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High-gain adaptive boundary stabilization for an axially moving string subject to unbounded boundary disturbance

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Abstract. In this paper, a vibration suppression scheme for an axially moving string under external disturbances is investigated. The disturbances are assumed to be increased exponentially. We employ the active disturbance rejection control (ADRC) approach to estimate the disturbance. We design a disturbance observer that has time-varying gain so that the disturbance can be estimated with an exponential way. In order to stabilize the closed loop system, we use a control constructed through a high-gain adaptive velocity feedback. The existence and uniqueness of solution of the closed loop system is proved through the use of semigroup theory. The Lyapunov method is employed to show the effectiveness of the boundary control for ensuring the vibration reduction. The obtained results improves certain previous results.

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1. Introduction

Axially moving continuous materials can be found in various engineering areas such as continuous material manufacturing lines and transport processes. Especially, the dynamics analysis and control for axially moving continuous materials have received a growing attention due to the entrance of new applications in flexible robotic manipulators and flexible space structures. The string model is used for continuously moving systems when ignoring the bending stiffness of the material such as threads and cables [1, 2, 3, 4]. Vibration suppression has been achieved through different types of control, see [5, 6, 7, 8, 9]. The adaptive control is very efficient for immobile vibrating systems, see [10, 11, 12, 13, 14]. It is designed in such a way to cope with varied or uncertain parameters. The control strategy with input $u(t)$ and output $y(t)$ is a relation of the form

$$\begin{cases} u(t) = \Phi(t, g(t), y(t)), \\ g'(t) = \Psi(t, g(t), y(t)), \ g(t_0) \in \mathbb{R} \end{cases}$$

where Φ, Ψ and g are functions to be determined.

The present paper deals with the stabilization of solutions of an axially moving string subject to a boundary disturbance by a control input at the right boundary,

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that is

$$\begin{cases} y_{tt} + 2vy_{xt} - \left(1 - v^2\right) y_{xx} = 0, \ \ x \in (0,1)\,, \ t > 0, \\ y(0,t) = 0, \quad t \geq 0, \\ y_x(1,t) = u(t) + d(t), \ t \geq 0, \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \ \ x \in (0,1)\,, \\ y_{out}(t) = \left(y_t + vy_x\right)(1,t), \ t > 0, \end{cases} \tag{1}$$

where we denote by $y(x,t)$ the transverse displacement of the string which is axially moving with a constant velocity v such that $0 < v < 1$. The functions y_0 and y_1 are respectively the initial displacement and the initial velocity. The function $u(t)$ is the boundary control (input) whereas $y_{out}(t)$ stands for the measured signal of the system at that free end (output). The function d represents the unknown external disturbance which is assumed to satisfy

$$d \in C\left(0,\infty\right) \text{ and } \left|d\left(t\right)\right|, \left|d'\left(t\right)\right| \leq C_d e^{a_0 t} \tag{2}$$

for some positive constants C_d and a_0.

In this paper, we shall apply the active disturbance rejection control (ADRC) technique. This approach that was initiated by Han [15] is destined to deal with the systems with large external disturbances or internal structure uncertainties. The most important advantage of ADRC is that the disturbance can be estimated via an extended state observer [16]) in actual time and is canceled in the feedback loop which reduces the control energy considerably. ADRC was applied first to stabilization of a one-dimensional anti-stable wave equation subject to boundary disturbance in [17] and then to stabilization of a one-dimensional anti-stable wave equation subject to general control matched disturbance in [18]. The application of this approach has been extended to multi-dimensional systems such as wave equation [19] and Kirchhoff plate [20].

All works aforementioned required that the disturbances are uniformly bounded. Since the disturbances comes mostly from the the external environment, we do not need to be bounded uniformly and the assumption (2) is reasonable.

The boundary stabilization of this system governed by equation (1) with nonlinearity of Kirchhoff type was discussed in [21] and with geometric nonlinearity in [22]. The adaptive control was also discussed by the present author in [23] where the considered control is of the following form

$$u(t) = -\xi(t)y_{out}(t), \ \dot{\xi}(t) = py_{out}^2(t), \xi(0) = \xi_0, p > 0.$$

The objective of this paper is to investigate the stabilization of (1) by using an adaptive control. For this aim, we design the following high gain adaptive output feedback controller

$$\begin{cases} u(t) = -k(t)y_{out}(t) + \vartheta\left(t\right), \\ k'(t) = py_{out}^2(t), \ p > 0, \ k(0) = k_0 > 0 \end{cases} \tag{3}$$

where the extra term $\vartheta(t)$ is a new input that will be specified later. The closed-loop system of (1) is given by

$$\begin{cases} y_{tt}+2vy_{xt}-\left(1-v^2\right)y_{xx}=0, \ \ x\in(0,1)\,, \ t>0, \\ y(0,t)=0, \quad t\geq 0, \\ y_x(1,t)=-k(t)\left(y_t+vy_x\right)(1,t)+\vartheta(t)+d(t), \ t\geq 0, \\ y(x,0)=y_0(x), \ y_t(x,0)=y_1(x), \ \ x\in(0,1)\,, \\ k'(t)=p\left(y_t+vy_x\right)^2(1,t), \ p>0, \ k(0)=k_0>0. \\ y_{out}(t)=\left(y_t+vy_x\right)(1,t), \ t>0. \end{cases} \tag{4}$$

Regarding the case of immobile string $(v=0)$ and without distribuances, Kobayashi [24] considered high-gain adaptive stabilization of undamped semilinear second-order hyperbolic systems. The adaptive stabilizer was constructed by a high-gain adaptive velocity feedback. The multiplier technique was used to adaptively stabilize these systems where some examples were given to illustrate the obtained result.

