1},-x_{2j},x_{1j+1},\dots,x_{1n}). \tag{10}$$

Comparing equations (9) and (10), we see that

$$f(x_{11},\dots,x_{1j-1},-x_{2j},x_{1j+1},\dots,x_{1n})=f(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n}),$$

and so f is even in the jth variable. Once more, fix $j\in\{1,\dots,n\}$. Setting $x_{2k}=0$ for all $k\in\{1,\dots,n\}\backslash\{j\}$, applying Lemma 2.1 and using the evenness of f in the jth variable, we find

$$2^{n-1}[f(x_{11},\dots,x_{1j-1},x_{1j}+ax_{2j},x_{1j+1},\dots,x_{1n}) \\ +f(x_{11},\dots,x_{1j-1},x_{1j}-ax_{2j},x_{1j+1},\dots,x_{1n})] \\ =2^n[f(x_{11},\dots,x_{1j-1},x_{1j},x_{1j+1},\dots,x_{1n})+a^2f(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n})]. \tag{11}$$

It follows from relation (11) that f is quadratic in the jth variable. Since j is arbitrary, we obtain the desired result. □

It is shown in [18, Propositon 2.1] that a mapping f satisfying functional equation (1) if and only if it satisfies $f(ax+y)+f(ax-y)=2a^2f(x)+2f(y)$. In this case, it is easy to check that f is an even function and thus equation (1) is valid for f if and only if it satisfies (3). This result, Theorem 3 from [25] and Proposition 2.2 lead us to the following corollary. Since the proof is a direct consequence of the mentioned results, is omitted.

Corollary 2.3. *A mapping $f:V^n\longrightarrow W$ satisfies functional equation* (4) *if and only if it satisfies* (2).

3. Stability results for (4)

In this section, we prove the generalized Hyers-Ulam stability of equation (4) in non-Archimedean spaces. We recall some basic facts concerning non-Archimedean spaces and some preliminary results. Let us recall that a metric d on a nonempty set X is said to be non-Archimedean (or an ultrametric) provided $d(x,z)\leq\max\{d(x,y),d(y,z)\}$ for $x,y,z\in X$. By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$, and $|r+s|\leq\max\{|r|,|s|\}$ for all $r,s\in\mathbb{K}$. Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$.

Let $\mathcal{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\|:\mathcal{X}\longrightarrow\mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|rx\| = |r|\|x\|, \quad (x \in \mathcal{X}, r \in \mathbb{K})$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, \|\cdot\|)$ is called a *non-Archimedean normed space*. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\} \quad (n \geq m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

In [13], Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

Let p be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. One should remember that if $p > 2$, then $|2^n| = 1$ in for all integers n.

We recall that for a field $\mathbb{K}$ with multiplicative identity 1, the characteristic of $\mathbb{K}$ is the smallest positive number n such that $\overbrace{1 + \cdots + 1}^{n-times} = 0$.

Throughout, for two sets A and B, the set of all mappings from A to B is denoted by B^A. We wish to prove the generalized Hyers-Ulam stability of equation (4) in non-Archimedean spaces. The proof is based on a fixed point result that can be derived from [8, Theorem 1]. To present it, we introduce the following three hypotheses:

(H1) E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $g_1, \ldots, g_j : E \longrightarrow E$ and $L_1, \ldots, L_j : E \longrightarrow \mathbb{R}_+$,

(H2) $\mathcal{T} : Y^E \longrightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \max_{i \in \{1,\ldots,j\}} L_i(x) \left\|\lambda(g_i(x)) - \mu(g_i(x))\right\|, \quad \lambda, \mu \in Y^E, x \in E,$$

(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through

$$\Lambda\delta(x) := \max_{i \in \{1,\ldots,j\}} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^E, x \in E.$$

Here, we highlight the following theorem which is a fundamental result in fixed point theory [8]. This result plays a key tool to obtain our objective in this paper.

