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Int-soft bi-hyperideals in ordered ternary semihypergroups

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Abstract. The aim of this paper is to study po-ternary semihypergroups in terms of the int-soft bi-hyperideals. We introduce the notion of int-soft bi-hyperideals in po-ternary semihypergroups and some properties of them are investigated. Characterizations of bi-hyperideals in terms of int-soft bi-hyperideals are obtained. We prove that every int-soft hyperideal is an int-soft bi-hyperideal, but the converse is not true. Examples are provided to illustrate the results.

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1. Introduction and preliminaries

The algebraic hyperstructures represent a natural generalization of classical algebraic structures which is based on the notion of hyperoperation introduced by the French mathematician Marty [11] in 1934. Hila et al. [6, 13, 14, 15] provided some results on ternary semihypergroups. Chavlina [3] introduced the concept of ordering hypergroups as a special class of hypergroups. In [5], Heidari and Davvaz introduced the concept of ordered semihypergroups, as a generalization of the concept of ordered semigroups.

Molodsov [12] introduced the parameterized family of sets, known as soft set theory, as a mathematical tool for dealing with hesitancy, fuzzyness and unsure articles. Moreover, several operations on soft sets were introduced by Maji et al. [10]. Anvariyeh et al. [1] introduced soft semihypergroups by using the concept soft set theory. Sezgin et al. [17] introduced int-soft interior ideals, as a new approach to the classical semigroup theory via soft set. Naz and Shabir [16] defined the basic properties of soft sets and compared soft sets to the related concepts of semihypergroups. Hila et al. [7, 8] studied ternary and m-ary semihypergroups in terms of soft sets. Some results on the applying of the int-soft theory in ordered semihypergroups have been obtained in [4, 9]. In [18, 19], int-soft hyperideals are introduced and studied in ordered ternary semihypergroups.

In this paper, we study po-ternary semihypergroups in terms of the int-soft bi-hyperideals. We introduce the notion of int-soft bi-hyperideals in po-ternary semihypergroups and some properties of them are investigated. Characterizations of bi-hyperideals in terms of int-soft bi-hyperideals are obtained. We prove that every int-soft hyperideal is an int-soft bi-hyperideals but the converse is not true. Examples are provided to illustrate the results.

Throughout the paper, int-soft bi-hyperideal will be denoted by ISBH and ordered ternary semihypergroup will be denoted by po-ternary semihypergroup. To develop

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our main results, we need the following notions. In this paper, the parameter set of the soft set is a po-ternary semihypergroup, whereas the universe set is any set.

For any non-empty set S, let $\mathcal{P}(S)$ be the set of all subsets of S and let $\mathcal{P}^*(S)$ be the set of all non-empty subsets of S. A map $\circ : S \times S \to \mathcal{P}^*(S)$ such that $\forall(a,b,c,d) \in S^4, a = c, b = d$ imply $a \circ b = c \circ d$ is called a hyperoperation on the set S and the couple $(S, \circ)$ is called a hypergroupoid. A hypergroupoid $(S, \circ)$ is called a semihypergroup if for all $(a,b,c) \in S^3$, we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{p \in a \circ b} p \circ c = \bigcup_{q \in b \circ c} a \circ q$$

If $a \in S$ and X and Y are non-empty subsets of S, then we denote

$$X \circ Y = \bigcup_{x \in X, y \in Y} x \circ y,\ a\ o\ X = \{a\} \circ X \text{ and } X\ o\ a = X \circ \{a\}.$$

A map $\circ :\ S \times S \times S \to \mathcal{P}^*(S)$ is called a ternary hyperoperation on the set S, where S is a non-empty set and $\mathcal{P}^*(S)$ denotes the set of all non-empty subsets of S.

A ternary hypergroupoid is called the pair $(S, \circ)$ where $\circ$ is a ternary hyperoperation on the set S.

If X, Y, Z are non-empty subsets of S, then we define

$$(X \circ Y \circ Z) = \bigcup_{x \in X, y \in Y, z \in Z} (x \circ y \circ z).$$

Definition 1.1. A ternary hypergroupoid $(S, \circ)$ is called a ternary semihypergroup if for all $a, b, c, d, e \in S$, we have

$$(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b \circ (c \circ d \circ e).$$

Definition 1.2. Let $(S, \circ)$ be a ternary semihypergroup and H a non-empty subset of S. Then H is called a ternary subsemihypergroup of S if and only if $(H \circ H \circ H) \subseteq H$.

Definition 1.3. A non-empty subset A of a ternary semihypergroup S is called a left (right, lateral) hyperideal of S if $(S \circ S \circ A) \subseteq A((A \circ S \circ S) \subseteq A, (S \circ A \circ S) \subseteq A)$.

A non-empty subset H of a ternary semihypergroup S is called a hyperideal of S if it is a left, right and lateral hyperideal of S. A non-empty subset A of a ternary semihypergroup S is called two-sided hyperideal of S if it is a left and right hyperideal of S. A lateral hyperideal A of a ternary semihypergroup S is called a proper lateral hyperideal of S if $A \neq S$.