In the control engineering, the correct method of the time differentiation of a functional is necessary for designing a controller in the Lyapunov method. The correct method for axially moving systems should be considered under Eulerian description. The time derivative of the energy of a system in axial movement is obtained using the Leibniz rule. This leads to consider the net rate of flow of mass across the boundary while computing the derivative (for more details, see [25]). If we denote the partial derivatives by $(.)_t=\frac{\partial(.)}{\partial t}$ and $(.)_x=\frac{\partial(.)}{\partial x}$ then, the total derivative operator with respect to time is given by

$$\frac{d}{dt}(.)=\dot{(.)}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}=(.)_t+v(.)_x. \tag{5}$$

The content of the remaining parts of this paper is structured into three parts. The first part is reserved to recall some preliminary results and to study the well posedness of the closed loop system. In the second part, we will use ADRC method to estimate the disturbance $d(t)$. In the last part, we will prove that under the adaptive control (3) and the results obtained in the second part that the system (4) is exponentially stable.

2. Existence result

In this section, we present an existence and uniqueness result for problem (4). We shall use the usual Lebesgue space $H=L^2(0,1)$ and Sobolev space $H^1(0,1)$. The scalar product and norm in H are denoted by $(.;.)$ and $\|.\|$, respectively. We introduce the following subspace

$$V=\left\{y\in H^1(0,1), y(0)=0\right\}.$$

equipped with the norm $\|w\|_V=\|w_x\|$. Clearly $V\subset H\subset V'$ where V' denotes the dual of V. The following inequalities will be utilized in this paper

2.1. Abstract setting. Let $A:D(A)\to H$, $Ay=-y_{xx}$ is self-adjoint operator with domain

$$D(A)=\left\{y\in H^1, \ y_{xx}\in H, \ y(0)=y_x(1)=0\right\}.$$

The operator A is a positive operator with a compact inverse in H. Thus, we have

$$A^{\frac{1}{2}}y:D(A)\to H, \ A^{\frac{1}{2}}y=y_x$$

with

$$D(A^{\frac{1}{2}}) = \left\{y \in H^1,\ y(0) = 0\right\} = V$$

and $A^{\frac{1}{2}}$ is a canonical isomorphism from V onto H (see [26]). An extension of A (which is denoted in the same way) is defined by

$$(Af, g)_{V' \times V} = \left(A^{\frac{1}{2}} f, A^{\frac{1}{2}} g\right)$$

for any $f, g \in V$. It holds that

$$(Ay, y) = (y_x, y_x) = \left\|A^{\frac{1}{2}} y\right\|. \tag{6}$$

By Poincaré inequality

$$\|y\| \leq \left\|A^{\frac{1}{2}} y\right\|. \tag{7}$$

The closed loop system (4) can be written as

$$\begin{cases} \frac{d^2y}{dt^2} + k(t)BB^* \frac{d}{dt} y + B\left(\vartheta + d\right) + Ay = 0,\ t > 0, \\ y(0) = y_0,\ y_t(0) = y_1, \\ \dot{k}(t) = pw^2\left(t\right),\ p > 0,\ k(0) = k_0 > 0,\ t > 0, \\ y_{out}(t) = B^* \dot{y}(t),\ t > 0. \end{cases} \tag{8}$$

The operator B is defined by $B : \mathbb{R} \to V'$, $B = \delta\left(x - 1\right)$ with $B^* = \left(\delta\left(x - 1\right), .\right)$ where δ is the Dirac distribution (see [27]). We denote by $Y\left(t\right) := \left(y\left(t\right), \frac{d}{dt} y\left(t\right), k\left(t\right)\right)$, then $\frac{d}{dt} Y := \left(\frac{d}{dt} y\left(t\right), \frac{d^2y}{dt^2}\left(t\right), k'\left(t\right)\right)$ and Y satisfies

$$\begin{cases} \frac{d}{dt} Y = \mathcal{A}Y + \mathcal{B}\left(d + \vartheta\right) \\ Y\left(0\right) = Y_0 = \left(y_0, y_1, k_0\right)^T \end{cases}$$

where

$$\mathcal{A}\begin{pmatrix} y \\ z \\ k \end{pmatrix} = \begin{pmatrix} z \\ -kBB^*z - Ay \\ p\left[B^*z\right]^2 \end{pmatrix} \tag{9}$$

with domain

$$D\left(\mathcal{A}\right) = \left\{\left(y, z, k\right) \in D\left(A\right) \times V \times \mathbb{R}^+, kBB^*z + Ay \in H\ \right\}$$

and $\mathcal{B} = \left(0, -B\right)^T$. Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = V \times H \times \mathbb{R}^+.$$

We equip $\mathcal{H}$ with the inner product

$$\left\langle \left(y_1, z_1, k_1\right)^T, \left(y_2, z_2, k_2\right)^T \right\rangle = \left(A^{\frac{1}{2}} y_1, A^{\frac{1}{2}} y_2\right) + \left(z_1, z_2\right) + \frac{1}{2p} k_1 k_2.$$

In order to show the existence and uniqueness of solution of (8), we use the nonlinear semigroup approach, see [28]. We will show that the operator $\mathcal{A}$ defined in (9) generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$ and since $\mathcal{B}$ is admissible to $e^{\mathcal{A}t}$ (see [17]). Therefore, the following well-posedness result holds (see [29], Proposition 4.2.5 p. 118).

