Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 48(1), 2021, Pages 127–136
ISSN: 1223-6934

On strong lacunary summability of order α with respect to modulus functions

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Abstract. This research paper focuses on defining the relationships between the sets of strongly lacunary summable and lacunary statistically convergent sequences of complex numbers by using different modulus functions f and g under certain conditions and different orders $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$. Furthermore, for some special modulus functions, we establish the relations between the sets of strongly f-lacunary summable sequences and strongly f-lacunary summable sequences of order α.

2010 Mathematics Subject Classification. Primary 40A05; Secondary 40C05, 46A45.
Key words and phrases. Lacunary sequence, strong lacunary summability, statistical convergence, modulus function.

1. Introduction

The principle of statistical convergence was relied on the first version of the monograph of Zygmund [30] in 1935, and its definition was implemented in a short note by Fast [13] and later implemented independently by Schoenberg [27] with some specific characteristics of statistical convergence. In recent decades, statistical convergence has been mentioned in many several fields and under different names, such as hopfield neural network, approximation theory, Banach spaces, measure theory, summability theory, locally convex spaces, turnpike theory, number theory, ergodic theory, Fourier analysis, optimization and trigonometric series. Subsequently, Connor [5], Fridy [15], Šalát [28], Rath and Tripathy [25], Et [11], and many others were further explored from the perspective of the spaces of sequence and referred to the theory of summability. Further details and applications of this principle are available in [4, 9, 10, 14, 16, 20, 23].

Gadjiev and Orhan [18] provided the order of statistical convergence of a sequence of operators and then Çolak [8] provided and studied the order of statistical convergence for a sequence of numbers.

In 1953, Nakano [24] presented the thought of a modulus function for the first time. By using a modulus function Bhardwaj and Singh [3], Connor [6], Çolak [7], Gosh and Srivastava [19], Maddox [21], Ruckle [26], Altin and Et [2] and others have constructed and discussed some sequence spaces. In 2014, with the benefit of an unbounded modulus function, Aizpuru et al. [1] characterized another density's idea, as an outcome, a new nonmatrix convergence principle was acquired.

Received July 30, 2020. Accepted August 10, 2020.

2. Preliminaries

In this paper, the symbols c and ℓ_∞ represent the spaces of convergent and bounded sequences, respectively, as well as the symbols $\mathbb{C}$ and $\mathbb{N}$ represent the sets of all complex and natural numbers, respectively.

The number $\delta(E)$ of a set $E \subset \mathbb{N}$ is identified via

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |E_n|$$

and is named a natural density of E, where $|E_n| = |\{i \leq n : i \in E\}|$, the number of the elements of indicated set. It is obvious $\delta(\mathbb{N}) = 1$ and $\delta(E) = 0$ if $E \subset \mathbb{N}$ is a finite set and $\delta(\mathbb{N}\backslash E) = \delta(\mathbb{N}) - \delta(E) = 1 - \delta(E)$.

A sequence (ζ_k) in $\mathbb{C}$ is named convergent statistically (or S-convergent) to some $\zeta \in \mathbb{C}$ if for each $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |\zeta_k - \zeta| \geq \varepsilon\}) = 0$.

Let $0 < \alpha \leq 1$ be given. A sequence (ζ_k) in $\mathbb{C}$ is named convergent statistically of order α (or S^α-convergent) to some $\zeta \in \mathbb{C}$ if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : |\zeta_k - \zeta| \geq \varepsilon\}| = 0.$$

We imply a lacunary sequence $\theta = (k_r)$ of nonnegative integer numbers with $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. And the intervals formed by θ shall be represented by $I_r = (k_{r-1}\ , k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ can be shortened by q_r (see [17]).

Orhan and Fridy [17] defined lacunary statistical convergence as the following expression.

Suppose $\theta = (k_r)$ is a lacunary sequence. A sequence (ζ_k) of numbers is named lacunary statistically convergent (or S_θ-convergent) to ζ, if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |\zeta_k - \zeta| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. In this particular situation, we write $x_k \to \zeta\,(S_\theta)$ or $S_\theta - \lim \zeta_k = \zeta$. Throughout the paper, the class of S_θ-convergent sequences would be symbolized by S_θ.

A function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is named modulus, if the conditions mentioned below hold:

(i) $f(u) = 0 \Leftrightarrow u = 0$,
(ii) $f(u_1 + u_2) \leq f(u_1) + f(u_2)$ for every $u_1, u_2 \in \mathbb{R}^+ \cup \{0\}$,
(iii) f is increasing,
(iv) f is continuous at 0, from the right.

According to these characteristics, it is evident that a modulus$f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is continuous. There are unbounded and bounded modulus functions. As an instance, let us take $f(u) = \log(u + 1)$ and $g(u) = \frac{u}{u+1}$, then f is an unbounded modulus, but g is a bounded modulus. We also have $f(nu) \leq nf(u)$ for every modulus f and each positive integer n from condition (ii).

Lemma 2.1. [22] *For any modulus f, $\lim_{u\to\infty} \frac{f(u)}{u} = \beta$ exists and $\lim_{u\to\infty} \frac{f(u)}{u} = \inf_{u\in(0\,,\infty)} \frac{f(u)}{u}$.*

The definition below was given in [1] by Aizpuru et al.

The number $\delta_f(E)$ of a set $E \subset \mathbb{N}$ is identified via

$$\delta_f(E) = \lim_{n\to\infty} \frac{1}{f(n)} f(|E_n|)$$

and is called the f-density of E, where f is an unbounded modulus.