Definition 1.4. Let $(S, \circ)$ be a ternary semihypergroup. A binary relation ϱ is called:

(1) compatible on the left if $a\ \varrho\ b$ and $x \in (x_1 \circ x_2 \circ a)$ imply that there exists $y \in (x_1 \circ x_2 \circ b)$ such that $x\ \varrho\ y$;
(2) compatible on the right if $a\ \varrho\ b$ and $x \in (a \circ x_1 \circ x_2)$ imply that there exists $y \in (b \circ x_1 \circ x_2)$ such that $x\ \varrho\ y$;
(3) compatible on the lateral if $a\ \varrho\ b$ and $x \in (x_1 \circ a \circ x_2)$ imply that there exists $y \in (x_1 \circ b \circ x_2)$ such that $x\ \varrho\ y$;
(4) compatible on the two-sided if $a_1\ \varrho\ b_1$, $a_2\ \varrho\ b_2$, and $x \in (a_1 \circ z \circ a_2)$ imply that there exists $y \in (b_1 \circ z \circ b_2)$ such that $x\ \varrho\ y$;
(5) compatible if $a_1\ \varrho\ b_1$, $a_2\ \varrho\ b_2$, $a_3\ \varrho\ b_3$ and $x \in (a_1 \circ a_2 \circ a_3)$ imply that there exists $y \in (b_1 \circ b_2 \circ b_3)$ such that $x\ \varrho\ y$.

Definition 1.5. A ternary semihypergroup $(S, \circ)$ is called a po-ternary semihypergroup if there exits a partially ordered relation $\leq$ on S such that $\leq$ are compatible on left, compatible on right, compatible on lateral and compatible.

Let $(S, \circ, \leq)$ be a po-ternary semihypergroup. Then for any subset R of a po-ternary semihypergroup S, we denote $(R] := \{s \in S | s \leq r \text{ for some } r \in R\}$. If $R = \{a\}$, we also write $(\{a\}]$ as $(a]$. If X and Y are non-empty subsets of S, then we say that $X \leq Y$ if for every $a \in X$, there exists $b \in Y$ such that $a \leq b$.

Definition 1.6. A non-empty subset T of a po-ternary semihypergroup $(S, \circ, \leq)$ is said to be a po-ternary subsemihypergroup of S if $(T \circ T \circ T] \subseteq T$.

Definition 1.7. A non-empty subset A of a po-ternary semihypergroup S is called a right (lateral, left) hyperideal of S if
(1) $(A \circ S \circ S) \subseteq A$ $((S \circ A \circ S) \subseteq A$, $(S \circ S \circ A) \subseteq A)$,
(2) If $a \in A$ and $s \leq a$, then $s \in A$ for every $s \in S$.

A non-empty subset A of a po-ternary semihypergroup S is called a hyperideal of S if it is a left, right and lateral hyperideal of S. A non-empty subset A of a po-ternary semihypergroup S is called two-sided hyperideal of S if it is a left and right hyperideal of S.

2. Main results

In what follows, we take $E = S$ as the set of parameters, which is a po-ternary semihypergroup and U is an initial universe set, unless otherwise specified.

Definition 2.1. [2] A soft set f_A over U is defined as $f_A : E \to \mathcal{P}(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Hence f_A is also called an approximation function. A soft set f_A over U can be represented by the set of ordered pairs $f_A = (x, fA(x))|x \in E, f_A(x) \in \mathcal{P}(U)$.

It is clear to see that a soft set is a parametrized family of subsets of the set U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2. [2] Let $f_A, f_B \in S(U)$. Then, f_A is called a soft subset of f_B and denoted by $f_A \sqsubseteq f_B$, if $f_A(x) \subseteq f_B(x)\ \forall\ x \in E$.

Definition 2.3. [2] Let $f_A, f_B \in S(U)$. Then, union of f_A and f_B, denoted by $f_A \sqcup f_B$, is defined as $f_A \sqcup f_B = f_{A \sqcup B}$, where $f_{A \sqcup B}(x) = f_A(x) \cup f_B(x), \forall x \in E$.

Definition 2.4. [2] Let $f_A, f_B \in S(U)$. Then, intersection of f_A and f_B, denoted by $f_A \sqcap f_B$, is defined as $f_A \sqcap f_B = f_{A \sqcap B}$, where $f_{A \sqcap B}(x) = f_A(x) \cap f_B(x), \forall x \in E$.