Theorem 3.1. *Let hypotheses* (H1)-(H3) *hold and the function* $\epsilon : E \longrightarrow \mathbb{R}_+$ *and the mapping* $\varphi : E \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \epsilon(x), \quad \lim_{l \to \infty} \Lambda^l \epsilon(x) = 0 \qquad (x \in E).$$

Then, for every $x \in E$*, the limit* $\lim_{l \to \infty} \mathcal{T}^l \varphi(x) =: \psi(x)$ *and the function* $\psi \in Y^E$*, defined in this way, is a fixed point of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq \sup_{l \in \mathbb{N}_0} \Lambda^l \epsilon(x) \qquad (x \in E).$$

Here and subsequently, for the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathfrak{D}_a f : V^n \times V^n \longrightarrow W$ by

$$\mathfrak{D}_a f(x_1, x_2) = \sum_{q\in\{-1,1\}^n} f(x_1 + qax_2) - 2^n \sum_{\substack{0\le s_i\le n\\ l_1,\ldots,l_n\in\{1,2\}}} a^{2s_i} f(x_{l_1 1}, x_{l_2 2}, \ldots, x_{l_n n}).$$

In the sequel, P stands for $\{1-a, 1+a\}^n$. Furthermore, $\begin{pmatrix} n \\ k \end{pmatrix}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}_0$ with $n \ge k$ by $n!/(k!(n-k)!)$. With these notations, we have the upcoming result.

Theorem 3.2. *Let V be a linear space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a function satisfying the equality*

$$\lim_{l\to\infty} \left(\frac{1}{|2(1+a^2)|^n}\right)^l \max_{p\in P} \varphi(p^l x_1, p^l x_2) = 0, \tag{12}$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping fulfilling the inequality

$$\|\mathfrak{D}_a f(x_1, x_2)\| \le \varphi(x_1, x_2) \tag{13}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multi-quadratic mapping $\mathcal{Q} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{Q}(x)\| \le \sup_{l\in\mathbb{N}} \left(\frac{1}{|2+2a^2|^n}\right)^l \max_{p\in P} \phi(p^l x, p^l x) \tag{14}$$

for all $x \in V^n$.

Proof. Putting $x = x_1 = x_2$ in (13), we have

$$\left\|\sum_{p\in P} f(px) - 2^n \sum_{s_i=0}^{n} \begin{pmatrix} n \\ s_i \end{pmatrix} a^{2s_i} f(x)\right\| \le \varphi(x, x) \tag{15}$$

for all $x \in V^n$. Since

$$\sum_{s_i=0}^{n} \begin{pmatrix} n \\ s_i \end{pmatrix} a^{2s_i} = \sum_{s_i=0}^{n} \begin{pmatrix} n \\ s_i \end{pmatrix} a^{2s_i} \times 1^{n-s_i} = (1+a^2)^n,$$

inequality (15) implies that

$$\left\| f(x) - \frac{1}{(2+2a^2)^n} \sum_{p\in P} f(px)\right\| \le \frac{1}{|2+2a^2|^n} \varphi(x, x) \qquad (x \in V^n). \tag{16}$$

For each $x \in V^n$, set

$$\mathcal{T}\xi(x) := \frac{1}{(2+2a^2)^n} \sum_{p\in P} \xi(px) \text{ and } \epsilon(x) = \frac{1}{|2+2a^2|^n} \varphi(x, x) \qquad (\xi \in W^{V^n}).$$

Now, inequality (16) can be rewritten as follows:

$$\|f(x) - \mathcal{T} f(x)\| \le \epsilon(x) \qquad (x \in V^n). \tag{17}$$

Define $\Lambda\eta(x) := \max_{p\in P} \frac{1}{|2+2a^2|^n} \eta(px)$ for all $\eta \in \mathbb{R}_+^{V^n}, x \in V^n$. It is easily seen that Λ has the form described in (H3) with $E = V^n$, $g_i(x) := g_p(x) = px$ for all $x \in V^n$

and $L_i(x) = \frac{1}{|2+2a^2|^n}$ for any i. In addition, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we obtain

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \max_{p\in P} \frac{1}{|2+2a^2|^n} \|\lambda(px) - \mu(px)\|.$$

