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A variational approach for nonlocal problems with variable exponent and nonhomogeneous Neumann conditions

Anderson L. A. De Araujo, Shapour Heidarkhani, Ghasem A. Afrouzi, and Shahin Moradi

Abstract. We study the existence of at least one weak solution for $p(x)$-Kirchhoff-type problems of nonhomogeneous Neumann conditions. Our technical approach is based on variational methods. Some examples are presented to demonstrate the application of our main results.

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1. Introduction

The aim of this paper is to establish the existence of at least one weak solution for the following nonlocal problem

$$\begin{cases} T(u) = \lambda f(x, u(x)), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = \lambda g(\gamma(u(x))), & \text{on } \partial\Omega \end{cases} \tag{P_λ^f}$$

where

$$T(u) = M\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)dx\Big)\big(-\Delta_{p(x)}u + \alpha(x)|u|^{p(x)-2}u\big),$$

$\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $M : [0, +\infty[\to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$, $p \in C(\bar{\Omega})$, $\lambda > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function, $\alpha \in L^\infty(\Omega)$ with $ess\inf_\Omega \alpha > 0$, v is the outer unit normal to $\partial\Omega$ and $\gamma : W^{1,p(x)}(\Omega) \to L^{p(x)}(\partial\Omega)$ is the trace operator. If $\Omega =]a, b[$ and $h : [a, b] \to \mathbb{R}$ is a continuous function, then $\int_{\partial\Omega} h(x)dx$ reads $h(b) + h(a)$.

Problems like (P_λ^f) are usually called nonlocal problem because of the presence of the integral over the entire domain, and this implies that the first equation in (P_λ^f) is no longer a pointwise identity. In fact, such kind of problem can be traced back to the work of Kirchhoff. In [31] Kirchhoff proposed the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The problem (P_λ^f) is related to the stationary analogue of the problem (1.1).

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Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Similar nonlocal problems also model several physical and biological systems where u describes a process which depends on the average of itself, for example the population density. Lions [33] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions, various equations of Kirchhoff-type have been studied extensively, for instance see [23, 44, 46].

The study of various mathematical problems with the variable exponent growth condition has received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. They can model various phenomena which arise in the study of nonlinear elasticity theory, electro-rheological fluids and so on (see [45, 49]). Materials which require such advanced theories have been under experimental studies from the 1950s onwards. The first important discovery on electrorheological fluids was contributed by Willis Winslow in 1949. The viscosity of these fluids depends upon the electric field of the fluids. He discovered that the viscosity of such fluids as instance lithium polymetachrylate in an electrical field is an inverse relation to the strength of the field. The field causes string-like formations in the fluid, parallel to the field. They can increase the viscosity five orders of magnitude. This event is called the Winslow effect. For a general account of the underlying physics see [24] and for some technical applications [39]. Electrorheological fluids also have functions in robotics and space technology. Many experimental researches have been done chiefly in the USA, as in NASA laboratories. The necessary framework for the study of these problems is represented by the function spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. The study of various mathematical problems with variable exponent has received considerable attention in recent years. For background and recent results, we refer the reader to [1, 7, 8, 15, 17, 26, 28, 34, 38, 40, 41, 42, 48] and the references therein for details. For example, Yao in [48] by using the variational method, under appropriate assumptions on f and g, obtained a number of results on existence and multiplicity of solutions for the nonlinear Neumann boundary value problem of the following form

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = \mu g(x,u), & \text{on } \partial\Omega \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$, $p \in C(\bar{\Omega})$ and $p(x) > 1$. Moschetto in [38] under suitable assumptions on the functions α, f, p and g, based on the Ricceri two-local-minima theorem, together with the Palais-Smale property, investigated the existence of at least three solutions for the following Neumann problem

$$\begin{cases} -\Delta_{p(x)}u + \alpha(x)|u|^{p(x)-2}u = \alpha(x)f(u) + \lambda g(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial v} = 0, & \text{on } \partial\Omega. \end{cases}$$

Bonanno and Chinnì, based on variational methods, under appropriate growth conditions on the nonlinearity, established the existence of multiple solutions for nonlinear elliptic Dirichlet problems with variable exponent. D'Aguì and Sciammetta in [15] based on variational methods established the existence of an unbounded sequence of weak solutions for a class of differential equations with $p(x)$-Laplacian and subject to small perturbations of nonhomogeneous Neumann conditions. In [28] based on the variational methods and critical-point theory the existence of at least three solutions for elliptic problems driven by a $p(x)$-Laplacian was established. The existence of at least one nontrivial solution was also proved.