Theorem 2.1. *For $Y_0 \in D\left(A\right)$, the system (4) admits a unique solution satisfying*

$$y \in C\left(\left[0, T\right), D\left(A\right)\right) \cap C^1\left(\left[0, T\right), \mathcal{H}\right)$$

Proof. We first show that $\mathcal{A}$ is dissipative. Let $(y,z,k)^T \in D(\mathcal{A})$, then

$$\langle \mathcal{A}(y_1,z_1,k_1) - \mathcal{A}(y_2,z_2,k_2), (y_1-y_2, z_1-z_2, k_1-k_2)\rangle$$
$$= \left(A^{\frac{1}{2}}(z_1-z_2), A^{\frac{1}{2}}(y_1-y_2)\right) + ((-k_1BB^*z_1 - Ay_1) - (-k_2BB^*z_2 - Ay_2), z_1-z_2)$$
$$+ \frac{1}{2}(k_1-k_2)\left([B^*z_1]^2 - [B^*z_2]^2\right)$$
$$= ((-k_1BB^*z_1) - (-k_1BB^*z_2), z_1-z_2) + \frac{1}{2}(k_1-k_2)\left([B^*z_1]^2 - [B^*z_2]^2\right)$$
$$= \left[\frac{1}{2}(k_1-k_2) - k_1\right][B^*z_1]^2 + (k_1+k_2)B^*z_1B^*z_2 + \left[\frac{1}{2}(k_2-k_1) - k_2\right][B^*z_2]^2.$$

It follows that

$$\langle \mathcal{A}(y_1,z_1,k_1) - \mathcal{A}(y_2,z_2,k_2), (y_1-y_2, z_1-z_2, k_1-k_2)\rangle$$
$$= -\frac{1}{2}(k_1+k_2)([B^*z_1] - [B^*z_2])^2.$$

This shows that $\mathcal{A}$ is dissipative in $\mathcal{H}$. Next, we show that $\lambda I - \mathcal{A}$ is surjective for some $\lambda > 0$. Given $(\phi, \varphi, \chi)^T \in \mathcal{H}$, we seek $(y,z,k)^T \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})\begin{pmatrix} y \\ z \\ k \end{pmatrix} = \begin{pmatrix} \phi \\ \varphi \\ \chi \end{pmatrix}.$$

This is equivalent to

$$\lambda y - z = \phi, \tag{10}$$
$$\lambda z + kBB^*z + Ay = \varphi, \tag{11}$$
$$\lambda k - p[B^*z]^2 = \chi. \tag{12}$$

We suppose that we have found y with the appropriate regularity. Then, by (10)

$$z = \lambda y - \phi \in V. \tag{13}$$

Moreover, from (10) and (13), k is given by

$$k = \frac{1}{\lambda}\left[\chi + p[B^*z]^2\right] = \frac{1}{\lambda}\left\{\chi + p[B^*(\lambda y - \phi)]^2\right\}.$$

It remains only to determine y. From (11) and (13), y satisfies

$$\lambda^2 y + \lambda kBB^*y + Ay = \varphi + \lambda\phi + \lambda kBB^*\phi. \tag{14}$$

We set

$$A_\lambda = \lambda^2 + \lambda kBB^* + A$$

and

$$w = \varphi + \lambda\phi + \lambda kBB^*\phi \in H \subset V'.$$

Let us introduce the operator

$$\Phi_\lambda : V \to V : y \to y - \frac{1}{\lambda^2}(A_\lambda y - w).$$

The existence of a fixed point for Φ_λ is clearly equivalent to the existence of a solution to equation (14). First, we need to establish some estimates. Using (6), we get

$$(\lambda kBB^*y + Ay, y)_{V',V} = \int_0^1 y_x^2 dx + \frac{\lambda}{k}y^2(1,t) \le (1+\lambda k)\int_0^1 y_x^2 dx = (1+\lambda k)\|y\|_V^2$$

and

$$(\lambda k B B^* y + Ay, y)_{V',V} = \int_0^1 y_x^2 dx + \lambda k y^2(1,t) \geq \|y\|_V^2 .$$

Using Cauchy Schwartz inequality, we see that

$$\left|(\lambda k B B^* y + Ay, u)_{V',V}\right| = \int_0^1 y_x u_x dx + \lambda k y(1,t) u(1,t) \leq (1+\lambda k) \|y\|_V \|u\|_V$$

which implies that

$$\|\lambda k B B^* y + Ay\|_{V'} \leq (1+\lambda k) \|y\|_V . \tag{15}$$

We therefore have to show that, for λ large enough Φ_λ is a contraction, since then Φ_λ will have a unique fixed point. For this, we calculate

$$\begin{aligned} \|\Phi_\lambda(y_1) - \Phi_\lambda(y_2)\|_V^2 = & \frac{1}{\lambda^4}\left(\lambda k B B^*(y_1 - y_2) + A(y_1 - y_2)\right., \\ & \left.\lambda k B B^*(y_1 - y_2) + A(y_1 - y_2)\right). \end{aligned}$$

From the estimate (15), we get

$$\|\Phi_\lambda(y_1) - \Phi_\lambda(y_2)\|_V^2 \leq \frac{1}{\lambda^4}(1+\lambda k)^2 \|y_1 - y_2\|_V^2 .$$

For sufficiently large λ, then Φ_λ is a contraction mapping from V into itself and has a unique fixed point y in V which is a solution of (14). Applying Crandall-Liggett theorem in ([28], Chapter 2) to get the existence and uniqueness result. □

