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On some inequalities related to fractional Hermite-Hadamard type for differentiable convex functions

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Abstract. In this paper, we shall offer some new inequalities related to Hermite-Hadamard inequalities for differentiable convex functions involving generalized fractional integrals. Some of our new results are the extension of the previously established results.

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1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[14], [20], [37, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalize, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The overall structure of the study takes the form of three sections including introduction. The remainder of this work is organized as follows: we first mention some works which focus on Hermite-Hadamard inequality.

In Section 2, we introduce the generalized fractional integrals defined by Sarikaya and Ertuğral along with the very first result. In Section 3 we prove an identity for differentiable functions and using this identity we prove some trapezoid type inequalities for differentiable mappings.

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Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1). For some examples, please refer to ([3], [5], [10], [11], [15], [21], [31], [35], [36], [40]).

The Classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

Definition 1.1. The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex function if the following inequality hold

$$f(tx + (1 - t)y) = f(y + t(x - y)) \leq tf(x) + (1 - t)f(y)$$

for all $x,\ y \in [a, b]$ and $t \in [0, 1]$.We can say that f is concave if $(-f)$ is convex.

In [15], Dragomir and Agarwal gave the following identity and inequality related to the right part of Hermite-Hadamard inequality (1).

Lemma 1.1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, with $a < b$. If $f' \in L[a, b]$, then the following identity holds:*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1 - 2t)f'(ta + (1 - t)b)dt. \qquad (2)$$

Theorem 1.2. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left(|f'(a)| + |f'(b)|\right). \qquad (3)$$

In [44], Sarikaya et al. establish the following identity and inequality for the Riemann-Liouville fractional integrals.

Lemma 1.3. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, with $a < b$. If $f' \in L[a, b]$, then the following identity holds:*

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[I^\alpha_{a+}f(b) + I^\alpha_{b-}f(a)] = \int_0^1 [(1-t)^\alpha - t^\alpha]f(ta + (1-t)b)dt. \quad (4)$$

Theorem 1.4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds for fractional integrals:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[I^\alpha_{a+}f(b) + I^\alpha_{b-}f(a)]\right| \leq \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\left(|f'(a)| + |f'(b)|\right). \qquad (5)$$

Remark 1.1. If we take $\alpha = 1$ in identity 4 and inequality 5, then we have identity 2 and inequality 3 respectively.

Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, please see ([4], [12], [17], [18], [22]-[28], [30], [32], [33], [34], [38], [41], [43], [45]-[50]). For more information about fraction calculus please refer to ([19], [29]).

In this paper, we obtain the new generalized trapezoid type inequality for the generalized fractional integrals mentioned in next section.

2. New generalized fractional integral operators

In this section we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [42].
Let's define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following condition :

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_{\varphi} f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t) dt, \quad x > a, \tag{6}$$

$$_{b^-}I_{\varphi} f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t) dt, \quad x < b. \tag{7}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (6) and (7) are mentioned below.
i) If we take $\varphi(t) = t$, the operator (6) and (7) reduce to the Riemann integral as follows:

$$I_{a^+} f(x) = \int_a^x f(t) dt, \quad x > a,$$

$$I_{b^-} f(x) = \int_x^b f(t) dt, \quad x < b.$$

ii) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, the operator (6) and (7) reduce to the Riemann-Liouville fractional integral as follows:

$$I^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

$$I^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b.$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$, the operator (6) and (7) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I^{\alpha}_{a^+,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x > a,$$

$$I^{\alpha}_{b^-,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad x < b$$

where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [32].

iv) If we take $\varphi(t) = t(x-t)^{\alpha-1}$, the operator (6) reduces to the conformable fractional operators as follows:

$$I_a^{\alpha} f(x) = \int_a^x t^{\alpha-1} f(t) dt = \int_a^x f(t) d_{\alpha} t, \quad x > a, \ \alpha \in (0,1)$$

is given by Khalil et.al in [28].
Sarikaya and Ertuğral also establish the following Hermite-Hadamard inequality for the generalized fractional integral operators:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities for fractional integral operators hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Lambda(1)} \left[{}_{a+}I_{\varphi} f(b) + {}_{b-} I_{\varphi} f(a) \right] \leq \frac{f(a)+f(b)}{2} \tag{8}$$

where the mapping $\Lambda : [0,1] \to \mathbb{R}$ is defined by

$$\Lambda(x) = \int_0^x \frac{\varphi((b-a)t)}{t} dt.$$

For more recent results related to generalized fractional integral inequalities see, ([1], [2], [8], [9], [16], [39], [42]).

3. Trapezoid type inequalities for generalized fractional integrals

Let's start with the following Lemma.