For any element a of S, we define

$$A_a = \{(x, y, z) \in S \times S \times S\ : a \preceq x \circ y \circ z\}$$

Definition 2.5. Let f_S, g_S and h_S be soft sets over the common universe U. Then, int-soft product $f_S \mathbin{\hat{\diamond}} g_S \mathbin{\hat{\diamond}} h_S$ is defined by

$$(f_S \mathbin{\hat{\diamond}} g_S \mathbin{\hat{\diamond}} h_S)(a) = \begin{cases} \bigcup\limits_{(x,y,z) \in A_a} \{f_S(x) \cap g_S(y) \cap h_S(z)\}, & \text{if } A_a \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 2.6. Let A be any non empty subset of S. Recall that, we denote by S_A the soft characteristic function of A and define as follows:

$$S_A : S \rightarrow \mathcal{P}(U) \ , \ a \mapsto \begin{cases} U, & \text{if } a \in A, \\ \emptyset, & \text{if } a \notin A. \end{cases}$$

It is obvious that the soft characteristic function is a soft set over U.

The soft set S_S, where $\forall\ a \in S, S_S(a) = U$, is called the identity soft set over U. We denote it by $S_S = \mathbb{S}$, that is, $\forall\ a \in S, \mathbb{S}(a) = U$.

Definition 2.7. Let S be a po-ternary semihypergroup and f_S be a soft set over U. Then, f_S is called

(f_1) an int-soft ternary subsemihypergroup of S,if for all $a, b, c \in S$, the following statements hold:
 (1) $\bigcap_{y \in a \circ b \circ c} f_S(y) \supseteq f_S(a) \cap f_S(b) \cap f_S(c)$;
 (2) $a \leq b$ implies $f_S(a) \supseteq f_S(b)$.

(f_2) an int-soft left hyperideal of S if for all $a, b, c \in S$, the following statements hold:
 (1) $\bigcap_{y \in a \circ b \circ c} f_S(y) \supseteq f_S(c)$;
 (2) $a \leq b$ implies $f_S(a) \supseteq f_S(b)$.

(f_3) an int-soft right hyperideal of S if for all $a, b, c \in S$, the following statements hold:
 (1) $\bigcap_{y \in a \circ b \circ c} f_S(y) \supseteq f_S(a)$;
 (2) $a \leq b$ implies $f_S(a) \supseteq f_S(b)$.

(f_4) an int-soft lateral hyperideal of S if for all $a, b, c \in S$, the following statements hold:
 (1) $\bigcap_{y \in a \circ b \circ c} f_S(y) \supseteq f_S(b)$;
 (2) $a \leq b$ implies $f_S(a) \supseteq f_S(b)$.

(f_5) an int-soft hyperideal of S if f_S is an int-soft left hyperideal, an int-soft lateral hyperideal and an int-soft right hyperideal of S.

Definition 2.8. An int-soft po-ternary semihypergroup f_S over U is called an ISBH of S over U if for all $a, b, c, d, e \in S$ the following statements hold:
 (1) $\bigcap_{y \in a \circ b \circ c \circ d \circ e} f_S(y) \supseteq f_S(a) \cap f_S(c) \cap f_S(e)$;
 (2) $a \leq b$ *implies* $f_S(a) \supseteq f_S(b)$.

Example 2.1. Let $(S, \circ, \leq)$ be a po-ternary semihypergroup on $S = \{a_1, a_2, a_3, a_4\}$ with the ternary hyperoperation $\circ$ is given by $(x \circ y \circ z) = (x \circ y) \circ z$, where $\circ$ is the binary hyperoperation given by the table

$\circ$	a_1	a_2	a_3	a_4
a_1	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$
a_2	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$
a_3	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_1, a_2\}$
a_4	$\{a_1\}$	$\{a_1\}$	$\{a_1, a_2\}$	$\{a_1, a_2, a_3\}$

Order relation is defined by $\leq := \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_2, a_3), (a_2, a_4)\}$. We give the covering relation "$\prec$" of S as follows:

$$\prec = \{(a_2, a_3), (a_2, a_4)\}.$$

Let $U = Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$, quaternion group of order 8, be the universal set. Define the soft set f_S over $U = Q_8$ such that $f_S(a_1) = Q_8, f_S(a_2) = \{1, -1, i, -i\}, f_S(a_3) = \{1, -1\}$ and $f_S(a_4) = \emptyset$. Then, f_S is an ISBH of S over U.

Theorem 2.1. [18] *Let f_S be a soft set over U. Then, f_S is an int-soft ternary semihypergroup over U if and only if for all $a, b \in S$, we have*
(1) $f_S \hat{\diamond} f_S \hat{\diamond} f_S \sqsubseteq f_S$.
(2) *If $a \leq b$ then $f_S(a) \supseteq f_S(b)$.*

Theorem 2.2. *Let f_S be a soft set over U. Then, f_S is an ISBH of S over U if and only if*
(1) $f_S \hat{\diamond} f_S \hat{\diamond} f_S \sqsubseteq f_S$ and $f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \sqsubseteq f_S$.
(2) *If $a \leq b$ then $f_S(a) \supseteq f_S(b)$ $\forall$ $a, b \in S$.*

