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On dual jet N-linear connections in the time-dependent Hamilton geometry

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Abstract. In this paper we study the local adapted components of the N-linear connections on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$, together with its local adapted torsion and curvature d-tensors.

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1. Introduction

According to Olver's opinion [7], we consider that the 1-jet spaces and their duals are the fundamental ambient mathematical spaces used in the study of classical and quantum field theories in their Lagrangian and Hamiltonian approaches. For this reason, we start our geometrical study considering a smooth real manifold M^n of dimension n, whose local coordinates are $(x^i)_{i=\overline{1,n}}$, and we construct the dual 1-jet vector bundle (as time-dependent phase space of momenta [2], [6])

$$J^{1*}(\mathbb{R}, M) \equiv \mathbb{R} \times T^*M \to \mathbb{R} \times M,$$

whose local coordinates are denoted by (t, x^i, p_i^1). The transformations of coordinates $(t, x^i, p_i^1) \longleftrightarrow (\tilde{t}, \tilde{x}^i, \tilde{p}_i^1)$ on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$ are

$$\tilde{t} = \tilde{t}\,(t)\,, \quad \tilde{x}^i = \tilde{x}^i\left(x^j\right), \quad \tilde{p}_i^1 = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{d\tilde{t}}{dt}p_j^1, \tag{1}$$

where $d\tilde{t}/dt \neq 0$ and $\det(\partial \tilde{x}^i/\partial x^j) \neq 0$. Consequently, in our dual jet geometrical approach, we use a "relativistic" time t. Comparatively, in Atanasiu, Miron and his co-workers' Hamiltonian approach (see [1], [4] and [5]), the authors use the trivial bundle $\mathbb{R} \times T^*M$ over the base cotangent space T^*M, whose coordinates induced by T^*M are $\left(t, x^i, p_i\right)$. Thus, the changes of coordinates on the trivial bundle

$$\mathbb{R} \times T^*M \to T^*M$$

are given by

$$\tilde{t} = t, \quad \tilde{x}^i = \tilde{x}^i\left(x^j\right), \quad \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i}p_j, \tag{2}$$

pointing out the absolute character of the time variable t.

In order to point out the more naturalness of our dual jet approach of time-dependent Hamilton geometry, we underline that, from a geometrical point of view,

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the time-dependent Lagrangian theory from [5] relies on the geometrical study of the energy action integral

$$\mathbf{E}_1(c(t)) = \int_a^b L(t, x^i(t), y^i = \dot{x}^i(t))dt$$

which has the impediment that it is dependent by the reparametrizations $t \longleftrightarrow \tilde{t}$ of the same curve c. This is because $L(t, x^i, y^i)$ is a function on the vector bundle $\mathbb{R} \times TM \to M$. This inconvenience is removed in the Finsler geometry by imposing the 1-positive homogeneity condition $L(t, x^i, \lambda y^i) = \lambda L(t, x^i, y^i)$, $\forall\, \lambda > 0$. The second way to remove this inconvenience of dependence of reparametrizations of the energy action integral is to use the 1-jet space $J^1(\mathbb{R}, M) \equiv \mathbb{R} \times TM$ and the energy action integral (see [3])

$$\mathbf{E}_2(c(t)) = \int_a^b L(t, x^i(t), y_1^i = \dot{x}^i(t))\sqrt{|h_{11}(t)|}dt,$$

where $L(t, x^i, y_1^i)$ is a function on the 1-jet vector bundle $J^1(\mathbb{R}, M) \longrightarrow \mathbb{R} \times M$ and h_{11} is a semi-Riemannian metric on the time manifold $\mathbb{R}$. Taking into account that, via the Legendre duality of the Hamilton spaces with the Lagrange spaces, in the book [5] is shown that the theory of Hamilton spaces has the same symmetry as the Lagrange geometry, giving thus a geometrical framework for the Hamiltonian theory of Analytical Mechanics, it follows that the more natural house for the study of the time-dependent Hamilton geometry is the dual 1-jet space $J^{1*}(\mathbb{R}, M)$ which provides an energy action integral independent by temporal reparametrizations of the same curve.

The subsequent development of the time-dependent Hamilton geometry relies on the following geometrical constructions: (1) the writing of the time dependent Hamiltonian H associated with the time-dependent Lagrangian function $L(t, x^i, y_1^i)$; (2) the producing of a natural dual jet Hamiltonian nonlinear connection N (provided only by the Hamiltonian H and intimately connected with the canonical nonlinear connection produced by the Lagrangian function L, via its Euler-Lagrange equations); (3) the construction of a natural Cartan canonical N-linear connection $C\Gamma(N)$ on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$; (4) the computations of the adapted components of the d-torsions and d-curvatures associated with the Cartan connection $C\Gamma(N)$. Consequently, the present paper is only a step in the forthcoming time-dependent Hamilton geometry, creating geometrical foundations for the subsequent theory.

In this way, as an example, we will study in a subsequent paper, the *dual jet time-dependent Hamiltonian of electrodynamics* (see [5] and [2])

$$H = \frac{1}{4mc}h_{11}(t)\varphi^{ij}(x)p_i^1 p_j^1 - \frac{e}{m^2 c}A_{(1)}^{(i)}(x)p_i^1 + \frac{e^2}{m^3 c}F(t, x), \tag{3}$$

where $A_{(1)}^{(i)}(x)$ is a d-tensor on $J^{1*}(\mathbb{R}, M)$ having the physical meaning of a potential d-tensor of an electromagnetic field, e is the charge of the test body and the function $F(t, x)$ is given by $F(t, x) = h^{11}(t)\varphi_{ij}(x)A_{(1)}^{(i)}(x)A_{(1)}^{(j)}(x)$. This Hamiltonian is important because it naturally generalizes (in a time-dependent way) the Hamiltonian that governs the physical domain of the autonomous (i.e., time-independent) electrodynamics. The geometrization associated with this time-dependent Hamiltonian will consists of a canonical nonlinear connection N, a Cartan canonical N-linear connection $C\Gamma(N)$ together with its adapted d-torsions and d-curvatures. All these geometrical objects are provided only by the initial time-dependent Hamiltonian (3).

2. Nonlinear connections and adapted bases

In what follows, in order to locally study the linear connections on the dual 1-jet space $J^{1*}(\mathbb{R},M)$, we recall that a pair of local functions $N=\left(\underset{1}{N}{}^{(1)}_{(k)1},\underset{2}{N}{}^{(1)}_{(k)i}\right)$ on $J^{1*}(\mathbb{R},M)$, which transform by the rules (see [6])

$$\begin{aligned}\underset{1}{\widetilde{N}}{}^{(1)}_{(j)1}&=\underset{1}{N}{}^{(1)}_{(k)1}\frac{\partial x^k}{\partial\tilde{x}^j}-\frac{dt}{d\tilde{t}}\frac{\partial\tilde{p}^1_j}{\partial t},\\ \underset{2}{\widetilde{N}}{}^{(1)}_{(j)r}&=\underset{2}{N}{}^{(1)}_{(k)i}\frac{d\tilde{t}}{dt}\frac{\partial x^k}{\partial\tilde{x}^j}\frac{\partial x^i}{\partial\tilde{x}^r}-\frac{\partial x^i}{\partial\tilde{x}^r}\frac{\partial\tilde{p}^1_j}{\partial x^i},\end{aligned}\tag{4}$$

is called a *nonlinear connection* on the dual 1-jet bundle $J^{1*}(\mathbb{R},M)$. Moreover, the geometrical entity $\underset{1}{N}=\left(\underset{1}{N}{}^{(1)}_{(j)1}\right)$ (respectively $\underset{2}{N}=\left(\underset{2}{N}{}^{(1)}_{(j)i}\right)$) is called a *temporal* (respectively *spatial*) *nonlinear connection* on $J^{1*}(\mathbb{R},M)$.