The above relation shows that the hypothesis (H2) holds. By induction on l, one can check that for any $l \in \mathbb{N}$ and $x \in V^n$ that

$$\Lambda^l \epsilon(x) := \left(\frac{1}{|2+2a^2|^n}\right)^l \max_{p\in P} \epsilon(p^l x) \tag{18}$$

for all $x \in V^n$. Indeed, by definition of Λ, it follows from equality (18) is true for $l = 1$. If now (18) holds for $l \in \mathbb{N}$, then

$$\begin{aligned}\Lambda^{l+1}\epsilon(x) &= \Lambda(\Lambda^l \epsilon(x)) = \Lambda\left(\left(\frac{1}{|2+2a^2|^n}\right)^l \max_{p\in P}\epsilon(p^l x)\right) \\ &= \left(\frac{1}{|2+2a^2|^n}\right)^l \max_{p\in P}\Lambda\left(\epsilon(p^l x)\right) = \left(\frac{1}{|2+2a^2|^n}\right)^{l+1} \max_{p\in P}\epsilon(p^{l+1}x)\end{aligned}$$

for all $x \in V^n$. Relations (17) and (18) imply that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a unique mapping $\mathcal{Q} : V^n \longrightarrow W$ such that $\mathcal{Q}(x) = \lim_{l\to\infty}(\mathcal{T}^l f)(x)$ for all $x \in V^n$, and(14) holds as well. We shall to show that

$$\|\mathfrak{D}_a(\mathcal{T}^l f)(x_1, x_2)\| \leq \left(\frac{1}{|2+2a^2|^n}\right)^l \max_{p\in P}\varphi(p^l x_1, p^l x_2) \tag{19}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}$. We argue by induction on l. For $l = 1$ and for all $x_1, x_2 \in V^n$, we have

$$\begin{aligned}&\|\mathfrak{D}_a(\mathcal{T}f)(x_1, x_2)\| \\ &= \Bigg\| \sum_{q\in\{-1,1\}^n} (\mathcal{T}f)(x_1 + qax_2) - 2^n \sum_{\substack{0\leq s_i\leq n \\ l_1,\ldots,l_n\in\{1,2\}}} a^{2s_i}(\mathcal{T}f)(x_{l_1 1}, x_{l_2 2}, \ldots, x_{l_n n})\Bigg\| \\ &= \Bigg\| \frac{1}{(2+2a^2)^n} \sum_{q\in\{-1,1\}^n} \sum_{p\in P} f(px_1 + pqax_2) \\ &\qquad - \frac{1}{(1+a^2)^n} \sum_{\substack{0\leq s_i\leq n \\ l_1,\ldots,l_n\in\{1,2\}}} a^{2s_i} \sum_{p\in P} f(px_{l_1 1}, px_{l_2 2}, \ldots, px_{l_n n})\Bigg\| \\ &= \Bigg\| \frac{1}{(2+2a^2)^n} \sum_{p\in P} \mathfrak{D}_a(f)(p(x_1, x_2))\Bigg\| \\ &\leq \frac{1}{|2+2a^2|^n} \max_{p\in P} \|\mathfrak{D}_a(f)(p(x_1, x_2))\| \\ &\leq \frac{1}{|2+2a^2|^n} \max_{p\in P}\varphi(p(x_1, x_2))\end{aligned}$$

for all $x_1, x_2 \in V^n$. Assume that (19) is true for an $l \in \mathbb{N}$. Then

$$\begin{aligned}
&\left\|\mathfrak{D}_a(\mathcal{T}^{l+1}f)(x_1,x_2)\right\| \\
&= \left\| \sum_{q\in\{-1,1\}^n} (\mathcal{T}^{l+1}f)(x_1+qax_2) - 2^n \sum_{\substack{0\le s_i\le n\\ l_1,\dots,l_n\in\{1,2\}}} a^{2s_i}(\mathcal{T}^{l+1}f)(x_{l_1 1},x_{l_2 2},\dots,x_{l_n n})\right\| \\
&= \left\| \frac{1}{(2+2a^2)^n} \sum_{q\in\{-1,1\}^n} \sum_{p\in P} (\mathcal{T}^l f)(px_1+pqax_2) \right. \\
&\qquad \left. - \frac{1}{(1+a^2)^n} \sum_{\substack{0\le s_i\le n\\ l_1,\dots,l_n\in\{1,2\}}} a^{2s_i} \sum_{p\in P} (\mathcal{T}^l f)(p(x_{l_1 1},x_{l_2 2},\dots,x_{l_n n}))\right\| \\
&= \left\| \frac{1}{(2+2a^2)^n} \sum_{p\in P} \mathfrak{D}_a(\mathcal{T}^l f)(p(x_1,x_2))\right\| \\
&\le \frac{1}{|2+2a^2|^n} \max_{p\in P} \left\|\mathfrak{D}_a(\mathcal{T}^l f)(p(x_1,x_2))\right\| \\
&\le \left(\frac{1}{|2+2a^2|^n}\right)^{l+1} \max_{p\in P}\varphi(p^{l+1}(x_1,x_2))
\end{aligned}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (19) and applying (12), we arrive at $\mathfrak{D}_a\mathcal{Q}(x_1,x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping satisfies (4) and the proof is now completed. □