Lemma 3.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely continuous mapping on I° such that $f' \in L([a,b])$, where $a, b \in I^{\circ}$ with $a < b$. Then the following equality for generalized fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a) \right] = \frac{b-a}{4\Lambda(1)} \left[I_1 - I_2 + I_3 - I_4 \right]$$

where

$$I_1 = \int_0^1 \Lambda\left(\frac{t}{2}\right) f'\left(\frac{a+b}{2}t + (1-t)a\right) dt,$$

$$I_2 = \int_0^1 \Lambda\left(\frac{2-t}{2}\right) f'\left(\frac{a+b}{2}t + (1-t)a\right) dt,$$

$$I_3 = \int_0^1 \Lambda\left(\frac{1+t}{2}\right) f'\left(bt + \frac{a+b}{2}(1-t)\right) dt,$$

and

$$I_4 = \int_0^1 \Lambda\left(\frac{1-t}{2}\right) f'\left(bt + \frac{a+b}{2}(1-t)\right) dt.$$

Proof. By the integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 \Lambda\left(\frac{t}{2}\right) f'\left(\frac{a+b}{2}t+(1-t)\,a\right)dt \\
&= \frac{2}{b-a}\Lambda\left(\frac{t}{2}\right) f\left(\frac{a+b}{2}t+(1-t)\,a\right)\Big|_0^1 \\
&\quad -\frac{1}{b-a}\int_0^1 \frac{\varphi\left(\frac{t}{2}\left(\frac{b-a}{a}\right)\right)}{\frac{t}{2}} f\left(\frac{a+b}{2}t+(1-t)\,a\right)dt \\
&= \frac{2}{b-a}\Lambda\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^{\frac{a+b}{2}} \frac{\varphi\left(\frac{x-a}{b-a}\cdot(b-a)\right)}{\frac{x-a}{b-a}} f(x)\,\frac{2}{b-a}dx \\
&= \frac{2}{b-a}\Lambda\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^{\frac{a+b}{2}} \frac{\varphi(x-a)}{x-a} f(x)\,dx. \qquad (9)
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
I_2 &= \int_0^1 \Lambda\left(\frac{2-t}{2}\right) f'\left(\frac{a+b}{2}t+(1-t)\,a\right)dt \qquad (10) \\
&= \frac{2}{b-a}\Lambda\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\Lambda(1)\,f(a)+\frac{2}{b-a}\int_a^{\frac{a+b}{2}} \frac{\varphi(b-x)}{b-x} f(x)\,dx,
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_0^1 \Lambda\left(\frac{1+t}{2}\right) f'\left(bt+\frac{a+b}{2}(1-t)\right)dt \qquad (11) \\
&= \frac{2}{b-a}\Lambda(1)\,f(b)-\frac{2}{b-a}\Lambda\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} \frac{\varphi(x-a)}{x-a} f(x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_0^1 \Lambda\left(\frac{1-t}{2}\right) f'\left(bt+\frac{a+b}{2}(1-t)\right)dt \\
&= -\frac{2}{b-a}\Lambda\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} \frac{\varphi(b-x)}{b-x} f(x)\,dx. \qquad (12)
\end{aligned}$$

By the equalities (9)-(12), we establish

$$
\begin{aligned}
I_1 - I_2 + I_3 - I_4 = & \frac{2}{b-a}\Lambda(1)\left[f(a)+f(b)\right] - \frac{2}{b-a}\int_a^b \frac{\varphi(b-x)}{b-x} f(x)\,dx \\
& - \frac{2}{b-a}\int_a^b \frac{\varphi(x-a)}{x-a} f(x)\,dx \\
= & \frac{2}{b-a}\Lambda(1)\left[f(a)+f(b)\right] - \frac{2}{b-a}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] \qquad (13)
\end{aligned}
$$

which completes the proof. □

Theorem 3.2. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L([a,b])$,where $a, b \in \overset{\circ}{I}$ with $a < b$. If the function $|f'|$ is convex on $[a,b]$, then we have the following inequality for generalized fractional integral operators*

$$
\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] \right| \\
& \leq \frac{b-a}{4\Lambda(1)} \left\{ \int_0^1 \left(\frac{2-t}{2}\right) \left| \Lambda\left(\frac{t}{2}\right) - \Lambda\left(\frac{2-t}{2}\right) \right| dt \right. \\
& \left. + \int_0^1 \frac{1-t}{2} \left| \Lambda\left(\frac{1+t}{2}\right) - \Lambda\left(\frac{1-t}{2}\right) \right| dt \right\} \left[|f'(a)| + |f'(b)| \right]. \qquad (14)
\end{aligned}
$$