Example 2.1. The pair of local functions $\overset{0}{N}=\left(\underset{1}{\overset{0}{N}}{}^{(1)}_{(i)1},\underset{2}{\overset{0}{N}}{}^{(1)}_{(i)j}\right)$, where

$$\underset{1}{\overset{0}{N}}{}^{(1)}_{(i)1}=H^1_{11}p^1_i,\qquad \underset{2}{\overset{0}{N}}{}^{(1)}_{(i)j}=-\gamma^k_{ij}p^1_k,\tag{5}$$

is called the *canonical nonlinear connection* on $J^{1*}(R,M)$, *associated with the pair of semi-Riemannian metrics* $(h_{11}(t),\varphi_{ij}(x))$. Note that $H^1_{11}(t)$ (respectively $\gamma^k_{ij}(x)$) are the Christoffel symbols attached to the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$.

The nonlinear connection $N=\left(\underset{1}{N}{}^{(1)}_{(k)1},\underset{2}{N}{}^{(1)}_{(k)i}\right)$ is useful in order to construct the *adapted bases of vector and covector fields*, namely

$$\left\{\frac{\delta}{\delta t},\frac{\delta}{\delta x^i},\frac{\partial}{\partial p^1_i}\right\}\subset\mathcal{X}\left(J^{1*}(\mathbb{R},M)\right),\qquad\left\{dt,dx^i,\delta p^1_i\right\}\subset\mathcal{X}^*\left(J^{1*}(\mathbb{R},M)\right),\tag{6}$$

where

$$\begin{gathered}\frac{\delta}{\delta t}=\frac{\partial}{\partial t}-\underset{1}{N}{}^{(1)}_{(j)1}\frac{\partial}{\partial p^1_j},\qquad\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-\underset{2}{N}{}^{(1)}_{(j)i}\frac{\partial}{\partial p^1_j},\\ \delta p^1_i=dp^1_i+\underset{1}{N}{}^{(1)}_{(i)1}dt+\underset{2}{N}{}^{(1)}_{(i)j}dx^j.\end{gathered}\tag{7}$$

Remark 2.1. The adapted bases of vector and covector fields (6) are important because, with respect to the coordinate transformations (1), their elements have the local transformation laws as tensorial ones. For this reason, all future geometrical objects from this paper, such as linear connections, torsions and curvatures, will be locally described in adapted bases.

Obviously, the Lie algebra of vector fields on $J^{1*}(\mathbb{R},M)$ decomposes in the direct sum $\mathcal{X}\left(J^{1*}(\mathbb{R},M)\right)=\mathcal{X}\left(\mathcal{H}_{\mathbb{R}}\right)\oplus\mathcal{X}\left(\mathcal{H}_M\right)\oplus\mathcal{X}\left(\mathcal{W}\right)$, where

$$\mathcal{X}\left(\mathcal{H}_{\mathbb{R}}\right)=Span\left\{\frac{\delta}{\delta t}\right\},\ \mathcal{X}\left(\mathcal{H}_M\right)=Span\left\{\frac{\delta}{\delta x^i}\right\},\ \mathcal{X}\left(\mathcal{W}\right)=Span\left\{\frac{\partial}{\partial p^1_i}\right\},$$

while the Lie algebra of covector fields on $J^{1*}(\mathbb{R},M)$ decomposes in the direct sum $\mathcal{X}^*\left(J^{1*}(\mathbb{R},M)\right)=\mathcal{X}^*\left(\mathcal{H}_{\mathbb{R}}\right)\oplus\mathcal{X}^*\left(\mathcal{H}_M\right)\oplus\mathcal{X}^*\left(\mathcal{W}\right)$, where

$$\mathcal{X}^*\left(\mathcal{H}_{\mathbb{R}}\right)=Span\left\{dt\right\},\ \mathcal{X}^*\left(\mathcal{H}_M\right)=Span\left\{dx^i\right\},\ \mathcal{X}^*\left(\mathcal{W}\right)=Span\left\{\delta p^1_i\right\}.$$

Definition 2.1. The distributions $\mathcal{H}_{\mathbb{R}}$ and $\mathcal{H}_M$ are called the *$\mathbb{R}$-horizontal distribution* and *M-horizontal distribution* on $J^{1*}(\mathbb{R},M)$. The distribution $\mathcal{W}$ is called the *vertical distribution* on $J^{1*}(\mathbb{R},M)$. Moreover, we denote by $h_{\mathbb{R}}$, h_M and w the corresponding projections associated with these distributions.

In applications, the Poisson brackets of the d-vector fields (6) are very important. Consequently, by a direct calculus, we obtain

Proposition 2.1. *The Poisson brackets of the d-vector fields of the adapted basis (6) are given by*

$$\begin{array}{ll}\left[\dfrac{\delta}{\delta t},\dfrac{\delta}{\delta t}\right]=0, & \left[\dfrac{\delta}{\delta t},\dfrac{\delta}{\delta x^k}\right]=R^{(1)}_{(i)1k}\dfrac{\partial}{\partial p^1_i},\\ \left[\dfrac{\delta}{\delta t},\dfrac{\partial}{\partial p^1_k}\right]=B^{(1)\ (k)}_{(i)1(1)}\dfrac{\partial}{\partial p^1_i}, & \left[\dfrac{\delta}{\delta x^j},\dfrac{\delta}{\delta x^k}\right]=R^{(1)}_{(i)jk}\dfrac{\partial}{\partial p^1_i},\\ \left[\dfrac{\delta}{\delta x^j},\dfrac{\partial}{\partial p^1_k}\right]=B^{(1)\ (k)}_{(i)j(1)}\dfrac{\partial}{\partial p^1_i}, & \left[\dfrac{\partial}{\partial p^1_j},\dfrac{\partial}{\partial p^1_k}\right]=0,\end{array}\tag{8}$$

where

$$\begin{array}{ll}R^{(1)}_{(i)1k}=\dfrac{\delta \underset{1}{N}{}^{(1)}_{(i)1}}{\delta x^k}-\dfrac{\delta \underset{2}{N}{}^{(1)}_{(i)k}}{\delta t}, & R^{(1)}_{(i)jk}=\dfrac{\delta \underset{2}{N}{}^{(1)}_{(i)j}}{\delta x^k}-\dfrac{\delta \underset{2}{N}{}^{(1)}_{(i)k}}{\delta x^j},\\ B^{(1)\ (k)}_{(i)1(1)}=\dfrac{\partial \underset{1}{N}{}^{(1)}_{(i)1}}{\partial p^1_k}, & B^{(1)\ (k)}_{(i)j(1)}=\dfrac{\partial \underset{2}{N}{}^{(1)}_{(i)j}}{\partial p^1_k},\end{array}\tag{9}$$

and $\underset{1}{N}{}^{(1)}_{(i)1}$ *and* $\underset{2}{N}{}^{(1)}_{(i)j}$ *are the coefficients of the given nonlinear connection* N.