3. Disturbance estimate

This section is reserved to to estimate the disturbance $d(t)$ in (4). We employ the active disturbance rejection control (ADRC) approach to investigate this problem (see [20]). At first, by the ADRC method, we design a disturbance observer that has time-varying gain so that the disturbance can be estimated. Multiplying the first equation in (4) by $g(x) = x$ and integrating over $(0,1)$, we get

$$\int_0^1 x \frac{d}{dt}(y_t + v y_x)\, dx = y_x(1,t) - \int_0^1 y_x dx = -y(1,t) + u(t) + d(t)$$

or

$$\frac{d}{dt}\int_0^1 x(y_t + v y_x)\, dx = \int_0^1 (y_t + v y_x)\, dx - y(1,t) + u(t) + \vartheta(t) + d(t). \tag{16}$$

Set

$$w(t) = \int_0^1 x(y_t + v y_x)\, dx, \;\; w_0(t) = \int_0^1 (y_t + v y_x)\, dx - y(1,t) .$$

Then, (16) becomes

$$\frac{d}{dt} w(t) = w_0(t) + u(t) + d(t). \tag{17}$$

The relation (17) is a simple ODE where the disturbance locates on the right side. This is the first step to estimate the disturbance using the technique introduced in [30] for lumped parameter systems. It is achieved through the following time varying high gain extended state observer for ODE system (17)

$$\begin{cases} \dot{\hat{w}}(t) = w_0(t) + u(t) + \hat{d}(t) - r(t)\left(\hat{w}(t) - w(t)\right), \\ \dot{\hat{d}}(t) = -r^2(t)\left(\hat{w}(t) - w(t)\right) \end{cases} \tag{18}$$

where $r \in C^1(\mathbb{R}_+)$ is a time varying gain that is required to satisfy the following conditions
(H1) $r(t), \dot{r}(t) > 0$ and $\sup_{t\geq 0} \frac{\dot{r}(t)}{r(t)} = M < \infty$,
(H2) $\lim_{t\to\infty} \frac{|d(t)|}{r(t)} = 0$.
The next result characterizes the convergence of extended state observer (18) for system (17).Accordingly, we can use $\hat{d}$ as an approximation of d. For this, we define the errors e_w and e_d as follows

$$e_w = -r(t)\left(w(t) - \hat{w}(t)\right),\ e_d = d(t) - \hat{d}(t),\ t \geq 0.$$

Lemma 3.1. *Let $d(t)$ satisfy (2), under the assumptions (**H1**) and (**H2**), the solution of (18) satisfies*

$$\lim_{t\to\infty} e_w = \lim_{t\to\infty} e_d = 0.$$

Furthermore, if $r(t) = Re^{r_0 t}$ with $R > C_d$ and $r_0 > a_0$, then there exists a positive constant B such that

$$|e_w|, |e_d| \leq Be^{-(r_0 - a_0)t},\ t \geq 0.$$

Proof. A differentiation of e_w gives

$$\dot{e}_w(t) = -\dot{r}(t)\left(w(t) - \hat{w}(t)\right) - r(t)\left(\dot{w}(t) - \dot{\hat{w}}(t)\right),\ t \geq 0. \tag{19}$$

Replacing $\dot{w}(t)$ and $\dot{\hat{w}}(t)$ by their expressions from (17) and (18) in (19), respectively, we obtain

$$\begin{aligned}\dot{e}_w(t) &= -\dot{r}(t)\left(w(t) - \hat{w}(t)\right)\\ &\quad - r(t)\left[w_0(t) + u(t) + d(t) - w_0(t) - u(t) - \hat{d}(t) + r(t)\left(\hat{w}(t) - w(t)\right)\right]\\ &= -\dot{r}(t)\left(w(t) - \hat{w}(t)\right) - r(t)\left(d(t) - \hat{d}(t)\right) + r^2(t)\left(w(t) - \hat{w}(t)\right)\\ &= \frac{\dot{r}(t)}{r(t)}e_w(t) - r(t)\left(e_w(t) + e_d(t)\right),\ t \geq 0. \end{aligned} \tag{20}$$

Differentiating now e_d, we get

$$\dot{e}_d = \dot{d}(t) - \dot{\hat{d}}(t),\ t \geq 0. \tag{21}$$

Replacing $\dot{\hat{d}}(t)$ by its expression from (18) in (21), we find

$$\dot{e}_d = \dot{d}(t) - r^2(t)(w(t) - \hat{w}(t)) = r(t)e_w(t) + \dot{d}(t),\ t \geq 0. \tag{22}$$

Then, it follows from (20) and (22) that the errors satisfy the following initial values problem

$$\begin{cases} \dot{e}_w(t) = -r(t)\left(e_w(t) + e_d(t)\right) + \frac{\dot{r}(t)}{r(t)}e_w(t),\ t \geq 0,\\ \dot{e}_d(t) = r(t)e_w(t) + \dot{d}(t),\ t \geq 0,\\ e_w(0) = e_{w,0},\ e_d(0) = e_{d,0} \end{cases} \tag{23}$$

where the initial values $e_{\eta,0}$ and $e_{d,0}$ are given. This system can be written as follows

$$\begin{cases} \dot{e}(t) = A(t)\,e(t) + f_d(t),\ t \geq 0,\\ e(0) = e_0 = (e_{\eta,0}, e_{d,0}) \end{cases} \tag{24}$$