Proof. Suppose that f_S is an ISBH of S over U. *(2)* is straightforward by Definition 2.8. We claim that $f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \sqsubseteq f_S$. To prove the claim, let $x \in S$. If $A_x \neq \emptyset$. Then there exist $a, b, c \in S$ such that $x \preceq a \circ b \circ c$, and let $(p, q, r) \in A_a$ i.e $a \preceq p \circ q \circ r$ for any $p, q, r \in S$. Then $x \preceq p \circ q \circ r \circ b \circ c$ and there exists $y \in p \circ q \circ r \circ b \circ c$ such that $x \leq y$. Since f_S is an ISBH of S over U, then $f_S(x) \supseteq f_S(y) \supseteq \bigcap_{y \in p \circ q \circ r \circ b \circ c} \supseteq f_S(p) \cap f_S(r) \cap f_S(c)$. Thus

$$\begin{aligned}
(f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(x) &= \bigcup_{x \preceq a \circ b \circ c} \{(f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(a) \cap \mathbb{S}(b) \cap f_S(c)\} \\
&= \bigcup_{x \preceq a \circ b \circ c} \{\{\bigcup_{a \preceq p \circ q \circ r} (f_S(p) \cap \mathbb{S}(q) \cap f_S(r))\} \cap \mathbb{S}(b) \cap f_S(c)\} \\
&= \bigcup_{x \preceq a \circ b \circ c} \{\{\bigcup_{a \preceq p \circ q \circ r} (f_S(p) \cap f_S(r))\} \cap f_S(c)\} \\
&= \bigcup_{x \preceq p \circ q \circ r \circ b \circ c} \{\bigcap_{y \in p \circ q \circ r \circ b \circ c} f_S(y)\} \\
&\subseteq f_S(x).
\end{aligned}$$

If $A_x = \emptyset$. Then, $(f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(x) = \emptyset \subseteq f_S(x)$.
Hence, $f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \sqsubseteq f_S$.

Conversely, assume that the given conditions hold. Let $a, b, c, d, e \in S$. Then it is sufficient to show that $f_S(y) \supseteq f_S(a) \cap f_S(c) \cap f_S(e)$, for any $y \in a \circ b \circ c \circ d \circ e$. Since $y \leq y$ and $y \in a \circ b \circ c \circ d \circ e$, then we have $y \preceq a \circ b \circ c \circ d \circ e$. Thus by hypothesis, we have

$$\begin{aligned}
f_S(y) &\supseteq (f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(y) \\
&= \bigcup_{y \preceq p \circ q \circ r} \{(f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(p) \cap \mathbb{S}(q) \cap f_S(r)\} \\
&\supseteq \bigcap_{x \in a \circ b \circ c} \{(f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(x) \cap \mathbb{S}(d) \cap f_S(e)\} \\
&= \bigcap_{x \in a \circ b \circ c} \{\{\bigcup_{x \preceq l \circ m \circ n} f_S(l) \cap \mathbb{S}(m) \cap f_S(n)\} \cap \mathbb{S}(d) \cap f_S(e)\} \\
&\supseteq f_S(a) \cap \mathbb{S}(b) \cap f_S(c) \cap \mathbb{S}(d) \cap f_S(e) \\
&= f_S(a) \cap f_S(c) \cap f_S(e) \textit{ for every } a, b, c, d, e \in S \text{ and } y \in a \circ b \circ c \circ d \circ e.
\end{aligned}$$

Hence, $\bigcap_{y\in a\circ b\circ c\circ d\circ e} f_S(y) \supseteq f_S(a)\cap f_S(c)\cap f_S(e)$ for all $a,b,c,d,e\in S$. The rest of the proof is the consequence of the Theorem 2.1. □

Theorem 2.3. *Let A be a nonempty subset of a po-ternary semihypergroup S. Then, A is a bi-hyperideal S if and only if S_A is an ISBH of S over U.*

Proof. Suppose A is a bi-hyperideal of S, that is, $A\circ A\circ A\subseteq A$ and $A\circ S\circ A\circ S\circ A\subseteq A$. Then, we have

$$S_A \hat{\diamond} S_A \hat{\diamond} S_A = S_{(A\circ A\circ A]} \sqsubseteq S_{(A]} = S_A.$$

Thus, S_A is an int-soft semihypergroup over U. Moreover;

$$S_A \hat{\diamond} \mathbb{S} \hat{\diamond} S_A \hat{\diamond} \mathbb{S} \hat{\diamond} S_A = S_A \hat{\diamond} S_S \hat{\diamond} S_A \hat{\diamond} S_S \hat{\diamond} S_A = S_{(A\circ S\circ A\circ S\circ A]} \sqsubseteq S_{(A]} = S_A.$$

Next, let $a,b\in S$ such that $a\le b$. If $b\notin A$, then $S_A(b)=\emptyset\subseteq S_A(a)$. If $b\in A$, since A is a bi-hyperideal of S, then we have $a\in A$ and so $S_A(a)=U\supseteq S_A(b)\ \forall\, a,b\in S$.

Hence, by Theorem 2.2, S_A is an ISBH of S over U.