3. N-linear connections on the dual 1-jet space $E^*=J^{1*}(\mathbb{R},M)$

A linear connection on $E^*=J^{1*}\left(\mathbb{R},M\right)$ is an application $D:\mathcal{X}\left(E^*\right)\times\mathcal{X}\left(E^*\right)\to\mathcal{X}\left(E^*\right)$, $(X,Y)\to D_XY$, having the properties: (1) $D_{X_1+X_2}Y=D_{X_1}Y+D_{X_2}Y$, (2) $D_{fX}Y=fD_XY$, (3) $D_X\left(Y_1+Y_2\right)=D_XY_1+D_XY_2$, (4) $D_X\left(fY\right)=X(f)Y+fD_XY$, where X, X_1, X_2, Y_1, Y_2, $Y\in\mathcal{X}\left(E^*\right)$ and $f\in\mathcal{F}\left(E^*\right)$. Obviously, the linear connection D on E^* can be uniquely determined by *twenty-seven* local coefficients, which are written in the adapted basis (6) in the form:

$$\begin{aligned}D_{\frac{\delta}{\delta t}}\frac{\delta}{\delta t}&=A^1_{11}\frac{\delta}{\delta t}+A^i_{11}\frac{\delta}{\delta x^i}+A^{(1)}_{(i)11}\frac{\partial}{\partial p^1_i},\\ D_{\frac{\delta}{\delta t}}\frac{\delta}{\delta x^j}&=A^1_{j1}\frac{\delta}{\delta t}+A^i_{j1}\frac{\delta}{\delta x^i}+A^{(1)}_{(i)j1}\frac{\partial}{\partial p^1_i},\\ -D_{\frac{\delta}{\delta t}}\frac{\partial}{\partial p^1_j}&=A^{1(j)}_{(1)1}\frac{\delta}{\delta t}+A^{i(j)}_{(1)1}\frac{\delta}{\delta x^i}+A^{(1)(j)}_{(i)(1)1}\frac{\partial}{\partial p^1_i},\end{aligned}\tag{10}$$

$$\begin{aligned} D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta t} &= H^1_{1k}\frac{\delta}{\delta t}+H^i_{1k}\frac{\delta}{\delta x^i}+H^{(1)}_{(i)1k}\frac{\partial}{\partial p^1_i}, \\ D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} &= H^1_{jk}\frac{\delta}{\delta t}+H^i_{jk}\frac{\delta}{\delta x^i}+H^{(1)}_{(i)jk}\frac{\partial}{\partial p^1_i}, \\ -D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial p^1_j} &= H^{1(j)}_{\ \ (1)k}\frac{\delta}{\delta t}+H^{i(j)}_{\ \ (1)k}\frac{\delta}{\delta x^i}+H^{(1)(j)}_{(i)(1)k}\frac{\partial}{\partial p^1_i}, \end{aligned} \tag{11}$$

$$\begin{aligned} D_{\frac{\partial}{\partial p^1_k}}\frac{\delta}{\delta t} &= C^{1(k)}_{1(1)}\frac{\delta}{\delta t}+C^{i(k)}_{1(1)}\frac{\delta}{\delta x^i}+C^{(1)\ (k)}_{(i)1(1)}\frac{\partial}{\partial p^1_i}, \\ D_{\frac{\partial}{\partial p^1_k}}\frac{\delta}{\delta x^j} &= C^{1(k)}_{j(1)}\frac{\delta}{\delta t}+C^{i(k)}_{j(1)}\frac{\delta}{\delta x^i}+C^{(1)\ (k)}_{(i)j(1)}\frac{\partial}{\partial p^1_i}, \\ -D_{\frac{\partial}{\partial p^1_k}}\frac{\partial}{\partial p^1_j} &= C^{1(j)(k)}_{\ \ (1)(1)}\frac{\delta}{\delta t}+C^{i(j)(k)}_{\ \ (1)(1)}\frac{\delta}{\delta x^i}+C^{(1)(j)(k)}_{(i)(1)(1)}\frac{\partial}{\partial p^1_i}. \end{aligned} \tag{12}$$

The big number of the adapted coefficients lead us to construct linear connections whose number of coefficients is less. In this direction, let us consider a nonlinear connection N on E^*.

Definition 3.1. A linear connection D on E^* is called an *N-linear connection* if it preserves by parallelism the $\mathbb{R}$-horizontal, M-horizontal and vertical distributions $\mathcal{H}_\mathbb{R}$, $\mathcal{H}_M$ and $\mathcal{W}$ on E^*.

It is obvious that now an N-linear connection is uniquely described by the adapted basis of vector fields on E^* with *nine* adapted coefficients given by the relations:

$$\begin{aligned} D_{\frac{\delta}{\delta t}}\frac{\delta}{\delta t} &= A^1_{11}\frac{\delta}{\delta t},\ D_{\frac{\delta}{\delta t}}\frac{\delta}{\delta x^j}=A^i_{j1}\frac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta t}}\frac{\partial}{\partial p^1_j}=-A^{(1)(j)}_{(i)(1)1}\frac{\partial}{\partial p^1_i}, \\ D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta t} &= H^1_{1k}\frac{\delta}{\delta t},\ D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j}=H^i_{jk}\frac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial p^1_j}=-H^{(1)(j)}_{(i)(1)k}\frac{\partial}{\partial p^1_i}, \\ D_{\frac{\partial}{\partial p^1_k}}\frac{\delta}{\delta t} &= C^{1(k)}_{1(1)}\frac{\delta}{\delta t},\ D_{\frac{\partial}{\partial p^1_k}}\frac{\delta}{\delta x^j}=C^{i(k)}_{j(1)}\frac{\delta}{\delta x^i},\ D_{\frac{\partial}{\partial p^1_k}}\frac{\partial}{\partial p^1_j}=-C^{(1)(j)(k)}_{(i)(1)(1)}\frac{\partial}{\partial p^1_i}. \end{aligned}$$

Definition 3.2. The local functions

$$D\Gamma(N)=\left(A^1_{11},A^i_{j1},-A^{(1)(j)}_{(i)(1)1},H^1_{1k},H^i_{jk},-H^{(1)(j)}_{(i)(1)k},\ C^{1(k)}_{1(1)},C^{i(k)}_{j(1)},-C^{(1)(j)(k)}_{(i)(1)(1)}\right) \tag{13}$$

are called the *adapted coefficients of the* N-**linear connection** D on E^*.

Taking into acount the tensorial transformation laws of the d-vector fields of the adapted basis (6), by direct calculations, we obtain

Theorem 3.1. (i) *With respect to the coordinate transformations (1) on E^*, the adapted coefficients of the N-linear connection $D\Gamma(N)$ obey the following transformation rules:*

$$(h_{\mathbb{R}})\begin{cases} A^1_{11} = \tilde{A}^1_{11}\dfrac{d\tilde{t}}{dt} + \dfrac{dt}{d\tilde{t}}\dfrac{d^2\tilde{t}}{dt^2}, \\ A^i_{j1} = \tilde{A}^k_{l1}\dfrac{\partial x^i}{\partial \tilde{x}^k}\dfrac{\partial \tilde{x}^l}{\partial x^j}\dfrac{d\tilde{t}}{dt}, \\ A^{(1)(j)}_{(i)(1)1} = \tilde{A}^{(1)(l)}_{(k)(1)1}\dfrac{\partial \tilde{x}^k}{\partial x^i}\dfrac{\partial x^j}{\partial \tilde{x}^l}\dfrac{d\tilde{t}}{dt} - \delta^j_i\dfrac{dt}{d\tilde{t}}\dfrac{d^2\tilde{t}}{dt^2}, \end{cases}$$

$$(h_M)\begin{cases} H^1_{1k} = \tilde{H}^1_{1l}\dfrac{\partial \tilde{x}^l}{\partial x^k}, \\ H^l_{jk} = \tilde{H}^i_{rs}\dfrac{\partial \tilde{x}^r}{\partial x^j}\dfrac{\partial \tilde{x}^s}{\partial x^k}\dfrac{\partial x^l}{\partial \tilde{x}^i} + \dfrac{\partial x^l}{\partial \tilde{x}^i}\dfrac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k}, \\ H^{(1)(j)}_{(i)(1)k} = \tilde{H}^{(1)(l)}_{(r)(1)s}\dfrac{\partial \tilde{x}^r}{\partial x^i}\dfrac{\partial x^j}{\partial \tilde{x}^l}\dfrac{\partial \tilde{x}^s}{\partial x^k} - \dfrac{\partial \tilde{x}^r}{\partial x^i}\dfrac{\partial \tilde{x}^s}{\partial x^k}\dfrac{\partial^2 x^j}{\partial \tilde{x}^r \partial \tilde{x}^s}, \end{cases}$$