with

$$A(t)=\begin{pmatrix} -r(t)+\frac{\dot r(t)}{r(t)} & -r(t) \\ r(t) & 0 \end{pmatrix},\ f_d(t)=\begin{pmatrix} 0 \\ \dot d(t) \end{pmatrix}$$

where $e(t)=(e_w(t),e_d(t))^T$ and $f_d(t)=\left(0,\dot d(t)\right)^T$. Since A and $f_d(t)$ are continuous for $t\geq 0$, then for every $e_0\in\mathbb{R}^2$, the linear Cauchy problem (24) has a unique global solution. Next, we construct a Lyapunov function for system (23)

$$\mathcal{L}(t)=e_w^2(t)+\frac{3}{2}e_d^2(t)+e_w(t)e_d(t),\ t\geq 0.$$

We see that

$$\frac{1}{2}\left(e_w^2(t)+e_d^2(t)\right)\leq\mathcal{L}(t)\leq 2\left(e_w^2(t)+e_d^2(t)\right),\ t\geq 0. \tag{25}$$

A differentiation of $\mathcal{L}(t)$ gives

$$\begin{aligned}\dot{\mathcal{L}}(t) &= 2\dot e_w(t)e_w(t)+3\dot e_d(t)e_d(t)+\dot e_w(t)e_d(t)+e_w(t)\dot e_d(t)\\ &\leq -r(t)\left(e_w^2(t)+e_d^2(t)\right)+2\frac{\dot r(t)}{r(t)}e_w^2(t)+\frac{\dot r(t)}{r(t)}e_w(t)e_d(t)\\ &\quad +\left|\dot d(t)\right|\left(3\left|e_d(t)\right|+\left|e_w(t)\right|\right),\ t\geq 0.\end{aligned}$$

System (23) and assumptions (2) imply that

$$\begin{aligned}\dot{\mathcal{L}}(t) &\leq \left(-r(t)+\frac{1}{2}\frac{\dot r(t)}{r(t)}\right)\left(e_w^2(t)+e_d^2(t)\right)+2\frac{\dot r(t)}{r(t)}e_w^2(t)+3C_d\left|d(t)\right|\left(\left|e_d(t)\right|+\left|e_w(t)\right|\right)\\ &\leq \left(-r(t)+\frac{5}{2}\frac{\dot r(t)}{r(t)}\right)\left(e_w^2(t)+e_d^2(t)\right)+3\sqrt{2}C_de^{a_0t}\sqrt{e_w^2(t)+e_d^2(t)},\ t\geq 0.\end{aligned}$$

Assumption (**H1**) imply for t large enough that

$$-r(t)+\frac{5}{2}\frac{\dot r(t)}{r(t)}<-r(t)+\frac{5}{2}M<0.$$

This is owing to

$$\dot{\mathcal{L}}(t)\leq\left(-r(t)+\frac{5}{2}M\right)\mathcal{L}(t)+3\sqrt{2}C_de^{a_0t}\sqrt{\mathcal{L}(t)},\ t\geq 0$$

which gives

$$\frac{d}{dt}\sqrt{\mathcal{L}(t)}\leq\frac{1}{2}\left(-r(t)+\frac{5}{2}M\right)\sqrt{\mathcal{L}(t)}+\frac{3}{\sqrt{2}}C_de^{a_0t},\ t\geq 0.$$

Integrating over (t_0,t), we get the following inequality

$$\sqrt{\mathcal{L}(t)}\leq\sqrt{\mathcal{L}(t_0)}e^{\frac{1}{2}\int_{t_0}^t\left(-r(s)+\frac{5}{2}M\right)ds}+\frac{3C_d}{\sqrt{2}}\int_{t_0}^t e^{a_0\tau}e^{\frac{1}{2}\int_\tau^t\left(-r(s)+\frac{5}{2}M\right)ds}d\tau,\ t>t_0.$$

L'Hospital rule together with assumption (**H2**) give

$$\begin{aligned}\lim_{t\to\infty}\int_{t_0}^t e^{a_0\tau}e^{\frac{1}{2}\int_\tau^t\left(-r(s)+\frac{5}{2}M\right)ds}d\tau &= \lim_{t\to\infty}\frac{\int_{t_0}^t e^{a_0\tau}e^{\frac{1}{2}\int_{t_0}^\tau\left(r(s)-\frac{5}{2}M\right)ds}d\tau}{e^{\frac{1}{2}\int_{t_0}^t\left(r(s)-\frac{5}{2}M\right)ds}}\\ &= 2\lim_{t\to\infty}\frac{e^{a_0t}}{2r(t)-5M}=0,\end{aligned}$$

then $\lim_{t\to\infty}\sqrt{\mathcal{L}(t)}=0$. This leads by the equivalence result (25) to

$$\lim_{t\to\infty}\left(e_{\eta}^{2}(t)+e_{d}^{2}(t)\right)=0.$$

This completes the proof of the first assertion of Lemma 3.1. Furthermore, since

$$\int_{t_0}^{t} e^{a_0\tau}e^{\frac{1}{2}\int_{\tau}^{t}\left(-r(s)+\frac{5}{2}M\right)ds}d\tau=\frac{\int_{t_0}^{t} e^{a_0\tau}e^{\frac{1}{2}\int_{t_0}^{\tau}\left(r(s)-\frac{5}{2}M\right)ds}d\tau}{e^{\frac{1}{2}\int_{t_0}^{t}\left(r(s)-\frac{5}{2}M\right)ds}},\ t>t_0.$$

