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An existence result for homoclinic solutions for a linear ordinary differential equation of second order

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Abstract. In this paper we consider the equation $\ddot{u} + a(t)\dot{u} + u = 0$, where $a : \mathbb{R} \to \mathbb{R}$ is a function of class C^1. We give sufficient conditions for which the above equation admits a solution $u : \mathbb{R} \to \mathbb{R}$ fulfilling a condition of type $\lim_{t\to\pm\infty} u(t) = \lim_{t\to\pm\infty} \dot{u}(t) = 0$. The result is obtained through the method of differential inequalities, by using a classical Lyapunov function. Recall that a solution non-identically zero fulfilling the mentioned conditions is known in the literature as **homoclinic solution**.

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1. Introduction

It is well known the interest to study the asymptotic behavior on the semiaxis $\mathbb{R}_+ = [0, +\infty)$ of the solutions of differential equations (especially the ones of second order). The appetite for this study has been opened also by the famous book of Belmann [7].

In general, to make easy the study of an equation of second order, this is transformed into a system with two equations of first order. The convenient choosing of the transformation which realizes this thing can substantially contributes to obtain certain consistent results. Such ingenuous transformations have been used in [8]) and [9]) to research of the asymptotic stability of a linear homogeneous equation of second order.

Less studied, but however important is the research of the asymptotic behavior of the solutions on the whole real axis $\mathbb{R}$. Through the classes of properties which could be interesting is the boundedness of solutions or the vanishing of solutions to $\pm\infty$; such behaviors has been studied last time for example in [1], [2], [3], [4], [5], [6].

In the present paper we consider the equation

$$\ddot{u} + a(t)\dot{u} + u = 0, \tag{1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 and we give sufficient conditions for which equation (1) admits at least one solution $u : \mathbb{R} \to \mathbb{R}$ fulfilling a condition of type

$$\beta(t) \le u^2(t) + \left(\dot{u}(t) + \frac{a(t)}{2}u(t)\right)^2 \le \alpha(t),\ t \in \mathbb{R}, \tag{2}$$

where $\alpha, \beta : \mathbb{R} \to (0, +\infty)$ are continuous functions depending on a. Next, we present sufficient conditions for which $\alpha(\pm\infty) := \lim_{t\to\pm\infty} \alpha(t) = 0$. This fact assures

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the existence of a solution equation (1), which is non-identically zero, such that

$$\lim_{t\to\pm\infty} u(t) = \lim_{t\to\pm\infty} \dot{u}(t) = 0. \tag{3}$$

The left side of inequality (2) is used to prove that the found solution, fulfilling (3), is non-identically zero.

To establish inequality (2), we shall use the method of the Lyapunov function and differential inequalities.

In [8], [9] equation (1) is considered on $\mathbb{R}_+ := [0,+\infty)$ and the existence of solutions satisfying the condition $u(+\infty) = \dot{u}(+\infty) = 0$ is researched.

To extend the cited results to the whole real axis, one uses the known method of Krasnoselskii (see [10]), which has been used by this author to the study of the existence of bounded and periodic solutions.

Remark that, since $u = 0$ is a solution to equation (1), then each solution which satisfies (3), is a homoclinic solution.

2. Preliminaries

We start with the presentation of the principal notations and hypotheses used throughout the paper.

Denote by $(\cdot,\cdot)$ and $|\cdot|$ respectively the inner product and the euclidean norm in $\mathbb{R}^2$. Define on $\mathbb{R}^2$ the Lyapunov function

$$V(x) = u^2 + v^2 = |x|^2, \ x = (u,v)^T \in \mathbb{R}^2.$$

We use the transformation

$$v := \dot{u} + \frac{a(t)}{2}u$$

and equation (1) becomes

$$\dot{x} = f(t,x), \tag{4}$$

where

$$f(t,x) = \begin{pmatrix} v - \frac{a(t)}{2}u \\ \left(\frac{\dot{a}(t)}{2} + \frac{a^2(t)}{4} - 1\right)u - \frac{a(t)}{2}v \end{pmatrix}$$

and $x = \begin{pmatrix} u \\ v \end{pmatrix}$.

The derivative $\dot{V}$ of V along system (4) is, by definition,

$$\dot{V}(x) = (\operatorname{grad} V, f).$$

Therefore, we obtain

$$\dot{V}(x(t)) = -a(t)\left(u^2+v^2\right) + 2uv\left(\frac{\dot{a}(t)}{2} + \frac{a^2(t)}{4}\right). \tag{5}$$

Let us consider $a : \mathbb{R} \to \mathbb{R}$ a function of class $C^1(\mathbb{R})$, satisfying the hypotheses:

a$_1$) $\int_{(\cdot)}^{\pm\infty} a(t)\,dt = +\infty$,

a$_2$) $\int_{(\cdot)}^{+\infty} \left|2\dot{a}(t) + a^2(t)\right| dt < +\infty$, $\int_{-\infty}^{(\cdot)} \left|2\dot{a}(t) + a^2(t)\right| dt < +\infty$.