Proof. By the Lemma 3.1, we obtain

$$
\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] \right| \\
& \leq \frac{b-a}{4\Lambda(1)} \left\{ \left| \int_0^1 \left[\Lambda\left(\frac{t}{2}\right) - \Lambda\left(\frac{2-t}{2}\right) \right] f'\left(\frac{a+b}{2}t + (1-t)a\right) dt \right| \right. \\
& \left. + \left| \int_0^1 \Lambda\left(\frac{1+t}{2}\right) - \Lambda\left(\frac{1-t}{2}\right) f'\left(tb + (1-t)\frac{a+b}{2}\right) dt \right| \right\} \\
& \leq \frac{b-a}{4\Lambda(1)} \left\{ \int_0^1 \left| \left[\Lambda\left(\frac{t}{2}\right) - \Lambda\left(\frac{2-t}{2}\right) \right] \right| \left| f'\left(\frac{a+b}{2}t + (1-t)a\right) \right| dt \right. \\
& \left. + \int_0^1 \left| \Lambda\left(\frac{1+t}{2}\right) - \Lambda\left(\frac{1-t}{2}\right) \right| \left| f'\left(tb + (1-t)\frac{a+b}{2}\right) \right| dt \right\}.
\end{aligned}
$$

Since $|f'|$ is convex, we get

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right| \\
&\leq \frac{b-a}{4\Lambda(1)}\left\{\int_0^1\left|\left[\Lambda\left(\frac{t}{2}\right)-\Lambda\left(\frac{2-t}{2}\right)\right]\right|\left[t\left|f'\left(\frac{a+b}{2}\right)\right|+(1-t)\left|f'(a)\right|\right]dt\right. \\
&\quad\left.+\int_0^1\left|\Lambda\left(\frac{1+t}{2}\right)-\Lambda\left(\frac{1-t}{2}\right)\right|\left[t\left|f'(b)\right|+(1-t)\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt\right\} \\
&\leq \frac{b-a}{4\Lambda(1)}\left\{\int_0^1\left|\left[\Lambda\left(\frac{t}{2}\right)-\Lambda\left(\frac{2-t}{2}\right)\right]\right|\left[t\frac{|f'(a)|+|f'(b)|}{2}+(1-t)\left|f'(a)\right|\right]dt\right. \\
&\quad\left.+\int_0^1\left|\Lambda\left(\frac{1+t}{2}\right)-\Lambda\left(\frac{1-t}{2}\right)\right|\left[t\left|f'(b)\right|+(1-t)\frac{|f'(a)|+|f'(b)|}{2}\right]dt\right\} \\
&= \frac{b-a}{4\Lambda(1)}\left\{\int_0^1\left(\frac{2-t}{2}\right)\left|\Lambda\left(\frac{t}{2}\right)-\Lambda\left(\frac{2-t}{2}\right)\right|dt\right. \\
&\quad\left.+\int_0^1\left(\frac{1-t}{2}\right)\left|\Lambda\left(\frac{1+t}{2}\right)+\Lambda\left(\frac{1-t}{2}\right)\right|dt\right\}\left|f'(a)\right| \\
&\quad+\frac{b-a}{4\Lambda(1)}\left\{\int_0^1\frac{t}{2}\left|\Lambda\left(\frac{t}{2}\right)-\Lambda\left(\frac{2-t}{2}\right)\right|dt\right. \\
&\quad\left.+\int_0^1\frac{1+t}{2}\left|\Lambda\left(\frac{1+t}{2}\right)-\Lambda\left(\frac{1-t}{2}\right)\right|dt\right\}\left|f'(b)\right| \\
&= \frac{b-a}{4\Lambda(1)}\left\{\int_0^1\left(\frac{2-t}{2}\right)\left|\Lambda\left(\frac{t}{2}\right)-\Lambda\left(\frac{2-t}{2}\right)\right|dt\right. \\
&\quad\left.+\int_0^1\frac{1-t}{2}\left|\Lambda\left(\frac{1+t}{2}\right)-\Lambda\left(\frac{1-t}{2}\right)\right|dt\right\}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].
\end{aligned}
$$

This completes the proof. □

Remark 3.1. If we choose $\varphi(t)=t$ in Theorem 3.2, then inequality (14) reduces to the inequality (3).

Remark 3.2. If we choose $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 3.2, then inequality (14) reduces to the inequality (5).

Remark 3.3. If we choose $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 3.2, then we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[J_{a+,k}f(b)+J_{b-,k}f(a)\right]\right|\\&\leq\frac{b-a}{2\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\end{aligned}$$

which is proved by Farid et al. in [18, Theorem 2.4].