Conversely, let S_A be an ISBH of S over U. It means that S_A is an int-soft semihypergroup over U. Let $a\in A\circ A\circ A$. Then,

$$U = S_{(A\circ A\circ A)}(a) = S_{(A\circ A\circ A]}(a) = S_A \hat{\diamond} S_A \hat{\diamond} S_A(a) \subseteq S_A(a).$$

and so $a\in A$. Thus, $A\circ A\circ A\subseteq A$ and A is a subsemihypergroup of S. Moreover; let $b\in A\circ S\circ A\circ S\circ A$. Thus,

$$\begin{aligned} U &= S_{(A\circ S\circ A\circ S\circ A)}(b) = S_{(A\circ S\circ A\circ S\circ A]}(b) = (S_A \hat{\diamond} S_S \hat{\diamond} S_A \hat{\diamond} S_S \hat{\diamond} S_A)(b) \\ &= (S_A \hat{\diamond} \mathbb{S} \hat{\diamond} S_A \hat{\diamond} \mathbb{S} \hat{\diamond} S_A)(b) \subseteq S_A(b). \end{aligned}$$

and so $b\in A$. Thus, $A\circ S\circ A\circ S\circ A\subseteq A$ and A is a bi-hyperideal of S. Next; let $a\in A$ such that $S\ni b\le a$. Since S_A is a ISBH of S, then we have $S_A(b)\supseteq S_A(a)=U$ and so $S_A(b)=U$. Thus, $b\in A$. Hence, A is a bi-hyperideal of S. □

Theorem 2.4. *Let S be a po-ternary semihypergroup and f_S be an int-soft hyperideal of S over U. Then f_S is an ISBH of S over U.*

Proof. Let $a,b,c,d,e\in S$. Since f_S is an int-soft hyperideal of S over U, then for any $y\in a\circ b\circ c\circ d\circ e$, we have $\bigcap_{y\in a\circ b\circ c\circ d\circ e} f_S(y) = \bigcap_{y\in (a\circ b\circ c)\circ d\circ e} f_S(y) \supseteq f_S(e) \supseteq f_S(a)\cap f_S(c)\cap f_S(e)$. Thus, f_S is an ISBH of S over U. The converse of the Proposition 2.4, is not true in general. We can illustrate it by the following example. □

Example 2.2. Let $(S,\circ,\le)$ be a po-ternary semihypergroup on $S=\{a_1,a_2,a_3,a_4\}$ with the ternary hyperoperation $\circ$ is given by $(a\circ b\circ c)=(a\circ b)\circ c$, where $\circ$ is the binary hyperoperation given by the table

$\circ$	a_1	a_2	a_3	a_4
a_1	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$
a_2	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$
a_3	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_1,a_2\}$
a_4	$\{a_1\}$	$\{a_1\}$	$\{a_1,a_2\}$	S

The order relation is defined by

$$\le := \{(a_1,a_1),(a_2,a_2),(a_3,a_3),(a_4,a_4),(a_1,a_2),(a_1,a_3),(a_1,a_4),(a_3,a_4)\}.$$

Let $U = D_3 = \{R_0, R_{120}, R_{240}, f_1, f_2, f_3\}$,dihedral group of order 6, be the universal set. If we construct a soft set such that $f_S(a_1) = D_3, f_S(a_2) = \{R_0, R_{120}, R_{240}\}, f_S(a_3) = \{R_0, f_1\}$ and $f_S(a_4) = \emptyset$. Then, f_S is an int-soft bi-hyperideal of S. But f_S is not an int-soft left hyperideal of S over U as: $a_4 \circ a_4 \circ a_3 = \{a_1, a_2, a_3, a_4\} \circ a_3 = a_1 \cup a_1 \cup a_1 \cup \{a_1, a_2\} = \{a_1, a_2\}$. Therefore, $\bigcap_{y \in a_4 \circ a_4 \circ a_3} f_S(y) = f_S(a_1) \cap f_S(a_2) = f_S(a_2) = \{R_0, R_{120}, R_{240}\} \nsupseteq \{R_0, f_1\} = f_S(a_3)$.

Definition 2.9. An element $a \in S$ is called regular if there exists an element $x \in S$ such that $a \preceq a \circ x \circ a$. If every element of S is regular, then S is called regular po-ternary semihypergroup.

Theorem 2.5. *The following conditions in a po-ternary semihypergroup S are equivalent:*
(1) *S is regular.*
(2) *$f_S = f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S$, for every ISBH f_S of S over U.*

Proof. Suppose (1) holds. Let f_S be any ISBH f_S of S over U and a be any element of S. Then, since S is regular, there exists an element $x \in S$ such that $a \preceq a \circ x \circ a$. So $a \preceq a \circ x \circ a \circ x \circ a$ and we have

$$\begin{aligned}
(f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(a) &= \bigcup_{a \preceq p \circ q \circ r} \{(f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(p) \cap \mathbb{S}(q) \cap f_S(r)\} \text{ for some } p, q, r \in S \\
&\supseteq \bigcap_{y \in a \circ x \circ a} \{(f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S)(y)\} \cap \mathbb{S}(x) \cap f_S(a) \\
&= \bigcap_{y \in a \circ x \circ a} \{ \bigcup_{y \preceq l \circ m \circ n} f_S(l) \cap \mathbb{S}(m) \cap f_S(n)\} \cap \mathbb{S}(x) \cap f_S(a) \\
&\supseteq f_S(a) \cap \mathbb{S}(x) \cap f_S(a) \cap \mathbb{S}(x) \cap f_S(a) \\
&= f_S(a) \cap f_S(a) \cap f_S(a) = f_S(a).
\end{aligned}$$