$$(w)\begin{cases} C^{1(k)}_{1(1)} = \tilde{C}^{1(r)}_{1(1)}\dfrac{\partial x^k}{\partial \tilde{x}^r}\dfrac{d\tilde{t}}{dt}, \\ C^{i(k)}_{j(1)} = \tilde{C}^{r(s)}_{l(1)}\dfrac{\partial x^i}{\partial \tilde{x}^r}\dfrac{\partial \tilde{x}^l}{\partial x^j}\dfrac{\partial x^k}{\partial \tilde{x}^s}\dfrac{d\tilde{t}}{dt}, \\ C^{(1)(j)(k)}_{(i)(1)(1)} = \tilde{C}^{(1)(r)(s)}_{(l)(1)(1)}\dfrac{\partial \tilde{x}^l}{\partial x^i}\dfrac{\partial x^j}{\partial \tilde{x}^r}\dfrac{\partial x^k}{\partial \tilde{x}^s}\dfrac{d\tilde{t}}{dt}. \end{cases}$$

(ii) *Conversely, to give an N-linear connection D on E^* is equivalent to give a set of nine local coefficients $D\Gamma(N)$ as in (13), which obey the rules described in* (i).

Example 3.1. Let us consider the canonical nonlinear connection $\overset{0}{N}$, which is given by (5), associated with the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$. Then, the local components

$$B\Gamma\left(\overset{0}{N}\right) = \left(H^1_{11},\ 0,\ -A^{(1)(j)}_{(i)(1)1},\ 0,\ \gamma^i_{jk},\ -H^{(1)(j)}_{(i)(1)k},\ 0,\ 0,\ 0\right) \tag{14}$$

where

$$A^{(1)(j)}_{(i)(1)1} = -\delta^j_i H^1_{11}, \qquad H^{(1)(j)}_{(i)(1)k} = \gamma^j_{ik}, \tag{15}$$

define an $\overset{0}{N}$-linear connection on E^*, which is called the *canonical $\overset{0}{N}$-linear Berwald connection attached to the metrics $h_{11}(t)$ and $\varphi_{ij}(x)$.*

Let us consider that D is a fixed N-linear connection on E^*, defined by the adapted coefficients (13). The linear connection $D\Gamma(N)$ naturally induces derivations on the set of d-tensor fields on the dual 1-jet space E^*. Starting from a d-vector field $X \in \mathcal{X}(E^*)$ and a d-tensor field T on E^*, locally expressed by

$$\begin{aligned} X &= X^1\frac{\delta}{\delta t} + X^i\frac{\delta}{\delta x^i} + X^{(1)}_{(i)}\frac{\partial}{\partial p^1_i}, \\ T &= T^{1i(k)(1)\dots}_{1j(1)(l)\dots}\frac{\delta}{\delta t}\otimes\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial p^1_l}\otimes dt\otimes dx^j\otimes\delta p^1_k\otimes\dots, \end{aligned}$$

we obtain

$$\begin{aligned} D_X T &= X^1 D_{\frac{\delta}{\delta t}} T + X^s D_{\frac{\delta}{\delta x^s}} T + X^{(1)}_{(s)} D_{\frac{\partial}{\partial p^1_s}} T \\ &= \left\{ X^1 T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots/1} + X^s T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots|s} + \right. \\ &\left. + X^{(1)}_{(s)} T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots} |^{(s)}_{(1)} \right\} \frac{\delta}{\delta t} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial p^1_l} \otimes dt \otimes dx^j \otimes \delta p^1_k \otimes \ldots, \end{aligned}$$

where

$$(h_{\mathbb{R}}) \begin{cases} T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots/1} = \dfrac{\delta T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots}}{\delta t} + T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots} A^1_{11} + \\ \quad + T^{1r(k)(1)\ldots}_{1j(1)(l)\ldots} A^i_{r1} + T^{1i(r)(1)\ldots}_{1j(1)(l)\ldots} A^{(1)(k)}_{(r)(1)1} + \ldots - \\ \quad - T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots} A^1_{11} - T^{1i(k)(1)\ldots}_{1r(1)(l)\ldots} A^r_{j1} - \\ \quad - T^{1i(k)(1)\ldots}_{1j(1)(r)\ldots} A^{(1)(r)}_{(l)(1)1} - \ldots, \end{cases}$$

$$(h_M) \begin{cases} T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots|s} = \dfrac{\delta T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots}}{\delta x^s} + T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots} H^1_{1s} + \\ \quad + T^{1r(k)(1)\ldots}_{1j(1)(l)\ldots} H^i_{rs} + T^{1i(r)(1)\ldots}_{1j(1)(l)\ldots} H^{(1)(k)}_{(r)(1)s} + \ldots - \\ \quad - T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots} H^1_{1s} - T^{1i(k)(1)\ldots}_{1r(1)(l)\ldots} H^r_{js} - \\ \quad - T^{1i(k)(1)\ldots}_{1j(1)(r)\ldots} H^{(1)(r)}_{(l)(1)s} - \ldots, \end{cases}$$

$$(w) \begin{cases} T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots} \Big|^{(s)}_{(1)} = \dfrac{\partial T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots}}{\partial p^1_s} + T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots} C^{1(s)}_{1(1)} + \\ \quad + T^{1r(k)(1)\ldots}_{1j(1)(l)\ldots} C^{i(s)}_{r(1)} + T^{1i(r)(1)\ldots}_{1j(1)(l)\ldots} C^{(1)(k)(s)}_{(r)(1)(1)} + \ldots - \\ \quad - T^{1i(k)(1)\ldots}_{1j(1)(l)\ldots} C^{1(s)}_{1(1)} - T^{1i(k)(1)\ldots}_{1r(1)(l)\ldots} C^{r(s)}_{j(1)} - \\ \quad - T^{1i(k)(1)\ldots}_{1j(1)(r)\ldots} C^{(1)(r)(s)}_{(l)(1)(1)} - \ldots. \end{cases}$$

Definition 3.3. The local derivative operators "$_{/1}$", "$_{|i}$" and "$|^{(i)}_{(1)}$" are called the $\mathbb{R}$-*horizontal covariant derivative*, the M-*horizontal covariant derivative* and the *vertical covariant derivative attached to the* N*-linear connection* $D\Gamma(N)$.

Remark 3.1. The operators "$_{/1}$", "$_{|i}$" and "$|^{(i)}_{(1)}$" have the properties:

(i) They are distributive with respect to the addition of the d-tensor fields of the same type.

(ii) They commute with the operation of contraction.

(iii) They verify the Leibniz rule with respect to the tensor product.