We compute

$$\lim_{t\to\infty}\frac{\int_{t_0}^{t} e^{a_0\tau}e^{\frac{1}{2}\int_{\tau}^{t}\left(-r(s)+\frac{5}{2}M\right)ds}d\tau}{e^{-(r_0-a_0)t}}=\lim_{t\to\infty}\frac{\int_{t_0}^{t} e^{a_0\tau}e^{\frac{1}{2}\int_{t_0}^{\tau}\left(r(s)-\frac{5}{2}M\right)ds}d\tau}{e^{-(r_0-a_0)t+\frac{1}{2}\int_{t_0}^{t}\left(r(s)-\frac{5}{2}M\right)ds}}.$$

Again L'Hospital rule

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_{t_0}^{t} e^{a_0\tau}e^{\frac{1}{2}\int_{\tau}^{t}\left(-r(s)+\frac{5}{2}M\right)ds}d\tau}{e^{-(r_0-a_0)t}} &= 2\lim_{t\to\infty}\frac{e^{a_0 t}}{\left(Re^{r_0t}-\frac{5}{2}M-2\left(r_0-a_0\right)\right)e^{-(r_0-a_0)t}}\\ &= 2\lim_{t\to\infty}\frac{1}{e^{-r_0t}\left(Re^{r_0t}-\frac{5}{2}M-2b\right)}=\frac{2}{R}.\end{aligned}$$

This shows that there exists a positive constant A_1 such that

$$\frac{3C_d}{\sqrt{2}}\int_{t_0}^{t} e^{a_0\tau}e^{\frac{1}{2}\int_{\tau}^{t}\left(-\frac{r(s)}{\beta}+\frac{5}{2}M\right)ds}d\tau\leq A_1e^{-(r_0-a_0)t},\ t>0. \tag{26}$$

Assumption (**H1**) implies for t sufficiently large that

$$e^{\frac{1}{2}\int_{t_0}^{t}\left(-\frac{r(s)}{\beta}+\frac{5}{2}M\right)ds}\leq e^{-(r_0-a_0)t},\ t>0. \tag{27}$$

Now, the identities (26) and (27) together lead to

$$\sqrt{\mathcal{L}(t)}\leq Ae^{-(r_0-a_0)t},\ t>0.$$

So,

$$\sqrt{e_w^2(t)+e_d^2(t)}\leq A\sqrt{2}e^{-(r_0-a_0)t},\ t>0.$$

This proves the second assertion in Lemma 3.1 with $B=A\sqrt{2}$. □

The next step of ADRC is to cancel disturbance in the feedback-loop. From Lemma 3.1, $\hat{d}(t)$ can be considered as an estimate of the disturbance $d(t)$. For this, we take the feedback control law $\vartheta(t)=-\hat{d}(t)$. Then, the new closed-loop system is written as

$$\begin{cases} y_{tt}+2vy_{xt}-\left(1-v^2\right)y_{xx}=0,\ \ x\in(0,1)\,,\ t>0,\\ y(0,t)=0,\quad t\geq 0,\\ y_x(1,t)=-k(t)\left(y_t+vy_x\right)(1,t)+e_d,\ t\geq 0,\\ \dot{e}_w(t)=-r(t)\left(e_w(t)+e_d(t)\right)+\frac{\dot{r}(t)}{r(t)}e_w(t),\\ \dot{e}_d(t)=r(t)e_w(t)+\dot{d}(t),\ t\geq 0,\\ y(x,0)=y_0(x),\ y_t(x,0)=y_1(x),\ \ x\in(0,1)\,,\\ e_w(0)=e_{w,0},\ e_d(0)=e_{d,0}.\\ k'(t)=p\left(y_t+vy_x\right)^2(1,t),\ p>0,\ k(0)=k_0>0.\end{cases} \tag{28}$$

4. Exponential stability

In this section, we consider the stability of the closed-loop system (28). First, we define the classical energy associated to (28) by

$$E(t) = E_y(t) + E_{w,d}(t), \ t \geq 0$$

where $E_y(t)$ and $E_{w,d}(t)$ are given by

$$E_y(t) = \frac{1}{2}\|y_t + vy_x\|^2 + \frac{1}{2}\|y_x\|^2, \ t \geq 0$$

and

$$E_{w,d}(t) = e_w^2(t) + e_d^2(t), \ t \geq 0.$$

According to Lemma 3.1, $E_{w,d}(t)$ converges to zero exponentially, so there remains to show that $E_y(t)$ converges to zero exponentially.

Theorem 4.1. *Under assumptions **(H1)** and **(H2)**, the solution of (28) satisfies* $(y_t + vy_x)(1) \in L^2(0,\infty)$ *and there exist two positive constants M and δ, independent of t such that*

$$E_y(t) \leq Me^{-\delta t}, \ t \geq 0.$$

Proof. Applying (5), it follows that

$$\frac{d}{dt}E(t) = \int_0^1 (y_t + vy_x)\left(y_{tt} + 2vy_{xt} + v^2 y_{xx}\right) dx + \int_0^1 y_x (y_{xt} + vy_{xx})\, dx, \ t \geq 0.$$

Substituting the second derivative of y from (28) into the previous identity and integrating by parts, we obtain

$$\frac{d}{dt}E(t) = [(y_t + vy_x)\, y_x]_0^1, \ t \geq 0. \tag{29}$$

The second boundary condition in (28) implies

$$\frac{d}{dt}E(t) = -k(t)\,(y_t + vy_x)^2(1) - vy_x^2(0) + (y_t + vy_x)(1)\, e_d(t), \ t \geq 0.$$