Thuerefore, $f_S \sqsubseteq f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S$ and hence by Theorem 2.2, $f_S = f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S \hat{\diamond} \mathbb{S} \hat{\diamond} f_S$ for every ISBH f_S of S over U.

Conversely, let (2) holds and consider f_S, g_S, h_S be respectively, int-soft right hyperideal, int-soft lateral hyperideal and int-soft left hyperideal of S over U. Then $f_S \sqcap g_S \sqcap h_S$ is an ISBH of S over U. Therefore, by (2),

$$\begin{aligned}
f_S \sqcap g_S \sqcap h_S &= (f_S \sqcap g_S \sqcap h_S) \hat{\diamond} \mathbb{S} \hat{\diamond} (f_S \sqcap g_S \sqcap h_S) \hat{\diamond} \mathbb{S} \hat{\diamond} (f_S \sqcap g_S \sqcap h_S) \\
&\sqsubseteq f_S \hat{\diamond} (\mathbb{S} \hat{\diamond} g_S \hat{\diamond} \mathbb{S}) \hat{\diamond} h_S \\
&\sqsubseteq f_S \hat{\diamond} g_S \hat{\diamond} h_S \subseteq (f_S \hat{\diamond} g_S \hat{\diamond} h_S]
\end{aligned}$$

Further by Theorem 2.23[18], $(f_S \hat{\diamond} g_S \hat{\diamond} h_S] \sqsubseteq f_S \sqcap g_S \sqcap h_S$. Therefore, $(f_S \hat{\diamond} g_S \hat{\diamond} h_S] = f_S \sqcap g_S \sqcap h_S$ and hence by Theorem 2.25[18], S is regular. □

Theorem 2.6. *Let $\{f_{S_i} \mid i \in I\}$ be a family of ISBHs of a po-ternary semihypergroup of S over U. Then $f_S = \bigcap_{i \in I} f_{S_i}$ is an ISBHs of a po-ternary semihypergroup of S over U where $(\bigcap_{i \in I} f_{S_i})(x) = \bigcap_{i \in I} (f_{S_i})(x)$.*

Proof. Let $a, b, c \in S$. Then, since each $\{f_{S_i} \mid i \in I\}$ is an ISBHs of S over U, so $\bigcap_{y \in a \circ b \circ c} f_{S_i}(y) \supseteq f_{S_i}(a) \cap f_{S_i}(b) \cap f_{S_i}(c)$. Thus for any $y \in a \circ b \circ c$, $f_{S_i}(y) \supseteq$

$f_{S_i}(a)\cap f_{S_i}(b)\cap f_{S_i}(c)$ and we have $f_S(y)=\big(\bigcap_{i\in I} f_{S_i}\big)(y)=\bigcap_{i\in I}\big(f_{S_i}(y)\big)\supseteq\bigcap_{i\in I}\big(f_{S_i}(a)\cap f_{S_i}(b)\cap f_{S_i}(c)\big)=\big(\bigcap_{i\in I}\big(f_{S_i}(a)\big)\big)\bigcap\big(\bigcap_{i\in I}\big(f_{S_i}(b)\big)\big)\bigcap\big(\bigcap_{i\in I}\big(f_{S_i}(c)\big)\big)=\big(\bigcap_{i\in I} f_{S_i}\big)(a)\bigcap\big(\bigcap_{i\in I} f_{S_i}\big)(b)\bigcap\big(\bigcap_{i\in I} f_{S_i}\big)(c)=f_S(a)\cap f_S(b)\cap f_S(c)$, which implies that $\bigcap_{y\in a\circ b\circ c} f_S(y)\supseteq f_S(a)\cap f_S(b)\cap f_S(c)$. Let $a,b,c,d,e\in S$ and $\bigcap_{x\in a\circ b\circ c\circ d\circ e} f_{S_i}(x)\supseteq f_{S_i}(a)\cap f_{S_i}(c)\cap f_{S_i}(e)$. Thus for any $x\in a\circ b\circ c\circ d\circ e$, $f_{S_i}(x)\supseteq f_{S_i}(a)\cap f_{S_i}(c)\cap f_{S_i}(e)$. Then $f_S(x)=\big(\bigcap_{i\in I} f_{S_i}\big)(x)=\bigcap_{i\in I}\big(f_{S_i}(x)\big)\supseteq\bigcap_{i\in I}\{f_{S_i}(a)\cap f_{S_i}(c)\cap f_{S_i}(e)\}=\big(\bigcap_{i\in I}\big(f_{S_i}(a)\big)\big)\cap\big(\bigcap_{i\in I}\big(f_{S_i}(c)\big)\big)\cap\big(\bigcap_{i\in I}\big(f_{S_i}(e)\big)\big)=\big(\bigcap_{i\in I} f_{S_i}\big)(a)\cap\big(\bigcap_{i\in I} f_{S_i}\big)(c)\cap\big(\bigcap_{i\in I} f_{S_i}\big)(e)=f_S(a)\cap f_S(c)\cap f_S(e)$. Thus $\bigcap_{x\in a\circ b\circ c\circ d\circ e} f_S(x)\supseteq f_S(a)\cap f_S(c)\cap f_S(e)$. Furthermore, if $a\leq b$, we will prove $f_S(a)\supseteq f_S(b)$. Since every $f_{S_i},(i\in I)$ is an ISBHs of a po-ternary semihypergroup of S over U, then it can be obtained that $f_{S_i}(a)\supseteq f_{S_i}(b)$ for all $i\in I$. Thus $f_S(a)=\big(\bigcap_{i\in I} f_{S_i}\big)(a)=\bigcap_{i\in I}\big(f_{S_i}(a)\big)\supseteq\bigcap_{i\in I}\big(f_{S_i}(b)=\big(\bigcap_{i\in I} f_{S_i}\big)(b)=f_S(b)$. Hence f_S is an ISBHs of a po-ternary semihypergroup of S over U. □