Remark 3.2. *(i)* If $T = f(t, x^k, p^1_k)$ is a function on E^*, then the following expressions of the local covariant derivatives are true:

$$f_{/1} = \frac{\delta f}{\delta t} = \frac{\partial f}{\partial t} - \underset{1}{N}{}^{(1)}_{(i)1} \frac{\partial f}{\partial p^1_i}, \quad f_{|j} = \frac{\delta f}{\delta x^j} = \frac{\partial f}{\partial x^j} - \underset{2}{N}{}^{(1)}_{(i)j} \frac{\partial f}{\partial p^1_i}, \quad f|^{(i)}_{(1)} = \frac{\partial f}{\partial p^1_i}.$$

(ii) If $T = Y$ is a d-vector field on E^*, locally expressed by

$$Y = Y^1 \frac{\delta}{\delta t} + Y^i \frac{\delta}{\delta x^i} + Y^{(1)}_{(i)} \frac{\partial}{\partial p^1_i},$$

then the following expressions of the local covariant derivatives are true:

$$(h_{\mathbb{R}})\begin{cases} Y^1{}_{/1} = \dfrac{\delta Y^1}{\delta t} + Y^1 A^1_{11}, \\ Y^i{}_{/1} = \dfrac{\delta Y^i}{\delta t} + Y^j A^i_{j1}, \\ Y^{(1)}_{(i)/1} = \dfrac{\delta Y^{(1)}_{(i)}}{\delta t} - Y^{(1)}_{(j)} A^{(1)(j)}_{(i)(1)1}, \end{cases} \qquad (h_M)\begin{cases} Y^1{}_{|k} = \dfrac{\delta Y^1}{\delta x^k} + Y^1 H^1_{1k}, \\ Y^i{}_{|k} = \dfrac{\delta Y^i}{\delta x^k} + Y^j H^i_{jk}, \\ Y^{(1)}_{(i)|k} = \dfrac{\delta Y^{(1)}_{(i)}}{\delta x^k} - Y^{(1)}_{(j)} H^{(1)(j)}_{(i)(1)k}, \end{cases}$$

$$(w)\begin{cases} Y^1 \mid^{(k)}_{(1)} = \dfrac{\partial Y^1}{\partial p^1_k} + Y^1 C^{1(k)}_{1(1)}, \\ Y^i \mid^{(k)}_{(1)} = \dfrac{\partial Y^i}{\partial p^1_k} + Y^j C^{i(k)}_{j(1)}, \\ Y^{(1)}_{(i)} \mid^{(k)}_{(1)} = \dfrac{\partial Y^{(1)}_{(i)}}{\partial p^1_k} - Y^{(1)}_{(j)} C^{(1)(j)(k)}_{(i)(1)(1)}. \end{cases}$$

(iii) If $T = \omega$ is a d-covector field on E^*, locally expressed by

$$\omega = \omega_1 dt + \omega_i dx^i + \omega^{(i)}_{(1)} \delta p^1_i,$$

then the following expressions of the local covariant derivatives are true:

$$(h_{\mathbb{R}})\begin{cases} \omega_{1/1} = \dfrac{\delta \omega_1}{\delta t} - \omega_1 A^1_{11}, \\ \omega_{i/1} = \dfrac{\delta \omega_i}{\delta t} - \omega_j A^j_{i1}, \\ \omega^{(i)}_{(1)/1} = \dfrac{\delta \omega^{(i)}_{(1)}}{\delta t} + \omega^{(j)}_{(1)} A^{(1)(i)}_{(j)(1)1}, \end{cases} \qquad (h_M)\begin{cases} \omega_{1|k} = \dfrac{\delta \omega_1}{\delta x^k} - \omega_1 H^1_{1k}, \\ \omega_{i|k} = \dfrac{\delta \omega_i}{\delta x^k} - \omega_j H^j_{ik}, \\ \omega^{(i)}_{(1)|k} = \dfrac{\delta \omega^{(i)}_{(1)}}{\delta x^k} + \omega^{(j)}_{(1)} H^{(1)(i)}_{(j)(1)k}, \end{cases}$$

$$(w)\begin{cases} \omega_1 \mid^{(k)}_{(1)} = \dfrac{\partial \omega_1}{\partial p^1_k} - \omega_1 C^{1(k)}_{1(1)}, \\ \omega_i \mid^{(k)}_{(1)} = \dfrac{\partial \omega_i}{\partial p^1_k} - \omega_j C^{j(k)}_{i(1)}, \\ \omega^{(i)}_{(1)} \mid^{(k)}_{(1)} = \dfrac{\partial \omega^{(i)}_{(1)}}{\partial p^1_k} + \omega^{(j)}_{(1)} C^{(1)(i)(k)}_{(j)(1)(1)}. \end{cases}$$

Notation. In the particular case of the canonical Berwald $\overset{0}{N}$-linear connection given by (5), (14) and (15), associated with the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$, the local covariant derivatives are denoted by "$_{//1}$", "$_{||i}$" and "$\|^{(i)}_{(1)}$".

Considering the canonical Liouville-Hamilton d-tensor field of momenta on E^*, which is given by

$$\mathbb{C}^* = \mathbb{C}^{(1)}_{(i)} \frac{\partial}{\partial p^1_i} = p^1_i \frac{\partial}{\partial p^1_i},$$

by direct computations, we can give an application of this paragraph.

Definition 3.4. The d-tensor fields

$$\Delta^{(1)}_{(i)1} = \mathbb{C}^{(1)}_{(i)/1}, \quad \Delta^{(1)}_{(i)j} = \mathbb{C}^{(1)}_{(i)|j}, \quad \vartheta^{(1)(j)}_{(i)(1)} = \mathbb{C}^{(1)}_{(i)}|^{(j)}_{(1)}, \tag{16}$$

are called the *momentum non-metrical deflection d-tensor fields attached to the N-linear connection $D\Gamma(N)$.*

Proposition 3.2. *The momentum deflection d-tensor fields on E^*, attached to the N-linear connection $D\Gamma(N)$, have the expressions:*

$$\Delta^{(1)}_{(i)1} = -\underset{1}{N}{}^{(1)}_{(i)1} - A^{(1)(k)}_{(i)(1)1} p^1_k, \quad \Delta^{(1)}_{(i)j} = -\underset{2}{N}{}^{(1)}_{(i)j} - H^{(1)(k)}_{(i)(1)j} p^1_k,$$
$$\vartheta^{(1)(j)}_{(i)(1)} = \delta^j_i - C^{(1)(k)(j)}_{(i)(1)(1)} p^1_k. \tag{17}$$

3.1. Torsion d-tensors. Let D be an N-linear connection on E^*. The torsion $\mathbf{T}$ of D is given by

$$\mathbf{T}(X,Y) = D_X Y - D_Y X - [X,Y], \quad \forall\ X,Y \in \mathcal{X}(E^*). \tag{18}$$

Let us suppose that the N-linear connection D is given in the adapted basis (6) by the coefficients $D\Gamma(N)$ from (13). In this context, we have

Theorem 3.3. *The local torsion d-tensors of the N-linear connection D on E^* have the expressions:*

$$h_{\mathbb{R}}\mathbf{T}\left(\frac{\delta}{\delta t},\frac{\delta}{\delta t}\right) = T^1_{11}\frac{\delta}{\delta t}, \quad h_M\mathbf{T}\left(\frac{\delta}{\delta t},\frac{\delta}{\delta t}\right) = T^k_{11}\frac{\delta}{\delta x^k},$$
$$w\mathbf{T}\left(\frac{\delta}{\delta t},\frac{\delta}{\delta t}\right) = T^{(1)}_{(r)11}\frac{\partial}{\partial p^1_r},$$
$$h_{\mathbb{R}}\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta t}\right) = T^1_{1j}\frac{\delta}{\delta t}, \quad h_M\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta t}\right) = T^k_{1j}\frac{\delta}{\delta x^k},$$
$$w\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta t}\right) = T^{(1)}_{(r)1j}\frac{\partial}{\partial p^1_r},$$
$$h_{\mathbb{R}}\mathbf{T}\left(\frac{\partial}{\partial p^1_j},\frac{\delta}{\delta t}\right) = P^{1(j)}_{1(1)}\frac{\delta}{\delta t}, \quad h_M\mathbf{T}\left(\frac{\partial}{\partial p^1_j},\frac{\delta}{\delta t}\right) = P^{k(j)}_{1(1)}\frac{\delta}{\delta x^k},$$
$$w\mathbf{T}\left(\frac{\partial}{\partial p^1_j},\frac{\delta}{\delta t}\right) = P^{(1)\ (j)}_{(r)1(1)}\frac{\partial}{\partial p^1_r},$$
$$h_{\mathbb{R}}\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right) = T^1_{ij}\frac{\delta}{\delta t}, \quad h_M\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right) = T^k_{ij}\frac{\delta}{\delta x^k},$$
$$w\mathbf{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right) = T^{(1)}_{(r)ij}\frac{\partial}{\partial p^1_r},$$
$$h_{\mathbb{R}}\mathbf{T}\left(\frac{\partial}{\partial p^1_j},\frac{\delta}{\delta x^i}\right) = P^{1(j)}_{i(1)}\frac{\delta}{\delta t}, \quad h_M\mathbf{T}\left(\frac{\partial}{\partial p^1_j},\frac{\delta}{\delta x^i}\right) = P^{k(j)}_{i(1)}\frac{\delta}{\delta x^k},$$
$$w\mathbf{T}\left(\frac{\partial}{\partial p^1_j},\frac{\delta}{\delta x^i}\right) = P^{(1)\ (j)}_{(r)i(1)}\frac{\partial}{\partial p^1_r},$$
$$h_{\mathbb{R}}\mathbf{T}\left(\frac{\partial}{\partial p^1_j},\frac{\partial}{\partial p^1_i}\right) = S^{1(i)(j)}_{\ (1)(1)}\frac{\delta}{\delta t}, \quad h_M\mathbf{T}\left(\frac{\partial}{\partial p^1_j},\frac{\partial}{\partial p^1_i}\right) = S^{k(i)(j)}_{\ (1)(1)}\frac{\delta}{\delta x^k},$$