Denote in addition

$$\begin{aligned} z(t) \ &: = \left|\frac{\dot{a}(t)}{2} + \frac{a^2(t)}{4}\right| - a(t), \\ w(t) \ &: = -\left|\frac{\dot{a}(t)}{2} + \frac{a^2(t)}{4}\right| - a(t), \end{aligned}$$

for all $t \in \mathbb{R}$.

3. The main result

We can state and prove the following result.

Theorem 3.1. *Suppose that $a : \mathbb{R} \to \mathbb{R}$ is a function of class $C^1(\mathbb{R})$, fulfilling hypotheses* a$_1$) *and* a$_2$). *Then, the equation* (1) *admits at least one homoclinic solution.*

Proof. Let $n \in \mathbb{N}^*$ be arbitrary.

Since $V : \mathbb{R}^2 \to \mathbb{R}_+$ is continuous, $V(0) = 0$, $\lim\limits_{|x|\to\infty} V(x) = \infty$ and $V(\mathbb{R})$ is connected, it follows that for every $r_0 > 0$, there exists $x_0 \in \mathbb{R}$, $x_0 \neq 0$ such that $V(x_0) = r_0$. Therefore one can consider $x_n = (u_n, v_n)^T$ the unique solution to system (4), for which

$$V(x_n(-n)) = e^{-\int_{-n}^{0} z(s)ds}. \tag{6}$$

From the obvious inequalities

$$-\left(\left|\frac{\dot{a}}{2} + \frac{a^2}{4}\right| + a\right)\left(u_n^2 + v_n^2\right) \le \dot{V}(x_n) \le \left(\left|\frac{\dot{a}}{2} + \frac{a^2}{4}\right| - a\right)\left(u_n^2 + v_n^2\right),$$

it follows that

$$w(t)V(x_n(t)) \le \dot{V}(x_n(t)) \le z(t)V(x_n(t)), \ (\forall)\ t \ge -n. \tag{7}$$

Since $x_n(-n) \neq 0$, we get $V(x_n(t)) > 0$, for all $t \ge -n$.

By (7) we obtain then

$$w(t) \le \frac{\dot{V}(x_n(t))}{V(x_n(t))} \le z(t), \ (\forall)\ t \ge -n$$

and therefore

$$V(x_n(-n))\, e^{\int_{-n}^{t} w(s)ds} \le V(x_n(t)) \le V(x_n(-n))\, e^{\int_{-n}^{t} z(s)ds}, \tag{8}$$

for all $t \ge -n$.

From relations (6) and (8) we deduce the following inequality

$$V(x_n(t)) \le e^{\int_0^t z(t)dt}, \ (\forall)\ t \ge -n.$$

Define the mapping $\alpha : \mathbb{R} \to (0, +\infty)$, by

$$\alpha(t) = e^{\int_0^t z(s)ds}. \tag{9}$$

By hypotheses a$_1$) and a$_2$), it follows that

$$\lim_{t\to\pm\infty} \alpha(t) = \lim_{t\to\pm\infty} e^{\int_0^t z(s)ds} = \lim_{t\to\pm\infty} e^{\int_0^t \left(\left|\frac{\dot{a}(s)}{2} + \frac{a(s)^2}{4}\right| - a(s)\right)ds} = 0.$$

The conclusion is that for every $n \in \mathbb{N}^*$, there exists a function x_n such that

$$\dot{x}_n(t) = f(t, x_n(t)), \ (\forall)\ t \in [-n, n] \tag{10}$$

and

$$V(x_n(t)) \le \alpha(t), \ (\forall)\ t \in [-n, n]. \tag{11}$$

We extend x_n to the whole real axis, by setting

$$\widetilde{x}_n(t) := \begin{cases} x_n(t), & \text{if } t \in [-n, n], \\ x_n(-n), & \text{if } t \le -n, \\ x_n(n), & \text{if } t \ge n. \end{cases}$$

Consider the function space

$$C_c := \left\{ x : \mathbb{R} \to \mathbb{R}^2,\ x \text{ continuous} \right\},$$

endowed with the topology of the uniform convergence on compact subsets of $\mathbb{R}$; as it is known, this topology can be defined through the following family of seminorms

$$|x|_n := \sup\left\{ |x(t)|,\ t \in [-n,n] \right\},\ n \in \mathbb{N}^*.$$

Furthermore, we know that a family $\mathcal{A} \subset C_c$ is relatively compact if and only if $\mathcal{A}$ is equi-continuous and uniformly bounded on compact subsets of $\mathbb{R}$ (the Ascoli-Arzela' Theorem).