In recent years, many authors looked for existence and multiplicity of solutions to $p(x)$-Kirchhoff-type problems, for an overview on this subject, we cite the papers [11, 12, 13, 16, 19, 25, 30, 47] and the reference therein. For example, Han and Dai in [25] dealt with the sub-supersolution method for the $p(x)$-Kirchhoff type equations. They established a sub-supersolution principle for the Dirichlet problems involving $p(x)$-Kirchhoff, and presented a strong comparison theorem for the $p(x)$-Kirchhoff type equations. They also gave some applications of the obtained abstract theorems to the eigenvalue problems for the $p(x)$-Kirchhoff type equations. Dai and Wei in [16] proved the existence of infinitely many non-negative solutions of the Dirichlet problem involving the $p(x)$-Kirchhoff-type by applying a general variational principle due to Ricceri and the theory of the variable exponent Sobolev spaces. In [12], Cammaroto and Vilasi by use variational nature and weak formulation takes place in suitable variable exponent Sobolev spaces, established the existence of three weak solutions for a nonlinear transmission problem involving degenerate nonlocal coefficients of $p(x)$-Kirchhoff type. Hssini et al. in [30] based on variational methods, obtained the existence and multiplicity of solutions for a class of $p(x)$-Kirchhoff type equations with Neumann boundary condition. In [17] multiplicity results for nonlocal problems with variable exponent and nonhomogeneous Neumann conditions were established. In fact, using variational methods and critical point theory the existence results for the problem under algebraic conditions with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear term were ensured. Furthermore, by combining two algebraic conditions on the nonlinear term which guarantees the existence of two solutions, applying the mountain pass theorem given by Pucci and Serrin the existence of third solution for the problem was proved.

The existence and multiplicity of solutions for stationary higher order problems of Kirchhoff type (in n-dimensional domains, $n \geq 1$) were also investigated in some recent papers, using variational methods like the symmetric mountain pass theorem in [14] and a three critical point theorem in [5]. Moreover, in [3, 4], some evolutionary higher order Kirchhoff problems were studied, mainly focusing on the qualitative properties of the solutions.

Motivated by the above facts, in the present paper, we study the existence of at least one non-trivial weak solution for the problem (P_λ^f) under an asymptotical behaviour of the nonlinear datum at zero, see Theorems 3.1. Example 3.1 illustrates Theorem 3.1. We give some remarks on our results. In Theorem 3.2 we present an application of Theorem 3.1. We present Example 3.2 to illustrate Theorem 3.2. As special cases of Theorem 3.1, we obtain Theorem 3.3 considering the case $p(x) = p > N$.

Compared to the previous results, we give some new assumptions to obtain the existence of at least one non-trivial weak solution of the problem (P_λ^f). Recent related works are generalized.

We refer to the recent monograph by Molica Bisci, Rădulescu and Servadei [36] for related problems concerning the variational analysis of solutions of some classes of nonlocal problems.

The paper is organized as follows. In Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to our abstract results.

2. Preliminaries

We shall prove the existence of at least one non-trivial weak solution to the problem (P_λ^f) applying the following version of Ricceri's variational principle [43, Theorem 2.1] as given by Bonanno and Molica Bisci in [9].

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for every $r > \inf_X \Phi$, let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

We refer the interested reader to the papers [2, 21, 22, 27, 29, 35, 37] in which Theorem 2.1 has been successfully employed to the existence of at least one non-trivial solution for boundary value problems.

Here and in the sequel, meas(Ω) denotes the Lebesgue measure of the set Ω, and we also assume that $p \in C(\bar{\Omega})$ verifies the following condition:

$$N < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty. \tag{2.1}$$

Let Ω be a bounded domain of $\mathbb{R}^N$, denote:

$$L^{p(x)}(\Omega) := \Big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < +\infty\Big\},$$

$$L^{p(x)}(\partial\Omega) := \Big\{u : \partial\Omega \to \mathbb{R} \text{ measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma < +\infty\Big\}.$$

We can introduce the norms on $L^{p(x)}(\Omega)$ and $L^{p(x)}(\partial\Omega)$ by:

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{\beta > 0 : \int_\Omega |\frac{u(x)}{\beta}|^{p(x)} dx \le 1\Big\},$$

$$\|u\|_{L^{p(x)}(\partial\Omega)} = \inf\Big\{\beta > 0 : \int_{\partial\Omega} |\frac{u(x)}{\beta}|^{p(x)} d\sigma \le 1\Big\}$$

where $d\sigma$ is the surface measure on $\partial\Omega$.

Let X be the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ defined by putting $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega)\Big\}$$

and it can be equipped with the norm:

$$\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}. \tag{2.2}$$

It is well known (see [20]) that, in view of (2.1), both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces.