$$w\mathbf{T}\left(\frac{\partial}{\partial p_j^1}, \frac{\partial}{\partial p_i^1}\right) = S_{(r)(1)(1)}^{(1)(i)(j)} \frac{\partial}{\partial p_r^1},$$

where

$$\begin{cases} T_{11}^1 = 0, \quad T_{11}^k = 0, \quad T_{(r)11}^{(1)} = 0, \\ T_{1j}^1 = H_{1j}^1, \quad T_{1j}^k = -A_{j1}^k, \quad T_{(r)1j}^{(1)} = R_{(r)1j}^{(1)}, \\ P_{1(1)}^{1(j)} = C_{1(1)}^{1(j)}, \quad P_{1(1)}^{k(j)} = 0, \quad P_{(r)1(1)}^{(1)\ (j)} = B_{(r)1(1)}^{(1)\ (j)} + A_{(r)(1)1}^{(1)(j)}, \end{cases} \tag{19}$$

$$\begin{cases} T_{ij}^1 = 0, \quad T_{ij}^k = H_{ij}^k - H_{ji}^k, \quad T_{(r)ij}^{(1)} = R_{(r)ij}^{(1)}, \\ T_{i(1)}^{1(j)} = 0, \quad P_{i(1)}^{k(j)} = C_{i(1)}^{k(j)}, \quad P_{(r)i(1)}^{(1)\ (j)} = B_{(r)i(1)}^{(1)\ (j)} + H_{(r)(1)i}^{(1)(j)}, \end{cases} \tag{20}$$

$$S_{(1)(1)}^{1(i)(j)} = 0, \quad S_{(1)(1)}^{k(i)(j)} = 0, \quad S_{(r)(1)(1)}^{(1)(i)(j)} = -\left(C_{(r)(1)(1)}^{(1)(i)(j)} - C_{(r)(1)(1)}^{(1)(j)(i)}\right) \tag{21}$$

and the distinguished tensors

$$R_{(r)1j}^{(1)}, \ R_{(r)ij}^{(1)}, \ B_{(r)1(1)}^{(1)\ (j)}, \ B_{(r)i(1)}^{(1)\ (j)}$$

are given by the formulas (9).

Proof. Taking into account the Poisson brackets formulas (8) and (9), together with the local description in the adapted basis (6) of the N-linear connection $D\Gamma(N)$ (see (13)), we successively obtain

$$h_{\mathbb{R}}\mathbf{T}\left(\frac{\delta}{\delta t}, \frac{\delta}{\delta t}\right) = h_{\mathbb{R}} D_{\frac{\delta}{\delta t}} \frac{\delta}{\delta t} - h_{\mathbb{R}} D_{\frac{\delta}{\delta t}} \frac{\delta}{\delta t} - h_{\mathbb{R}}\left[\frac{\delta}{\delta t}, \frac{\delta}{\delta t}\right] = 0.$$

Consequently, the first equality from (19) is true. In the sequel, we have

$$h_M\mathbf{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t}\right) = h_M D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta t} - h_M D_{\frac{\delta}{\delta t}} \frac{\delta}{\delta x^j} - h_M\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t}\right] = -A_{j1}^k \frac{\delta}{\delta x^k},$$

and the fifth equality from (19) is correct. Then, for example, we have

$$\begin{aligned} w\mathbf{T}\left(\frac{\partial}{\partial p_j^1}, \frac{\delta}{\delta t}\right) &= wD_{\frac{\partial}{\partial p_j^1}} \frac{\delta}{\delta t} - wD_{\frac{\delta}{\delta t}} \frac{\partial}{\partial p_j^1} - w\left[\frac{\partial}{\partial p_j^1}, \frac{\delta}{\delta t}\right] = \\ &= \left(A_{(r)(1)1}^{(1)(j)} + B_{(r)1(1)}^{(1)\ (j)}\right) \frac{\partial}{\partial p_r^1}, \end{aligned}$$

and the ninth equality from (19) is true. In the same manner, we obtain the other equalities. □

Corollary 3.4. *The torsion* $\mathbf{T}$ *of an arbitrary* N*-linear connection* D *on* E^* *is determined by* ***ten*** *effective local* ***d-tensors of torsion****, arranged in the following table:*

	$h_{\mathbb{R}}$	h_M	w
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_M h_{\mathbb{R}}$	T^1_{1j}	T^k_{1j}	$R^{(1)}_{(r)1j}$
$w h_{\mathbb{R}}$	$P^{1(j)}_{1(1)}$	0	$P^{(1)\ (j)}_{(r)1(1)}$
$h_M h_M$	0	T^k_{ij}	$R^{(1)}_{(r)ij}$
$w h_M$	0	$P^{k(j)}_{i(1)}$	$P^{(1)\ (j)}_{(r)i(1)}$
ww	0	0	$S^{(1)(i)(j)}_{(r)(1)(1)}$

Example 3.2. For the canonical Berwald $\overset{0}{N}$-linear connection given by (5), (14) and (15), associated with the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$, all d-tensors of torsion vanish, except $R^{(1)}_{(r)ij} = -\mathcal{R}^s_{rij}p^1_s$, where $\mathcal{R}^s_{rij}(x)$ are the local components of the curvature tensor of the semi-Riemannian metric $\varphi_{ij}(x)$.