We want to show that the family $\{\widetilde{x}_n\}_{n\in\mathbb{N}^*}$ is relatively compact on compact subsets of $\mathbb{R}$.

To this aim, let us consider $[-k,k] \subset \mathbb{R}$ an arbitrary compact subset of $\mathbb{R}$, $k \in \mathbb{N}^*$. It is obviously that there exists $n_0 \in \mathbb{N}^*$, such that $[-k,k] \subset [-n,n]$, for all $n \in \mathbb{N}^*$, $n \geq n_0$, and therefore,

$$\widetilde{x}_n(t) = x_n(t),\ \ (\forall)\ \ t \in [-k,k],\ n \geq n_0.$$

Since

$$V(x_n(t)) = u_n(t)^2 + v_n(t)^2 = |x_n(t)|^2$$

and

$$V(x_n(t)) \leq \alpha(t),\ \ (\forall)\ \ t \in [-k,k],$$

it follows that

$$|x_n(t)| \leq M_k,\ \ (\forall)\ \ t \in [-k,k],\ n \geq n_0,$$

where

$$M_k := \sup\left\{ \alpha(t),\ t \in [-k,k] \right\}^{\frac{1}{2}}.$$

Hence, the family $\{\widetilde{x}_n\}_{n\in\mathbb{N}^*}$ is uniformly bounded on $[-k,k]$.

Next, by setting

$$L_k := \sup\left\{ |f(t,x)|,\ t \in [-k,k],\ \ |x| \leq M_k \right\},$$

it follows that

$$|\widetilde{x}_n'(t)| = |f(t, \widetilde{x}_n(t))| \leq L_k,\ \ (\forall)\ \ t \in [-k,k],\ n \geq n_0.$$

Then the family $\{\widetilde{x}_n\}_{n\in\mathbb{N}^*}$ is equi-continuous on $[-k,k]$, having the family of derivatives, $\{\widetilde{x}_n'\}_{n\in\mathbb{N}^*}$, uniformly bounded on $[-k,k]$.

Hence, by passing to subsequences, one may suppose that

$$\widetilde{x}_n \to x,\ \text{ in } C_c.$$

From $\widetilde{x}_n'(t) = f(t, \widetilde{x}_n(t))$, it follows that $\{\widetilde{x}_n'(t)\}_{n\in\mathbb{N}^*}$ converges uniformly on $[-k,k]$ to $\dot{x}$. So,

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ V(x(t)) \leq \alpha(t) \end{cases},\ \ (\forall)\ \ t \in [-k,k]. \tag{12}$$

But since each $t \in \mathbb{R}$ belongs to an interval $[-k,k]$, it follows that (12) is true for every $t \in \mathbb{R}$.

To end the proof of Theorem 3.1, it remains to show that $x(t)$ is non-identically zero.

From the inequalities

$$w(t) \leq \frac{\dot{V}(x_n(t))}{V(x_n(t))} \leq z(t),\ \ (\forall)\ \ t \geq -n.$$

we get

$$V(x_n(t)) \geq V(x_n(-n))\, e^{\int_{-n}^{t} w(s)ds} = e^{-\int_{-n}^{0} z(s)ds + \int_{-n}^{t} w(s)ds}.$$

By setting $t = 0$ in the previous inequality, we obtain then

$$V(x_n(0)) \geq e^{\int_{-n}^{0}(w(s)-z(s))ds} = e^{-2\int_{-n}^{0}\left|\frac{\dot{a}(s)}{2}+\frac{a(s)^2}{4}\right|ds}. \tag{13}$$

If, be means of contradiction, x would be identically zero, then, since $\lim_{n\to\infty} x_n(0) = x(0)$, it would result that

$$\lim_{n\to\infty} V(x_n(0)) = 0.$$

Therefore, by passing to limit as $n \to \infty$ in relation (13), it would follow that

$$0 \geq e^{-2\int_{-\infty}^{0}\left|\frac{\dot{a}(s)}{2}+\frac{a(s)^2}{4}\right|ds}$$

or, equivalently,

$$\int_{-\infty}^{0}\left|\frac{\dot{a}(s)}{2}+\frac{a(s)^2}{4}\right|ds = +\infty,$$

which contradicts the hypothesis a_2). □

4. Final remarks

The inequality appearing in (12) allows us to estimate the "speed" of the convergence to zero of the homoclinic solution $x(\cdot)$. For example, by setting

$$a(t) = \frac{t}{t^2+1},$$

which fulfills the hypotheses a_1) and a_2), we obtain the estimation

$$|x(t)| \leq \frac{k}{1+t^2},$$

where k is a positive constant.

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