Moreover, since $\alpha \in L^\infty(\Omega)$, with $\alpha^- := ess\inf_{x\in\Omega}\alpha(x) > 0$ is assumed, then the following norm

$$\|u\|_\alpha = \inf\Big\{\sigma > 0 : \int_\Omega \big(\alpha(x)|\frac{u(x)}{\sigma}|^{p(x)} + |\frac{\nabla u(x)}{\sigma}|^{p(x)}\big)dx \le 1\Big\},$$

on $W^{1,p(x)}(\Omega)$ is equivalent to that introduce in (2.2). Since $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ (see [20] or [32]) and $p^- > N$, $W^{1,p(x)}(\Omega)$ is continuously embedded in $C^0(\bar\Omega)$ and one has

$$\|u\|_{C^0(\bar\Omega)} \le k_{p^-}\|u\|_{W^{1,p^-}(\Omega)}.$$

When Ω is convex, an explicit upper bound for the constant k_{p^-} is

$$k_{p^-} \le 2^{\frac{p^--1}{p^-}} \max\left\{(\frac{1}{\|\alpha\|_1})^{\frac{1}{p^-}}, \frac{d}{N^{\frac{1}{p^-}}}(\frac{p^--1}{p^--N}\mathrm{meas}(\Omega))^{\frac{p^--1}{p^-}}\frac{\|\alpha\|_\infty}{\|\alpha\|_1}\right\}$$

where $\|\alpha\|_1 = \int_\Omega \alpha(x)dx$ and $\|\alpha\|_\infty = \sup_{x\in\Omega}\alpha(x)$ and $d = diam(\Omega)$ (see [6, Remark 1]). On the other hand, taking into account that $p^- \le p(x)$, [32, Theorem 2.8] ensures that $L^{p(x)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ and the constant of such embedding does not exceed $1 + \mathrm{meas}(\Omega)$. So, one has

$$\|u\|_{W^{1,p^-}(\Omega)} \le (1 + \mathrm{meas}(\Omega))\|u\|_{W^{1,p(x)}(\Omega)} \le (1 + \mathrm{meas}(\Omega))\|u\|_\alpha.$$

In conclusion, put

$$c = k_{p^-}(1 + \mathrm{meas}(\Omega)),$$

it results

$$\|u\|_{C^0(\bar\Omega)} \le c\|u\|_\alpha \tag{2.3}$$

for each $u \in W^{1,p(x)}(\Omega)$.

Remark 2.1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, that means:

(a) $t \mapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$,
(b) $x \mapsto f(x,t)$ is continuous for a.e. $x \in \Omega$,
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1(\Omega)$ such that

$$\sup_{|t|\le\rho}|f(x,t)| \le l_\rho(x)$$

for a.e. $x \in \Omega$.

Corresponding to the functions f, g and M, we introduce the functions $F : \Omega\times\mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ and $\widetilde{M} : [0, +\infty[\to \mathbb{R}$, respectively, as follows

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

$$G(t) = \int_0^t g(\xi)d\xi \quad \text{for all } t \in \mathbb{R}$$

and

$$\widetilde{M}(t) = \int_0^t M(\xi)d\xi \quad \text{for all } t \ge 0.$$

We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of the problem (P_λ^f) if

$$M\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)dx\Big)$$
$$\int_\Omega \big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) + \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)\big)dx$$
$$-\lambda\Big(\int_\Omega f(x,u(x))v(x)dx + \int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))d\sigma\Big) = 0$$

for every $v \in W^{1,p(x)}(\Omega)$.

Proposition 2.2 ([18, Proposition 2.4]). *Let $\rho_\alpha(u) = \int_\Omega[|\nabla u|^{p(x)} + \alpha(x)|u|^{p(x)}]dx$ for $u \in W^{1,p(x)}(\Omega)$, we have*
(1) $\|u\|_\alpha \geq 1 \Longrightarrow \|u\|_\alpha^{p^-} \leq \rho_\alpha(u) \leq \|u\|_\alpha^{p^+}$,
(2) $\|u\|_\alpha \leq 1 \Longrightarrow \|u\|_\alpha^{p^+} \leq \rho_\alpha(u) \leq \|u\|_\alpha^{p^-}$.

3. Main results

We state our main result as follows.