3.2. Curvature d-tensors.

Let D be an N-linear connection on E^*. The curvature $\mathbf{R}$ of D is given by

$$\mathbf{R}(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \qquad \forall\ X,Y,Z \in \mathcal{X}(E^*). \tag{22}$$

We will express $\mathbf{R}$ by his adapted components, taking into account the adapted local decomposition of the vector fields on E^*. In this direction, firstly we prove

Theorem 3.5. *The curvature tensor field* $\mathbf{R}$ *of the* N*-linear connection* D *on* E^* *has the properties:*

$$\begin{array}{lll} h_{\mathbb{R}}\mathbf{R}(X,Y)Z^{\mathcal{H}_M} = 0, & h_{\mathbb{R}}\mathbf{R}(X,Y)Z^{\mathcal{W}} = 0, & h_M\mathbf{R}(X,Y)Z^{\mathcal{H}_{\mathbb{R}}} = 0, \\ h_M\mathbf{R}(X,Y)Z^{\mathcal{W}} = 0, & w\mathbf{R}(X,Y)Z^{\mathcal{H}_{\mathbb{R}}} = 0, & w\mathbf{R}(X,Y)Z^{\mathcal{H}_M} = 0, \end{array} \tag{23}$$

$$\mathbf{R}(X,Y)Z = h_{\mathbb{R}}\mathbf{R}(X,Y)Z^{\mathcal{H}_{\mathbb{R}}} + h_M\mathbf{R}(X,Y)Z^{\mathcal{H}_M} + w\mathbf{R}(X,Y)Z^{\mathcal{W}}. \tag{24}$$

Proof. Because the N-linear connection D preserves by parallelism the $\mathcal{H}_{\mathbb{R}}$-horizontal, $\mathcal{H}_M$-horizontal and vertical distributions, via the formula (22), the operator $\mathbf{R}(X,Y)$ carries $h_{\mathbb{R}}$-horizontal (resp. h_M-horizontal) vector fields into $h_{\mathbb{R}}$-horizontal (resp. h_M-horizontal) vector fields and the vertical vector fields into vertical vector fields. Thus, the first six equations from (23) are true. The next one is an easy consequence of the first six. □

Taking into account the preceding geometrical result, by straightforward calculus, we obtain

Theorem 3.6. *The curvature tensor* $\mathbf{R}$ *of the* N*-linear connection* D *is completely determined by* ***fifteen*** *local* ***d-tensors of curvature:***

$$\mathbf{R}\left(\frac{\delta}{\delta t},\frac{\delta}{\delta t}\right)\frac{\delta}{\delta t} = 0, \quad \mathbf{R}\left(\frac{\delta}{\delta t},\frac{\delta}{\delta t}\right)\frac{\delta}{\delta x^i} = 0, \quad \mathbf{R}\left(\frac{\delta}{\delta t},\frac{\delta}{\delta t}\right)\frac{\partial}{\partial p^1_i} = 0,$$

$$\mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t}\right)\frac{\delta}{\delta t} = R^1_{11k}\frac{\delta}{\delta t}, \quad \mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t}\right)\frac{\delta}{\delta x^i} = R^l_{i1k}\frac{\delta}{\delta x^l},$$

$$\mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t}\right)\frac{\partial}{\partial p^1_i} = -R^{(1)(i)}_{(l)(1)1k}\frac{\partial}{\partial p^1_l},$$

$$\mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\delta}{\delta t}\right)\frac{\delta}{\delta t}=P^{1\ (k)}_{11(1)}\frac{\delta}{\delta t},\quad \mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\delta}{\delta t}\right)\frac{\delta}{\delta x^i}=P^{l\ (k)}_{i1(1)}\frac{\delta}{\delta x^l},$$

$$\mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\delta}{\delta t}\right)\frac{\partial}{\partial p_i^1}=-P^{(1)(i)\ (k)}_{(l)(1)1(1)}\frac{\partial}{\partial p_l^1},$$

$$\mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta t}=R^1_{1jk}\frac{\delta}{\delta t},\quad \mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i}=R^l_{ijk}\frac{\delta}{\delta x^l},$$

$$\mathbf{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial p_i^1}=-R^{(1)(i)}_{(l)(1)jk}\frac{\partial}{\partial p_l^1},$$

$$\mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta t}=P^{1\ (k)}_{1j(1)}\frac{\delta}{\delta t},\quad \mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i}=P^{l\ (k)}_{ij(1)}\frac{\delta}{\delta x^l},$$

$$\mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial p_i^1}=-P^{(1)(i)\ (k)}_{(l)(1)j(1)}\frac{\partial}{\partial p_l^1},$$

$$\mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\partial}{\partial p_j^1}\right)\frac{\delta}{\delta t}=S^{1(j)(k)}_{1(1)(1)}\frac{\delta}{\delta t},\quad \mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\partial}{\partial p_j^1}\right)\frac{\delta}{\delta x^i}=S^{l(j)(k)}_{i(1)(1)}\frac{\delta}{\delta x^l},$$

$$\mathbf{R}\left(\frac{\partial}{\partial p_k^1},\frac{\partial}{\partial p_j^1}\right)\frac{\partial}{\partial p_i^1}=-S^{(1)(i)(j)(k)}_{(l)(1)(1)(1)}\frac{\partial}{\partial p_l^1},$$

which we can arrange in the following table:

	$h_{\mathbb{R}}$	h_M	w
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_Mh_{\mathbb{R}}$	R^1_{11k}	R^l_{i1k}	$-R^{(1)(i)}_{(l)(1)1k}$
$wh_{\mathbb{R}}$	$P^{1\ (k)}_{11(1)}$	$P^{l\ (k)}_{i1(1)}$	$-P^{(1)(i)\ (k)}_{(l)(1)1(1)}$
h_Mh_M	R^1_{1jk}	R^l_{ijk}	$-R^{(1)(i)}_{(l)(1)jk}$
wh_M	$P^{1\ (k)}_{1j(1)}$	$P^{l\ (k)}_{ij(1)}$	$-P^{(1)(i)\ (k)}_{(l)(1)j(1)}$
ww	$S^{1(j)(k)}_{1(1)(1)}$	$S^{l(j)(k)}_{i(1)(1)}$	$-S^{(1)(i)(j)(k)}_{(l)(1)(1)(1)}$

(25)

Theorem 3.7. *The fifteen local curvature d-tensors from the Table (25) are given by the following formulas:*

1. $R^1_{11k} = \dfrac{\delta A^1_{11}}{\delta x^k}-\dfrac{\delta H^1_{1k}}{\delta t}+C^{1(r)}_{1(1)}R^{(1)}_{(r)1k},$

2. $R^l_{i1k} = \dfrac{\delta A^l_{i1}}{\delta x^k}-\dfrac{\delta H^l_{ik}}{\delta t}+A^r_{i1}H^l_{rk}-H^r_{ik}A^l_{r1}+C^{l(r)}_{i(1)}R^{(1)}_{(r)1k},$

3. $R^{(1)(i)}_{(l)(1)1k} = \dfrac{\delta A^{(1)(i)}_{(l)(1)1}}{\delta x^k}-\dfrac{\delta H^{(1)(i)}_{(l)(1)k}}{\delta t}+A^{(1)(r)}_{(l)(1)1}H^{(1)(i)}_{(r)(1)k}- -H^{(1)(r)}_{(l)(1)k}A^{(1)(i)}_{(r)(1)1}+C^{(1)(i)(r)}_{(l)(1)(1)}R^{(1)}_{(r)1k},$

4. $P^{1\ (k)}_{11(1)} = \dfrac{\partial A^1_{11}}{\partial p^1_k}-C^{1(k)}_{1(1)/1}+C^{1(r)}_{1(1)}P^{(1)\ (k)}_{(r)1(1)},$