Definition 2.10. Let f_S be a soft set of a po-ternary semihypergroup S over U and $\delta\in U$. Then δ-inclusion of f_S, denoted by $\mathcal{U}(f_S,\delta)$, is defined as

$$\mathcal{U}(f_S,\delta)=\{x\in S\ :\ f_S(x)\supseteq\delta\}.$$

Theorem 2.7. *Let f_S be a soft set of a po-ternary semihypergroup S over U and $\delta\in\mathcal{P}(U)$. Then f_S is an ISBH of S over U if and only if each nonempty δ-inclusive set $\mathcal{U}(f_S,\delta)$ is a bi-hyperideal of S.*

Proof. Suppose f_S is an ISBH of S over U. Let $\delta\in\mathcal{P}(U)$ such that $\mathcal{U}(f_S,\delta)\neq\emptyset$. Let $a,b,c\in\mathcal{U}(f_S,\delta)$. Then $f_S(a)\supseteq\delta, f_S(b)\supseteq\delta$ and $f_S(c)\supseteq\delta$. By hypothesis, we have $\bigcap_{y\in a\circ b\circ c} f_S(y)\supseteq f_S(a)\cap f_S(b)\cap f_S(c)\supseteq\delta\cap\delta\cap\delta=\delta$. Thus for any $y\in a\circ b\circ c$, we have $f_S(y)\supseteq\delta$, implies that $y\in\mathcal{U}(f_S,\delta)$. It follows that $a\circ b\circ c\subseteq\mathcal{U}(f_S,\delta)$. Hence $\mathcal{U}(f_S,\delta)$ is a ternary subsemihypergroup of S. Let $a,b,c\in\mathcal{U}(f_S,\delta)$ and $x,z\in S$. Then $f_S(a)\supseteq\delta, f_S(b)\supseteq\delta, f_S(c)\supseteq\delta$. Since f_S is an ISBH of S over U, then $\bigcap_{y\in a\circ x\circ b\circ z\circ c} f_S(y)\supseteq f_S(a)\cap f_S(b)\cap f_S(c)\supseteq\delta\cap\delta\cap\delta=\delta$. Hence $f_S(y)\supseteq\delta$ for any $y\in a\circ x\circ b\circ z\circ c$ implies that $y\in\mathcal{U}(f_S,\delta)$. Thus $A\circ\mathcal{U}(f_S,\delta)\circ A\circ\mathcal{U}(f_S,\delta)\circ A\subseteq\mathcal{U}(f_S,\delta)$. Let $a\in\mathcal{U}(f_S,\delta)$ *and* $b\in S$ *with* $b\leq a$. Then $\delta\subseteq f_S(a)\subseteq f_S(b)$, we get $b\in\mathcal{U}(f_S,\delta)$. Therefore $\mathcal{U}(f_S,\delta)$ is a bi-hyperideal of S.