Theorem 3.3. *Let $f: I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function on I° such that $f'\in L_q([a,b])$, where $a,b\in \overset{\circ}{I}$ with $a<b$. If the function $|f'|^q$, $q>1$, is convex on $[a,b]$, then we have the following inequality for generalized fractional integral operators*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right|\\&\leq\frac{b-a}{4\Lambda(1)}\left\{\left(\int_0^1\left|\Lambda\left(\frac{t}{2}\right)-\Lambda\left(\frac{2-t}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_0^1\left|\Lambda\left(\frac{1+t}{2}\right)-\Lambda\left(\frac{1-t}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the Lemma 3.1 and Hölder inequality, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right|\\&\leq\frac{b-a}{4\Lambda(1)}\left\{\int_0^1\left|\left[\Lambda\left(\frac{t}{2}\right)-\Lambda\left(\frac{2-t}{2}\right)\right]\right|\left|f'\left(\frac{a+b}{2}t+(1-t)a\right)\right|dt\right.\\&\left.+\int_0^1\left|\Lambda\left(\frac{1+t}{2}\right)-\Lambda\left(\frac{1-t}{2}\right)\right|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right\}\\&\leq\frac{b-a}{4\Lambda(1)}\left\{\left(\int_0^1\left|\Lambda\left(\frac{t}{2}\right)-\Lambda\left(\frac{2-t}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{a+b}{2}t+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_0^1\left|\Lambda\left(\frac{1+t}{2}\right)-\Lambda\left(\frac{1-t}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{15}$$

Since $|f'|^q$ is convex, we have

$$\begin{aligned}\int_0^1 \left|f'\left(\frac{a+b}{2}t+(1-t)a\right)\right|^q dt &\leq \int_0^1 \left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f'(a)\right|^q\right] \\ &= \frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2} \\ &\leq \frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4} \qquad (16)\end{aligned}$$

and similarly

$$\int_0^1 \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq \frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}. \qquad (17)$$

If we substitute (16) and (17) in (15), we establish the desired result. □

Corollary 3.4. *If we choose $\varphi(t)=t$ in Theorem 3.3, then we obtain*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\} \\ &\leq \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \qquad (18)\end{aligned}$$

Proof. The proof of the first inequality in (18) is obvious by choosing $\varphi(t)=t$. For the proof of second inequality, let $a_1=3\left|f'(a)\right|^q$, $b_1=\left|f'(b)\right|^q$, $a_2=\left|f'(a)\right|^q$ and $b_2=3\left|f'(b)\right|^q$. Using the facts that,

$$\sum_{k=1}^n (a_k+b_k)^s \leq \sum_{k=1}^n a_k^s+\sum_{k=1}^n b_k^s,\ \ 0\leq s<1 \qquad (19)$$

and $1+3^{\frac{1}{q}}\leq 4$. the desired result can be obtained straightforwardly. □

Corollary 3.5. *If we choose $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 3.3, then we obtain*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}f(b)+J_{b-}f(a)\right]\right| \\ &\leq \frac{b-a}{4}\left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}} \\ &\quad\times\left\{\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\} \\ &\leq \frac{b-a}{4}\left(\frac{8}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\end{aligned}$$

Proof. First inequality can be obtained by the inequality

$$(A-B)^q \leq A^q - B^q,$$

for any $A > B \geq 0$ and $q \geq 1$. The second inequality is obvious from the inequality (19). □

Corollary 3.6. *If we choose* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem 3.3, then we obtain*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[J_{a+,k} f(b) + J_{b-,k} f(a) \right] \right|$$
$$\leq \frac{b-a}{4} \left(\frac{2k}{\alpha p + k} \right)^{\frac{1}{p}} \left(1 - \frac{1}{2^{\frac{\alpha p}{k}}} \right)^{\frac{1}{p}}$$
$$\times \left\{ \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} \right\}$$
$$\leq \frac{b-a}{4} \left(\frac{8k}{\alpha p + k} \right)^{\frac{1}{p}} \left(1 - \frac{1}{2^{\frac{\alpha p}{k}}} \right)^{\frac{1}{p}} \left[|f'(a)| + |f'(b)| \right].$$

4. Conclusion

In the development of this work, Hermite-Hadamard inequalities for differentiable convex functions involving generalized fractional integrals have been deduced. We also give several results capturing Riemann-

Liouville fractional integrals and k-Riemann-Liouville fractional integrals as special cases. The authors hope that these results will serve as a motivation for future work in this fascinating area. As future directions, one may finds the similar inequalities through different types of convexities.

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