5. $P^{l\ (k)}_{i1(1)} = \dfrac{\partial A^l_{i1}}{\partial p^1_k}-C^{l(k)}_{i(1)/1}+C^{l(r)}_{i(1)}P^{(1)\ (k)}_{(r)1(1)},$

$$\mathbf{6}.\quad P_{(l)(1)1(1)}^{(1)(i)\ (k)} = \frac{\partial A_{(l)(1)1}^{(1)(i)}}{\partial p_k^1} - C_{(l)(1)(1)/1}^{(1)(i)(k)} + C_{(l)(1)(1)}^{(1)(i)(r)} P_{(r)1(1)}^{(1)\ (k)},$$

$$\mathbf{7}.\quad R_{1jk}^1 = \frac{\delta H_{1j}^1}{\delta x^k} - \frac{\delta H_{1k}^1}{\delta x^j} + C_{1(1)}^{1(r)} R_{(r)jk}^{(1)},$$

$$\mathbf{8}.\quad R_{ijk}^l = \frac{\delta H_{ij}^l}{\delta x^k} - \frac{\delta H_{ik}^l}{\delta x^j} + H_{ij}^r H_{rk}^l - H_{ik}^r H_{rj}^l + C_{i(1)}^{l(r)} R_{(r)jk}^{(1)},$$

$$\mathbf{9}.\quad R_{(l)(1)jk}^{(1)(i)} = \frac{\delta H_{(l)(1)j}^{(1)(i)}}{\delta x^k} - \frac{\delta H_{(l)(1)k}^{(1)(i)}}{\delta x^j} + H_{(l)(1)j}^{(1)(r)} H_{(r)(1)k}^{(1)(i)} - \\ - H_{(l)(1)k}^{(1)(r)} H_{(r)(1)j}^{(1)(i)} + C_{(l)(1)(1)}^{(1)(i)(r)} R_{(r)jk}^{(1)},$$

$$\mathbf{10}.\quad P_{1j(1)}^{1\ (k)} = \frac{\partial H_{1j}^1}{\partial p_k^1} - C_{1(1)|j}^{1(k)} + C_{1(1)}^{1(r)} P_{(r)j(1)}^{(1)\ (k)},$$

$$\mathbf{11}.\quad P_{ij(1)}^{l\ (k)} = \frac{\partial H_{ij}^l}{\partial p_k^1} - C_{i(1)|j}^{l(k)} + C_{i(1)}^{l(r)} P_{(r)j(1)}^{(1)\ (k)},$$

$$\mathbf{12}.\quad P_{(l)(1)j(1)}^{(1)(i)\ (k)} = \frac{\partial H_{(l)(1)j}^{(1)(i)}}{\partial p_k^1} - C_{(l)(1)(1)|j}^{(1)(i)(k)} + C_{(l)(1)(1)}^{(1)(i)(r)} P_{(r)j(1)}^{(1)\ (k)},$$

$$\mathbf{13}.\quad S_{1(1)(1)}^{1(j)(k)} = \frac{\partial C_{1(1)}^{1(j)}}{\partial p_k^1} - \frac{\partial C_{1(1)}^{1(k)}}{\partial p_j^1},$$

$$\mathbf{14}.\quad S_{i(1)(1)}^{l(j)(k)} = \frac{\partial C_{i(1)}^{l(j)}}{\partial p_k^1} - \frac{\partial C_{i(1)}^{l(k)}}{\partial p_j^1} + C_{i(1)}^{r(j)} C_{r(1)}^{l(k)} - C_{i(1)}^{r(k)} C_{r(1)}^{l(j)},$$

$$\mathbf{15}.\quad S_{(l)(1)(1)(1)}^{(1)(i)(j)(k)} = \frac{\partial C_{(l)(1)(1)}^{(1)(i)(j)}}{\partial p_k^1} - \frac{\partial C_{(l)(1)(1)}^{(1)(i)(k)}}{\partial p_j^1} + C_{(l)(1)(1)}^{(1)(r)(j)} C_{(r)(1)(1)}^{(1)(i)(k)} - \\ - C_{(l)(1)(1)}^{(1)(r)(k)} C_{(r)(1)(1)}^{(1)(i)(j)}.$$

Proof. The local decomposition in the adapted basis (6) of the N-linear connection $D\Gamma(N)$ (see (13)), together with the formulas (8) and (9), lead us to, for example,

$$\mathbf{R}\left(\frac{\partial}{\partial p_k^1}, \frac{\delta}{\delta t}\right)\frac{\partial}{\partial p_i^1} = -P_{(l)(1)1(1)}^{(1)(i)\ (k)} \frac{\partial}{\partial p_l^1} =$$

$$= D_{\frac{\partial}{\partial p_k^1}} D_{\frac{\delta}{\delta t}} \frac{\partial}{\partial p_i^1} - D_{\frac{\delta}{\delta t}} D_{\frac{\partial}{\partial p_k^1}} \frac{\partial}{\partial p_i^1} - D_{\left[\frac{\partial}{\partial p_k^1}, \frac{\delta}{\delta t}\right]} \frac{\partial}{\partial p_i^1}$$

$$= -D_{\frac{\partial}{\partial p_k^1}}\left(A_{(r)(1)1}^{(1)(i)} \frac{\partial}{\partial p_r^1}\right) + D_{\frac{\delta}{\delta t}}\left(C_{(r)(1)(1)}^{(1)(i)(k)} \frac{\partial}{\partial p_r^1}\right) + B_{(r)1(1)}^{(1)\ (k)} D_{\frac{\partial}{\partial p_r^1}} \frac{\partial}{\partial p_i^1}$$

$$= -\frac{\partial A_{(l)(1)1}^{(1)(i)}}{\partial p_k^1} \frac{\partial}{\partial p_l^1} + A_{(r)(1)1}^{(1)(i)} C_{(l)(1)(1)}^{(1)(r)(k)} \frac{\partial}{\partial p_l^1} \\ + \frac{\delta C_{(l)(1)(1)}^{(1)(i)(k)}}{\delta t} \frac{\partial}{\partial p_l^1} - C_{(r)(1)(1)}^{(1)(i)(k)} A_{(l)(1)1}^{(1)(r)} \frac{\partial}{\partial p_l^1} - B_{(r)1(1)}^{(1)\ (k)} C_{(l)(1)(1)}^{(1)(i)(r)} \frac{\partial}{\partial p_l^1}.$$

Therefore, we have

$$P^{(1)(i)\ (k)}_{(l)(1)1(1)} = \frac{\partial A^{(1)(i)}_{(l)(1)1}}{\partial p^1_k} - \underline{\underline{A^{(1)(i)}_{(r)(1)1} C^{(1)(r)(k)}_{(l)(1)(1)}}} - \underline{\underline{-\frac{\delta C^{(1)(i)(k)}_{(l)(1)(1)}}{\delta t} + C^{(1)(i)(k)}_{(r)(1)(1)} A^{(1)(r)}_{(l)(1)1}}} + B^{(1)\ (k)}_{(r)1(1)} C^{(1)(i)(r)}_{(l)(1)(1)}.$$

Now, using the formula of the $\mathbb{R}$-horizontal covariant derivative, we get

$$C^{(1)(i)(k)}_{(l)(1)(1)/1} = \underline{\underline{\frac{\delta C^{(1)(i)(k)}_{(l)(1)(1)}}{\delta t} - C^{(1)(i)(k)}_{(r)(1)(1)} A^{(1)(r)}_{(l)(1)1} + C^{(1)(r)(k)}_{(l)(1)(1)} A^{(1)(i)}_{(r)(1)1}}} + C^{(1)(i)(r)}_{(l)(1)(1)} A^{(1)(k)}_{(r)(1)1},$$

and, consequently, interchanging the underlined terms, it follows that

$$P^{(1)(i)\ (k)}_{(l)(1)1(1)} = \frac{\partial A^{(1)(i)}_{(l)(1)1}}{\partial p^1_k} - C^{(1)(i)(k)}_{(l)(1)(1)/1} + C^{(1)(i)(r)}_{(l)(1)(1)} P^{(1)\ (k)}_{(r)1(1)},$$

where we also used the last formula from (19). Obviously, this is the $6^{\text{-}th}$ relation of the above set of identities.

The other equalities are given in the same manner. □

Example 3.3. For the canonical Berwald $\overset{0}{N}$-linear connection given by (5), (14) and (15), associated with the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$, all curvature d-tensors vanish, except $R^l_{ijk} = R^{(1)(l)}_{(i)(1)jk} = \mathcal{R}^l_{ijk}$, where $\mathcal{R}^l_{ijk}(x)$ are the local curvature tensors of the semi-Riemannian metric $\varphi_{ij}(x)$.

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