Note that from Lemma 3.1 if $r(t) = Re^{r_0 t}$ with $R > C_d$ and $r_0 > a_0$, then there exist a positive constant B such that

$$\frac{d}{dt}E(t) \leq -k(t)\,(y_t + vy_x)^2(1) - vy_x^2(0) + B\,|(y_t + vy_x)(1)|\, e^{-(r_0 - a_0)t}, \ t \geq 0. \tag{30}$$

It follows by Young inequality that

$$\frac{d}{dt}E(t) \leq \frac{1}{2}(k_0 - 2k(t))\,(y_t + vy_x)^2(1) - vy_x^2(0,t) + \frac{2B^2}{k_0}e^{-2(r_0-a_0)t}, \ t \geq 0.$$

In order to show that the output $w(t)$ is well defined in $L^2(0;T)$ for any $T > 0$, we consider the Lyapunov functional $L(t)$ for system (4) as follows

$$L(t) = E(t) + \frac{1}{2p}k^2(t), \ t \geq 0.$$

Along solutions of (4), the total derivative of $L(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}L(t) &= \frac{d}{dt}E(t) + \frac{1}{2}k(t)\,(y_t + vy_x)^2(1)\\ &\leq \frac{1}{2}(k_0 - k(t))\,(y_t + vy_x)^2(1) - vy_x^2(0) + \frac{2B^2}{k_0}e^{-2(r_0-a_0)t}, \ t \geq 0.\end{aligned}$$

Since k is increasing, we get

$$\frac{d}{dt}L(t) \leq \frac{2B^2}{k_0}e^{-2(r_0-a_0)t},\ t \geq 0.$$

which implies by integrating over $(0,t)$ that

$$L(t) \leq E(0) + \frac{B^2}{(r_0-a_0)\,k_0} = L_1,\ t \geq 0.$$

This implies that

$$\sup_{t\geq 0}\left[E(t) + \frac{1}{2p}k^2\,(t)\right] \leq L_1,\ t \geq 0. \tag{31}$$

It results that

$$k\,(t) < \sqrt{2pL_1},\ t \geq 0. \tag{32}$$

Since $k'(t) = pw^2(t)$, $r > 0$, it results by integrating over $(0,t)$ for all $t \geq 0$ and the previous relation (31) that $w(t) \in L^2(0,\infty)$. In order to prove the rest of the assertions, we introduce the functional

$$\mathcal{V}(t) = E(t) + \epsilon\Phi(t),\ t \geq 0$$

where

$$\Phi(t) = \int_0^1 xy_x\,(y_t + vy_x)\,dx,\ t \geq 0$$

for some positive constants ϵ to be determined later. First, we establish an equivalence result between $\mathcal{V}(t)$ and $E(t)$. Applying Young and Poincaré inequalities to the functionals Φ, we find

$$\begin{aligned} |\Phi(t)| &\leq \int_0^1 |y_x\,(y_t + vy_x)|\,dx \\ &\leq \frac{1}{2}\left(\|y_t + vy_x\|^2 + \|y_x\|^2\right) \leq E(t),\ t \geq 0. \end{aligned}$$

Then, the following relation holds

$$\beta_1 E(t) \leq \mathcal{V}(t) \leq \beta_2 E(t),\ t \geq 0 \tag{33}$$

where $\beta_1 = 1-\epsilon$ and $\beta_2 = 1+\epsilon$ with $\epsilon < 1$. A differentiation of Φ gives

$$\begin{aligned} \frac{d}{dt}\Phi(t) &= \int_0^1 x\,(y_{xt} + vy_{xx})\,(y_t + vy_x)\,dx + v\int_0^1 y_x\,(y_t + vy_x)\,dx \\ &\quad + \int_0^1 xy_x\left(y_{tt} + 2vy_{xt} + v^2 y_{xx}\right)dx,\ t \geq 0. \end{aligned} \tag{34}$$

Substituting the second derivative from (4) into (34), we find

$$\begin{aligned} \frac{d}{dt}\Phi_1(t) &= \int_0^1 x\,(y_{xt} + vy_{xx})\,(y_t + vy_x)\,dx + v\int_0^1 y_x\,(y_t + vy_x)\,dx \\ &\quad + \int_0^1 xy_x y_{xx}dx,\ t \geq 0. \end{aligned} \tag{35}$$

Taking into account the boundary conditions, performing integration by parts and using Young inequality allow us to estimate the different terms in (35) as follows

$$\int_0^1 x\,(y_{xt} + vy_{xx})\,(y_t + vy_x)\,dx = \frac{1}{2}\,(y_t + vy_x)^2\,(1) - \frac{1}{2}\,\|y_t + vy_x\|^2\,,\ t \geq 0, \tag{36}$$

$$\int_0^1 y_x (y_t + vy_x)\, dx \le \frac{1}{2} \|y_t + vy_x\|^2 + \frac{1}{2} \|y_x\|^2,\ t \ge 0, \tag{37}$$

$$\int_0^1 xy_x y_{xx} dx = \frac{1}{2} y_x^2(1,t) - \frac{1}{2} \|y_x\|^2,\ t \ge 0. \tag{38}$$

Gathering the estimates (36)-(38) in (35), we obtain

$$\frac{d}{dt}\Phi(t) \le -\frac{1}{2}(1-v)\|y_t + vy_x\|^2 - \frac{1}{2}(1-v)\|y_x\|^2 + \frac{1}{2}(y_t+vy_x)^2(1) + \frac{1}{2}y_x^2(1),\ t \ge 0.$$