Theorem 3.1. *Assume that*

$$\sup_{\gamma\geq c}\frac{\gamma^{p^-}}{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)dx + a(\partial\Omega)G(\gamma)} > \frac{p^+c^{p^-}}{m_0} \qquad (D_F)$$

where $a(\partial\Omega) = \int_{\partial\Omega} d\sigma$ and c is the constant defined in (2.3), and there are a non-empty open set $D \subseteq \Omega$ and $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi\to 0^+}\frac{ess\inf_{x\in B} F(x,\xi)}{|\xi|^{p^-}} = +\infty \qquad (3.1)$$

and

$$\liminf_{\xi\to 0^+}\frac{ess\inf_{x\in D} F(x,\xi)}{|\xi|^{p^-}} > -\infty. \qquad (3.2)$$

Then, for each

$$\lambda \in \Lambda = \Big(0, \frac{m_0}{p^+c^{p^-}}\sup_{\gamma\geq c}\frac{\gamma^{p^-}}{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)dx + a(\partial\Omega)G(\gamma)}\Big)$$

the problem (P_λ^f) admits at least one non-trivial weak solution $u_\lambda \in X$. Moreover, one has

$$\lim_{\lambda\to 0^+}\|u_\lambda\|_\alpha = 0$$

and the real function

$$\lambda \to \widetilde{M}\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla u_\lambda(x)|^{p(x)} + \alpha(x)|u_\lambda(x)|^{p(x)}\big)dx\Big)$$
$$-\lambda\Big(\int_\Omega F(x,u_\lambda(x))dx + \int_{\partial\Omega} G(\gamma(u_\lambda(x)))d\sigma\Big)$$

is negative and strictly decreasing in Λ.

Proof. Our aim is to apply Theorem 2.1 to the problem (P_λ^f). Consider the functionals Φ, Ψ for every $u \in X$, defined by

$$\Phi(u) = \widetilde{M}\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)dx\Big) \tag{3.3}$$

and

$$\Psi(u) = \int_\Omega F(x, u(x))dx + \int_{\partial\Omega} G(\gamma(u(x)))d\sigma, \tag{3.4}$$

and put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)dx + \int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))d\sigma$$

for every $v \in X$, as well as is sequentially weakly upper semicontinuous. Due to Proposition 2.2, we have

$$\Phi(u) \geq \frac{m_0}{p^+}\|u\|_\alpha^{p^-} \tag{3.5}$$

for all $u \in X$ such that $\|u\|_\alpha > 1$, and since $p^- > 1$, it follows that Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)(v) =& M\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)dx\Big)\\ &\int_\Omega \big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) + \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)\big)dx\end{aligned}$$

for every $v \in X$. Furthermore, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested in Theorem 2.1, are verified. Note that the critical points of the functional I_λ are the weak solutions of the problem (P_λ^f). We now look on the existence of a critical point of the functional I_λ in X. By using the condition (D_F), there exists $\bar{\gamma} \geq c$ such that

$$\frac{\bar{\gamma}^{p^-}}{\int_\Omega \sup_{|t|\leq\bar{\gamma}} F(x, t)dx + a(\partial\Omega)G(\bar{\gamma})} > \frac{p^+c^{p^-}}{m_0}. \tag{3.6}$$

Choose

$$r = \frac{m_0}{p^+}\Big(\frac{\bar{\gamma}}{c}\Big)^{p^-}.$$

Moreover, for all $u \in X$ with $\Phi(u) < r$, then, owing to [10, Proposition 2.2], one has

$$\|u\|_\alpha \leq \max\big\{(p^+r)^{\frac{1}{p^+}}, (p^+r)^{\frac{1}{p^-}}\big\}.$$

So, due to the embedding $X \hookrightarrow C^0(\Omega)$ (see (2.3)), one has $\|u\|_\infty \leq c\|u\|_\alpha$. From the definition of r, it follows that

$$\Phi^{-1}(-\infty, r) = \{u \in X; \Phi(u) < r\} \subseteq \{u \in X; |u| \leq \bar{\gamma}\},$$

and this ensures

$$\begin{aligned}\Psi(u) \leq& \sup_{u\in\Phi^{-1}(-\infty,r)} \int_\Omega F(x, u(x))dx + \sup_{u\in\Phi^{-1}(-\infty,r)} \int_{\partial\Omega} G(\gamma(u(x)))d\sigma\\ &\leq \int_\Omega \sup_{|t|\leq\bar{\gamma}} F(x, t)dx + \int_{\partial\Omega} \max_{|t|\leq\bar{\gamma}} G(t)d\sigma\end{aligned}$$

$$\leq \int_{\Omega} \sup_{|t|\leq\bar{\gamma}} F(x,t)dx + a(\partial\Omega)G(\bar{\gamma})$$

for every $u \in X$ such that $\Phi(u) < r$. Then

$$\sup_{\Phi(u)<r} \Psi(u) \leq \int_{\Omega} \sup_{|t|\leq\bar{\gamma}} F(x,t)dx + a(\partial\Omega)G(\bar{\gamma}).$$