Conversely, suppose that $\mathcal{U}(f_S,\delta)\neq\emptyset$ is a bi-hyperideal of S. If $\bigcap_{y\in a\circ b\circ c} f_S(y)\subset f_S(a)\cap f_S(b)\cap f_S(c)$ for some $a,b,c\in S$, then there exists $\delta\in\mathcal{P}(U)$ such that $\bigcap_{y\in a\circ b\circ c} f_S(y)\subset\delta\subseteq f_S(a)\cap f_S(b)\cap f_S(c)$, which implies that $a,b,c\in\mathcal{U}(f_S,\delta)$ and $a\circ b\circ c\nsubseteq\mathcal{U}(f_S,\delta)$. It contradicts the fact that $\mathcal{U}(f_S,\delta)$ is a bi-hyperideal of S. Consequently, $\bigcap_{y\in a\circ b\circ c} f_S(y)\supseteq f_S(a)\cap f_S(b)\cap f_S(c)$ for all $a,b,c\in S$. Next we show that $\bigcap_{y\in a\circ b\circ c\circ d\circ e} f_S(y)\supseteq f_S(a)\cap f_S(c)\cap f_S(e)$ for all $a,b,c,d,e\in S$. Choose $f_S(a)\cap$

$f_S(c) \cap f_S(e) = \delta$, then $\delta \subseteq f_S(a), \delta \subseteq f_S(c), \delta \subseteq f_S(e)$, which implies that $a \in \mathcal{U}(f_S, \delta), c \in \mathcal{U}(f_S, \delta), e \in \mathcal{U}(f_S, \delta)$. Since $\mathcal{U}(f_S, \delta)$ is a bi-hyperideal of S, then we get $a \circ b \circ c \circ d \circ e \subseteq \mathcal{U}(f_S, \delta)$. Then for every $y \in a \circ b \circ c \circ d \circ e$, we have $f_S(y) \supseteq \delta$ and so $f_S(a) \cap f_S(c) \cap f_S(e) = \delta \subseteq \bigcap_{y \in a \circ b \circ c \circ d \circ e} f_S(y)$. Let $a, b \in S$ such that $a \leq b$. If $f_S(b) = \delta$ then $b \in \mathcal{U}(f_S, \delta)$. Since $\mathcal{U}(f_S, \delta)$ is a bi-hyperideal of S, then we get $a \in \mathcal{U}(f_S, \delta)$. So $f_S(a) \supseteq \delta = f_S(b)$. Therefore f_S is a ISBH of S over U. □

Example 2.3. Let $S = \{x, y, z, w\}$ be the po-ternary semihypergroup in Example 2.1 and f_S be a soft set over $U = \{a, b, c\}$. If we define a soft set f_S over U such that $f_S(x) = U, f_S(y) = \{a, b\}, f_S(z) = \{b, c\}$ *and* $f_S(w) = \emptyset$. Then, f_S is an ISBH of S over U. Then

$$\mathcal{U}(f_S, \delta) = \begin{cases} \{x, y\}, & \text{if } \delta = \{a\} \\ \{x, y, z\}, & \text{if } \delta = \{b\} \\ \{x, z\}, & \text{if } \delta = \{c\} \\ \{x, y\}, & \text{if } \delta = \{a, b\} \\ \{x\}, & \text{if } \delta = \{a, c\} \\ \{x, z\}, & \text{if } \delta = \{b, c\} \\ \{x\}, & \text{if } \delta = \{a, b, c\}. \end{cases}$$

So by Theorem 2.7, each δ-inclusive set $\mathcal{U}(f_S, \delta)$ is a bi-hyperideal of S.

For any $a \in S$, let S be a po-ternary semihypergroup and f_S be a soft set over U. We denote by I_a the subset of S defines as follows:

$$I_a = \{b \in S : f_S(b) \supseteq f_S(a)\}.$$

Theorem 2.8. *Let S be a po-ternary semihypergroup and f_S be an ISBH of S over U. Then I_a is a bi-hyperideal of S for every $a \in S$.*

Proof. Let $a \in S$. First of all $\emptyset \neq I_a \subseteq S$. since $a \in I_a$. Let $p, q \in S$ and $x, y, z \in I_a$. Since f_S is an ISBH of S over U and $p, q, x, y, z \in S$, then we have $\bigcap_{w \in x \circ p \circ y \circ q \circ z} f_S(w) \supseteq f_S(x) \cap f_S(y) \cap f_S(z)$. Since $x, y, z \in I_a$, then it follows that $f_S(x) \supseteq f_S(a), f_S(y) \supseteq f_S(a)$ and $f_S(z) \supseteq f_S(a)$. Thus $\bigcap_{w \in x \circ p \circ y \circ q \circ z} f_S(w) \supseteq f_S(a)$, implies that $f_S(w) \supseteq f_S(a)$, so $w \in I_a$ and so $x \circ p \circ y \circ q \circ z \subseteq I_a$. Let $b \in I_a$ and $c \in S$ with $c \leq b$. Since f_S is an ISBH of S over U and $b, c \in S$ with $c \leq b$, then we have $f_S(c) \supseteq f_S(b) \supseteq f_S(a)$. So $c \in I_a$, hence I_a is a bi-hyperideal of S for every $a \in S$. □

Example 2.4. Let $S = \{x, y, z, w\}$ be the po-ternary semihypergroup and f_S is an ISBH of S over U in Example 2.1. Then

$$I_x = \{x\},\ I_y = \{x, y\},\ I_z = \{x, y, z\},\ I_w = S.$$

So by Theorem 2.8, each $I_a, a \in S$ is a bi-hyperideal of S.

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