From the second boundary condition, it iholds that

$$y_x^2(1,t) = 2k^2(t)(y_t+vy_x)^2(1) + 2e_d^2 \le 2k^2(t)(y_t+vy_x)^2(1) + 2B^2 e^{-2(r_0-a_0)t},\ t \ge 0.$$

Then, we get

$$\begin{aligned}\frac{d}{dt}\Phi(t) \quad \le \quad & -\frac{1}{2}(1-v)\|y_t+vy_x\|^2 - \frac{1}{2}(1-v)\|y_x\|^2 + \frac{1}{2}\left(1+2k^2(t)\right)(y_t+vy_x)^2(1) \\ & + B^2 e^{-2(r_0-a_0)t},\ t \ge 0.\end{aligned} \tag{39}$$

Now, taking into account the relations (30) and (39), it holds

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) \le -\frac{\epsilon}{2}(1-v)\|y_t+vy_x\|^2 - \frac{\epsilon}{2}(1-v)\|y_x\|^2 - \left[k(t) - \frac{\epsilon}{2}\left(1+2k^2(t)\right)\right] \\ \times (y_t+vy_x)^2(1) + \epsilon B^2 e^{-2(r_0-a_0)t} + B\left|(y_t+vy_x)(1)\right| e^{-(r_0-a_0)t},\ t \ge 0.\end{aligned} \tag{40}$$

Applying the Young inequality to the last term in (40), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) \quad \le \quad & -\epsilon(1-v)E(t) - \left[k(t) - \frac{\epsilon}{2}\left(1+2k^2(t)\right) - \eta\right](y_t+vy_x)^2(1) \\ & + \left(\epsilon + \frac{1}{4\eta}\right) B^2 e^{-2(r_0-a_0)t},\ t \ge 0.\end{aligned} \tag{41}$$

for some $\eta > 0$. Since $k(t) < \sqrt{2pL_1}$ (see (32)) and $k(t) \ge k_0$, it follows that

$$k(t) - \frac{\epsilon}{2}\left(1+2k^2(t)\right) \ge k_0 - \frac{\epsilon}{2}(1+4pL_1),\ t \ge 0. \tag{42}$$

By virtue of (33), it results from (41) and (42) that

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) \quad \le \quad & -\frac{\epsilon}{\beta_2}\mathcal{V}(t) - \left[k_0 - \frac{\epsilon}{2}(1+4pL_1) - \eta\right](y_t+vy_x)^2(1) \\ & + \left(\epsilon + \frac{1}{4\eta}\right) B^2 e^{-2(r_0-a_0)t},\ t \ge 0.\end{aligned} \tag{43}$$

Choosing ϵ and η small enough, we end up with

$$\frac{d}{dt}\mathcal{V}(t) \le -\frac{\epsilon}{\beta_2}\mathcal{V}(t) + \left(\epsilon + \frac{1}{4\eta}\right) B^2 e^{-2(r_0-a_0)t},\ t \ge 0. \tag{44}$$

Integrating (44) over $(0,t)$, we entail that

$$\begin{aligned}\mathcal{V}(t) &\le \mathcal{V}(0)e^{-\frac{\epsilon}{\beta_2}t} + \left(\epsilon + \frac{1}{4\eta}\right) B^2 e^{-\frac{\epsilon}{\beta_2}t} \int_0^t e^{\left[-2(r_0-a_0)+\frac{\epsilon}{\beta_2}\right]s} ds \\ &\le \mathcal{V}(0)e^{-\frac{\epsilon}{\beta_2}t} + \left(\epsilon + \frac{1}{4\eta}\right) \frac{B^2}{\left[2(r_0-a_0) - \frac{\epsilon}{\beta_2}\right]}\left(e^{-\frac{\epsilon}{\beta_2}t} - e^{-2(r_0-a_0)t}\right),\ t \ge 0.\end{aligned} \tag{45}$$

Choosing again ϵ sufficiently small so that $2\left(r_0 - a_0\right) - \frac{\epsilon}{\beta_2} > 0$ and exploiting (33), we obtain

$$E_y(t) \leq Me^{-\delta t},\ t \geq 0.$$

where $M = \frac{\mathcal{V}(0)}{\beta_1} + \left(\epsilon + \frac{1}{4\eta}\right) \frac{B^2}{\beta_1\left[2(r_0 - a_0) - \frac{\epsilon}{\beta_2}\right]}$ and $\delta = \frac{\epsilon}{\beta_2}$. Then, the assertion of the theorem is established. □

Remark 4.1. The obtained results are valid for the immobile case ($v = 0$) and the present work improves the results of Kobayashi [24] for the linear case.

5. Conclusion

Throughout this study, we have dealt with the stabilization of an axially moving string subject to a boundary disturbances. A boundary control force containing two parts is applied. The first part is adaptive and is formulated using the high-gain output feedback, which ensures the exponential stabilization of the system without disturbances. The second part is destined to remove disturbances, which is constructed by following the ADRC approach. It is shown that a state feedback estimator can estimate exponentially the disturbance in real time. Thereafter, the disturbance is canceled in the feedback loop. The exponential stability of the closed loop system is shown with the help of a Lyapunov-type functional. This functional is equal to the energy modified by an appropriate functional. Our future concerns are to examine the impact of the above considerations on other axially moving systems, namely moving beams, such as Euler-Bernoulli beams and Timoshenko beams.

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