The following corollaries are the direct consequence of Theorem 3.2 concerning the stability of (4) when the norm $\mathfrak{D}_a f(x_1,x_2)$ is controlled by the sum and product of the norms powers of variables.

Corollary 3.3. *Let $\alpha \in \mathbb{R}$ fulfills $\alpha > \log_{(1+a)}^{2(1+a^2)}$ for the fixed positive integer a. Let also V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2 such that $|2| < 1$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathfrak{D}_a f(x_1,x_2)\| \le \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|^{n\alpha}$$

for all $x_1, x_2 \in V^n$, then, there exists a unique multi-quadratic mapping $\mathcal{Q} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{Q}(x)\| \le 2\left|\frac{(1+a)^\alpha}{2+2a^2}\right|^n \sum_{j=1}^{n} \|x_{1j}\|^{n\alpha}$$

for all $x = x_1 \in V^n$.

Proof. Putting $\phi(x_1,x_2)=\sum_{k=1}^{2}\sum_{j=1}^{n}\|x_{kj}\|^{\alpha}$, for $x=x_1$ we have $\phi(p^lx,p^lx)=2|p|^{ln\alpha}\sum_{j=1}^{n}\|x_{1j}\|^{n\alpha}$. By assumptions, we have $\frac{(1+a)^{\alpha}}{2+2a^2}>1$ and so

$$\sup_{l\in\mathbb{N}}\left(\frac{1}{|2+2a^2|^n}\right)^l\max_{p\in P}\phi(p^lx,p^lx)=2\sup_{l\in\mathbb{N}}\left|\frac{(1+a)^{\alpha}}{2+2a^2}\right|^{ln}\sum_{j=1}^{n}\|x_{1j}\|^{n\alpha}$$
$$=2\left|\frac{(1+a)^{\alpha}}{2+2a^2}\right|^{n}\sum_{j=1}^{n}\|x_{1j}\|^{n\alpha}.$$

Therefore, one can obtain the desired result by Theorem 3.2. □

Corollary 3.4. *Suppose that $p_{kj}>0$ for $k\in\{1,2\}$ and $j\in\{1,\dots,n\}$ fulfill $\alpha=\sum_{k=1}^{2}\sum_{j=1}^{n}p_{kj}>\log_{(1+a)}^{2(1+a^2)}$ for the fixed positive integer a. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2 such that $|2|<1$. If $f:V^n\longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathfrak{D}_af(x_1,x_2)\|\leq\prod_{k=1}^{2}\prod_{j=1}^{n}\|x_{kj}\|^{p_{kj}}$$

for all $x_1,x_2\in V^n$, then there exists a unique multi-quadratic mapping $\mathcal{Q}:V^n\longrightarrow W$ such that

$$\|f(x)-\mathcal{Q}(x)\|\leq\left|\frac{(1+a)^{\alpha}}{2+2a^2}\right|^{n}\prod_{j=1}^{n}\|x_{1j}\|^{p_{1j}+p_{2j}}$$

for all $x=x_1\in V^n$.

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(Abasalt Bodaghi) Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran
E-mail address: `abasalt.bodaghi@gmail.com`

(Somaye Salimi) Department of Mathematics, Qazvin Branch, Islamic Azad University, Qazvin, Iran
E-mail address: `somayesalimy@gmail.com`

(Ghasem Abbasi) Department of Mathematics, Qazvin Branch, Islamic Azad University, Qazvin, Iran
E-mail address: `g.abbasi@qiau.ac.ir`