By simple calculations and from the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)} \leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{r} \\
&\leq \frac{\int_{\Omega} \sup_{|t|\leq\bar{\gamma}} F(x,t)dx + \int_{\partial\Omega} \max_{|t|\leq\bar{\gamma}} G(t)d\sigma}{\frac{m_0}{p^+}(\frac{\bar{\gamma}}{c})^{p^-}} \\
&\leq \frac{\int_{\Omega} \sup_{|t|\leq\bar{\gamma}} F(x,t)dx + a(\partial\Omega)G(\bar{\gamma})}{\frac{m_0}{p^+}(\frac{\bar{\gamma}}{c})^{p^-}}.
\end{aligned}$$

Hence, putting

$$\lambda^* = \frac{m_0}{p^+c^{p^-}} \sup_{\gamma\geq c} \frac{\gamma^{p^-}}{\int_{\Omega} \sup_{|t|\leq\gamma} F(x,t)dx + a(\partial\Omega)G(\gamma)}.$$

Theorem 2.1 ensures that for every $\lambda \in (0, \lambda^*) \subseteq (0, \frac{1}{\varphi(r)})$, the functional I_λ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$. We will show that the function u_λ cannot be trivial.
Let us prove that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.7}$$

Owing to the assumptions (3.1) and (3.2), we can consider a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma > 0$) such that

$$\lim_{n\to+\infty} \frac{ess\inf_{x\in B} F(x,\xi_n)}{|\xi_n|^{p^-}} = +\infty$$

and

$$ess \inf_{x\in D} F(x,\xi) \geq \kappa|\xi|^{p^-}$$

for every $\xi \in [0, \sigma]$. We consider a set $\mathcal{G} \subset B$ of positive measure and a function $v \in X$ such that

(k_1) $v(x) \in [0, 1]$ for every $x \in \Omega$,
(k_2) $v(x) = 1$ for every $x \in \mathcal{G}$,
(k_3) $v(x) = 0$ for every $x \in \Omega \setminus D$.

Hence, fix $M > 0$ and consider a real positive number η with

$$M < \frac{\eta \operatorname{meas}(\mathcal{G}) + \kappa \int_{D\setminus\mathcal{G}} |v(x)|^{p^-} dx}{\frac{m_1}{p^-}\|v\|_\alpha^{p^-}}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and

$$ess\inf_{x\in B} F(x,\xi_n) \geq \eta|\xi_n|^{p^-}$$

for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is $0 \leq \xi_n v(x) < \sigma$ for n large enough), we have

$$\begin{aligned}
\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} =& \frac{\int_{\mathcal{G}} F(x,\xi_n)dx + \int_{D\setminus\mathcal{G}} F(x,\xi_n v(x))dx + \int_{\partial\Omega} G(\gamma(\xi_n v(x)))d\sigma}{\Phi(\xi_n v)}\\
\geq& \frac{\int_{\mathcal{G}} F(x,\xi_n)dx + \int_{D\setminus\mathcal{G}} F(x,\xi_n v(x))dx}{\Phi(\xi_n v)}\\
>& \frac{\eta\ \mathrm{meas}(\mathcal{G}) + \kappa\int_{D\setminus\mathcal{G}} |v(x)|^{p^-}dx}{\frac{m_1}{p^-}\|v\|_\alpha^{p^-}} > M.
\end{aligned}$$

Since M could be arbitrarily large, it is concluded that

$$\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty,$$

from which (3.7) clearly follows. Hence, there exists a sequence $\{w_n\} \subset X$ strongly converging to zero such that, for n large enough, $w_n \in \Phi^{-1}(-\infty, r)$ and

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we obtain

$$I_\lambda(u_\lambda) < 0, \tag{3.8}$$

hence that u_λ is not trivial. From (3.8) we easily observe that the map

$$(0,\lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.9}$$

is negative. Also, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\alpha = 0.$$

Indeed, bearing in mind that Φ is coercive and for every $\lambda \in (0,\lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\|_\alpha \leq L$ for every $\lambda \in (0,\lambda^*)$. After that, it is easy to see that there exist positive constants N and N' such that

$$\Big|\int_\Omega f(x,u_\lambda(x))u_\lambda(x)dx\Big| \leq N\|u_\lambda\|_\alpha \leq NL \tag{3.10}$$

and

$$\Big|\int_{\partial\Omega} g(\gamma(u_\lambda(x)))\gamma(u_\lambda(x))d\sigma\Big| \leq N'\|u_\lambda\|_\alpha \leq N'L \tag{3.11}$$

for every $\lambda \in (0,\lambda^*)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for every $v \in X$ and every $\lambda \in (0,\lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda\Big(\int_\Omega f(x,u_\lambda(x))u_\lambda(x)dx + \int_{\partial\Omega} g(\gamma(u_\lambda(x)))\gamma(u_\lambda(x))d\sigma\Big) \tag{3.12}$$

for every $\lambda \in (0,\lambda^*)$. For $\|u_\lambda\|_\alpha \geq 1$, owing to Proposition 2.2, we have

$$0 \leq m_0\|u_\lambda\|_\alpha^{p^-} \leq \Phi'(u_\lambda)(u_\lambda),$$

from (3.12), we have

$$0 \le m_0 \|u_\lambda\|_\alpha^{p^-} \le \lambda\Big(\int_\Omega f(x,u_\lambda(x))u_\lambda(x)dx + \int_{\partial\Omega} g(\gamma(u_\lambda(x)))\gamma(u_\lambda(x))d\sigma\Big) \quad (3.13)$$

for any $\lambda \in (0,\lambda^*)$. Letting $\lambda \to 0^+$, by (3.13) together with (3.10) and (3.11), we get

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\alpha = 0.$$

The proof of the case $\|u_\lambda\|_\alpha \le 1$ is similar to case $\|u_\lambda\|_\alpha \ge 1$. Then, we have obviously the desired conclusion. Finally, we have to show that the map

$$\lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing in $(0,\lambda^*)$. For our goal we see that for any $u \in X$, one has

$$I_\lambda(u) = \lambda\Big(\frac{\Phi(u)}{\lambda} - \Psi(u)\Big). \quad (3.14)$$

Now, let us consider $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1,2$. Also, set

$$m_{\lambda_i} = \Big(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i})\Big) = \inf_{v\in\Phi^{-1}(-\infty,r)} \Big(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\Big)$$

for every $i = 1,2$.
Clearly, (3.9) together with (3.14) and the positivity of λ imply that

$$m_{\lambda i} < 0 \text{ for } i = 1,2. \quad (3.15)$$

Moreover

$$m_{\lambda_2} \le m_{\lambda_1}, \quad (3.16)$$

due to the fact that $0 < \lambda_1 < \lambda_2$. Then, by (3.14)-(3.16) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0,\lambda^*)$. The arbitrariness of $\lambda < \lambda^*$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0,\lambda^*)$. The proof is complete. $\square$

Here we present an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. Consider the problem

$$\begin{cases} M\big(\int_\Omega \frac{1}{p(x,y)}\big(|\nabla u(x)|^{p(x,y)} + \alpha(x)|u(x)|^{p(x,y)}\big)dx\big)(-\Delta_{p(x,y)}u + \alpha(x)|u|^{p(x,y)-2}u) \\ = \lambda f(x,y,u), \qquad \text{in } \Omega, \\ |\nabla u|^{p(x,y)-2}\frac{\partial u}{\partial v} = \lambda g(\gamma(u(x))), \text{ on } \partial\Omega \end{cases} \quad (3.17)$$

where $M(t) = \frac{3}{2} + \frac{\cos(t)}{2}$ for every $t \in [0,+\infty)$, $p(x,y) = x^2+y^2+3$ for every $x,y \in \Omega$, $\alpha(x,y) = x^2+y^2+1$ for every $x,y \in \Omega$,

$$f(x,y,t) = \frac{4(x^2+y^2)}{10^3c^3}\big(3t^2+2t+\frac{4t^3}{t^4+1}\big)$$

for every $(x,y,t) \in \Omega\times\mathbb{R}$ and $g(t) = \frac{3}{10^3c^3}t^2$ for every $t \in \mathbb{R}$. By the expression of f, we have

$$F(x,y,t) = \frac{4(x^2+y^2)}{10^3c^3}(t^3+t^2+\ln(t^4+1))$$

for every $(x, y, t) \in \Omega \times \mathbb{R}$. By simple calculations, we obtain $a(\partial\Omega) = 2\pi$, $m_0 = 1$, $p^- = 3$ and $p^+ = 4$. Since

$$\sup_{\gamma \geq c} \frac{\gamma^3}{\int \int_\Omega \sup_{|t| \leq \gamma} F(x, y, t) dx dy + a(\partial\Omega) G(\gamma)} > 4c^3 = \frac{p^+ c^{p^-}}{m_0},$$

we observe that all assumptions of Theorem 3.1 are fulfilled. Hence, Theorem 3.1 implies that for each $\lambda \in (0, \frac{10^3}{16\pi})$ the problem (3.17) admits at least one non-trivial weak solution $u_\lambda \in X$. Moreover, one has

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\alpha = 0$$

and the real function

$$\begin{aligned} \lambda \to & \widetilde{M}\Big(\int_\Omega \frac{1}{p(x,y)} \big(|\nabla u_\lambda(x)|^{p(x,y)} + \alpha(x, y)|u_\lambda(x)|^{p(x,y)}\big) dx\Big) \\ & - \lambda\Big(\int \int_\Omega F(x, y, u_\lambda(x)) dx dy + \int_{\partial\Omega} G(\gamma(u_\lambda(x))) d\sigma\Big) \end{aligned}$$

is negative and strictly decreasing in $(0, \frac{10^3}{16\pi})$.

Now, we give some remarks of our results.

Remark 3.1. In Theorem 3.1 we searched for the critical points of the functional I_λ naturally associated with the problem (P_λ^f). We note that, generally I_λ can be unbounded from the following in X. Indeed, for example, if we take $f(x, \xi) = 1 + |\xi|^{\varepsilon - p^+} \xi^{p^+ - 1}$ for $(x, \xi) \in \Omega \times \mathbb{R}$ and $g(\xi) = |\xi|^{\varepsilon - 1}$ for $\xi \in \mathbb{R}$ with $\varepsilon > p^+$, for any fixed $u \in X \backslash \{0\}$ and $\iota \in \mathbb{R}$, we obtain

$$\begin{aligned} I_\lambda(\iota u) = & \Phi(\iota u) - \lambda\Big(\int_\Omega F(x, \iota u(x)) dx + \int_{\partial\Omega} G(\gamma(u(x))) d\sigma\Big) \\ \leq & \iota^{p^+} \frac{m_1}{p^-} \|u\|^{p^+} - \lambda \iota \|u\|_{L^1(\Omega)} - \lambda \frac{\iota^\varepsilon}{\varepsilon} \|u\|^\varepsilon_{L^\varepsilon(\Omega)} - \lambda \frac{\iota^\varepsilon}{\varepsilon} \|u\|^\varepsilon_{L^\varepsilon(\partial\Omega)} \to -\infty \end{aligned}$$

as $\iota \to +\infty$. Hence, we can not use direct minimization to find critical points of the functional I_λ.

Remark 3.2. For fixed $\bar{\gamma} \geq c$ let

$$\frac{\bar{\gamma}^{p^-}}{\int_\Omega \sup_{|t| \leq \bar{\gamma}} F(x, t) dx + a(\partial\Omega) G(\bar{\gamma})} > \frac{p^+ c^{p^-}}{m_0}.$$

Then the result of Theorem 3.1 holds with $\|u_\lambda\|_\infty \leq \bar{\gamma}$.

Remark 3.3. We observe that Theorem 3.1 is a bifurcation result in the sense that the pair $(0, 0)$ belongs to the closure of the set

$$\{(u_\lambda, \lambda) \in X \times (0, +\infty) : u_\lambda \text{ is a non-trivial weak solution of } (P_\lambda^f)\}$$

in $X \times \mathbb{R}$. Indeed, by Theorem 3.1 we have that

$$\|u_\lambda\|_\alpha \to 0 \quad \text{as} \quad \lambda \to 0.$$

Hence, there exist two sequences $\{u_j\}$ in X and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that

$$\lambda_j \to 0^+ \quad \text{and} \quad \|u_j\|_\alpha \to 0,$$

as $j \to +\infty$. Moreover, we emphasis that due to the fact that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing, for every $\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, the weak solutions u_{λ_1} and u_{λ_2} ensured by Theorem 3.1 are different.

When f doesn't depend upon x, we obtain the following autonomous version of Theorem 3.1.

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$. Assume that*

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{|\xi|^{p^-}} = +\infty.$$

Then, for each

$$\lambda \in \Lambda = \Big(0, \frac{m_0}{p^+ c^{p^-}} \sup_{\gamma \geq c} \frac{\gamma^{p^-}}{meas(\Omega)F(\gamma) + a(\partial\Omega)G(\gamma)}\Big)$$

where c is the constant defined in (2.3), the problem

$$\begin{cases} T(u) = \lambda f(u(x)), & on\ \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = \lambda g(\gamma(u(x))), & in\ \partial\Omega \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in X$ such that

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\alpha = 0$$

and the real function

$$\begin{aligned} \lambda \to & \widetilde{M}\Big(\int_\Omega \frac{1}{p(x)} \big(|\nabla u_\lambda(x)|^{p(x)} + \alpha(x)|u_\lambda(x)|^{p(x)}\big) dx\Big) \\ & -\lambda\Big(\int_\Omega F(u_\lambda(x))dx + \int_{\partial\Omega} G(\gamma(u_\lambda(x)))d\sigma\Big) \end{aligned}$$

is negative and strictly decreasing in Λ.

We now present the following example to illustrate Theorem 3.2.

Example 3.2. We consider the autonomous problem

$$\begin{cases} M\big(\int_\Omega \frac{1}{p(x,y)} \big(|\nabla u(x)|^{p(x,y)} + \alpha(x)|u(x)|^{p(x,y)}\big) dx\big)\big(- \Delta_{p(x,y)} u + \alpha(x)|u|^{p(x,y)-2}u\big) \\ = \lambda f(u), \qquad \text{in } \Omega, \\ |\nabla u|^{p(x,y)-2}\frac{\partial u}{\partial v} = \lambda g(\gamma(u(x))), \ \text{on } \partial\Omega \end{cases} \tag{3.18}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$, $M(t) = \frac{3}{2} + \frac{\sin(t)}{2}$ for every $t \in [0, +\infty)$, $p(x, y) = x^2 + y^2 + 5$ for every $x, y \in \Omega$, $\alpha(x, y) = x^2 + y^2 + 2$ for every $x, y \in \Omega$,

$$f(t) = \frac{1}{10^4 c^5}\big(5t^4 + 3t^2\big)$$

and $g(t) = \frac{3}{10^4 c^5} t^2$ for every $t \in \mathbb{R}$. By simple calculations, we obtain

$$F(t) = \frac{1}{10^4 c^5}(t^5 + t^3)$$

for every $t \in \mathbb{R}$. Direct calculations give $a(\partial\Omega) = 4\pi$, $m_0 = 1$, $p^- = 5$ and $p^+ = 9$. We observe that all assumptions of Theorem 3.2 are fulfilled. Hence, Theorem 3.2 implies that for each Then, for each $\lambda \in \left(0, \frac{10^4}{36\pi}\right)$ the problem (3.18) admits at least one non-trivial weak solution $u_\lambda \in X$. Moreover, one has

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\alpha = 0$$

and the real function

$$\lambda \to \widetilde{M}\Big(\int_\Omega \frac{1}{p(x,y)}\big(|\nabla u_\lambda(x)|^{p(x,y)} + \alpha(x,y)|u_\lambda(x)|^{p(x,y)}\big)dx\Big)$$
$$-\lambda\Big(\int_\Omega F(u_\lambda(x))dx + \int_{\partial\Omega} G(\gamma(u_\lambda(x)))d\sigma\Big)$$

is negative and strictly decreasing in $\left(0, \frac{10^4}{36\pi}\right)$.

We end this paper by presenting the following version of Theorem 3.1, in the case $p(x) = p$ for every $x \in \Omega$.

Theorem 3.3. *Assume that*

$$\sup_{\gamma \geq c} \frac{\gamma^p}{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)dx + a(\partial\Omega)G(\gamma)} > \frac{pc^p}{m_0}$$

and there are a non-empty open set $D \subseteq \Omega$ and $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi \to 0^+} \frac{ess\inf_{x\in B} F(x,\xi)}{|\xi|^p} = +\infty$$

and

$$\liminf_{\xi \to 0^+} \frac{ess\inf_{x\in D} F(x,\xi)}{|\xi|^p} > -\infty.$$

Then, for each

$$\lambda \in \Lambda = \left(0, \frac{m_0}{pc^p} \sup_{\gamma \geq c} \frac{\gamma^p}{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)dx + a(\partial\Omega)G(\gamma)}\right)$$

the problem

$$\begin{cases} M\big(\int_\Omega \frac{1}{p}\big(|\nabla u(x)|^p + \alpha(x)|u(x)|^p\big)dx\big)(-\Delta_p u + \alpha(x)|u|^{p-2}u) & \\ = \lambda f(x, u(x)), & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial v} = \lambda g(\gamma(u(x))), & \text{on } \partial\Omega \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in W^{1,p}(\Omega)$. Moreover, one has

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_\alpha = 0$$

and the real function

$$\lambda \to \frac{\widetilde{M}}{p}\Big(\int_\Omega \big(|\nabla u_\lambda(x)|^p + \alpha(x)|u_\lambda(x)|^p\big)dx\Big)$$
$$-\lambda\Big(\int_\Omega F(x, u_\lambda(x))dx + \int_{\partial\Omega} G(\gamma(u_\lambda(x)))d\sigma\Big)$$

is negative and strictly decreasing in Λ.

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(Anderson L. A. De Araujo) Departamento de Matemática, Universidade Federal de Viçosa, 36570-000, Viçosa (MG), Brazil
E-mail address: anderson.araujo@ufv.br

(Shapour Heidarkhani) Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
E-mail address: sh.heidarkhani@razi.ac.ir

(Ghasem A. Afrouzi) Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: afrouzi@umz.ac.ir

(Shahin Moradi) Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: shahin.moradi86@yahoo.com