A sequence (ζ_k) in $\mathbb{C}$ is named f-statistically convergent (or $S(f)$-convergent) to some $\zeta \in \mathbb{C}$ if for any $\varepsilon > 0$, $\delta_f(\{k \in \mathbb{N} : |\zeta_k - \zeta| \geq \varepsilon\}) = 0$. $S(f)$ symbolizes the class of $S(f)$-convergent sequences throughout the paper.

Definition 2.1. Suppose f is an unbounded modulus, $\alpha \in (0, 1]$, and suppose $\theta = (k_r)$ is a lacunary sequence. Then the sequence (ζ_k) in $\mathbb{C}$ is named f-lacunary statistically convergent of order α (or $S_\theta^\alpha(f)$-convergent) to some $\zeta \in \mathbb{C}$, if

$$\lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} f(|k \in I_r : |\zeta_k - \zeta| \geq \varepsilon|) = 0$$

for every $\varepsilon > 0$. We write $\zeta_k \to \zeta\,(S_\theta^\alpha(f))$ or $S_\theta^\alpha(f) - \lim \zeta_k = \zeta$ in this particular situation and $S_\theta^\alpha(f)$ represents the class of $S_\theta^\alpha(f)$-convergent sequences throughout the paper. That is,

$$S_\theta^f = \left\{ (\zeta_k) : \lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |\zeta_k - \zeta| \geq \varepsilon\}|) = 0 \text{ for every } \varepsilon > 0 \right\}.$$

In the case $f(u) = u$, the concepts of $S_\theta^\alpha(f)$-convergence and S_θ^α-convergence are the same, that is, $S_\theta^\alpha(f)$ will reduce to S_θ^α, and in the particular case $f(u) = u$ and $\alpha = 1$, the concepts of $S_\theta^\alpha(f)$-convergence and S_θ-convergence are the same, that is, $S_\theta^\alpha(f)$ will reduce to S_θ.

Remark 2.1. It is easy to show that the $S_\theta^\alpha(f)$-convergence is not well defined for $\alpha > 1$.

Lemma 2.2. *The $S_\theta^\alpha(f)$-limit of an $S_\theta^\alpha(f)$-convergent sequence is unique.*

Theorem 2.3. *Suppose $\theta = (k_r)$ is a lacunary sequence and $\alpha \in (0, 1]$. Then*
(i) $S_\theta^\alpha(f) \subset S_\theta(f)$ for any unbounded modulus f.
(ii) $S_\theta^\alpha \subset S_\theta$.

The proof is clear, so it is omitted.

3. Main results

In this part of the paper, we are establishing the relations between $N_\theta^\alpha(g)$ and $N_\theta^\beta(f)$, $N_\theta^\beta(g)$ and $N_\theta^\alpha(f)$, $S_\theta^\beta(g)$ and $N_\theta^\alpha(f)$, $N_\vartheta^\beta(g)$ and $\ell_\infty \cap S_\theta^\alpha(f)$, where f and g are modulus functions under certain conditions and $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$. Moreover, the relations between N_θ and S_θ, S_θ and S are already established (see [17]).

Definition 3.1. Suppose f is a modulus function, $\theta = (k_r)$ is a lacunary sequence, and suppose $\alpha \in (0, 1]$. Then the sequence (ζ_k) in $\mathbb{C}$ is named strongly f-lacunary summable of order α (or strongly $N_\theta^\alpha(f)$-summable) to some $\zeta \in \mathbb{C}$, if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} f(|\zeta_k - \zeta|) = 0.$$

If the sequence (ζ_k) is strongly $N_\theta^\alpha(f)$-summable to ζ, we write $\zeta_k \to \zeta\left(N_\theta^\alpha(f)\right)$ or $N_\theta^\alpha(f) - \lim \zeta_k = \zeta$. The class of strongly $N_\theta^\alpha(f)$-summable sequences would be symbolized by $N_\theta^\alpha(f)$. That is,

$$N_\theta^\alpha(f) = \left\{(\zeta_k) : \lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} f(|\zeta_k - \zeta|) = 0 \text{ for some number } \zeta\right\}.$$

Note that this definition does not require the modulus function f to be unbounded.

The strong $N_\theta^\alpha(f)$-summability will reduce to the strong N_θ^α-summability if we take $f(u) = u$, and in the particular case $\alpha = 1$ and $f(u) = u$, the strong $N_\theta^\alpha(f)$-summability will reduce to the strong N_θ-summability.

Theorem 3.1. *Suppose f and g are modulus functions, $\alpha, \beta \in (0, 1]$ such that $\alpha \le \beta$, and suppose $\theta = (k_r)$ is a lacunary sequence. If*

$$\sup_{u\in(0,\infty)} \frac{f(u)}{g(u)} < \infty,$$

then $N_\theta^\alpha(g) \subset N_\theta^\beta(f)$.

Proof. Let $p = \sup_{u\in(0,\infty)} \frac{f(u)}{g(u)} < \infty$. Then we have $0 < \frac{f(u)}{g(u)} \le p$ and so $f(u) \le pg(u)$ for any $u \in \mathbb{R}^+ \cup \{0\}$. Now it is clear that $p > 0$ and if (ζ_k) is strongly $N_\theta^\alpha(g)$-summable to $\zeta \in \mathbb{C}$, then

$$\frac{1}{h_r^\alpha} \sum_{k\in I_r} f(|\zeta_k - \zeta|) \le \frac{1}{h_r^\alpha} \sum_{k\in I_r} pg(|\zeta_k - \zeta|).$$

Since $\alpha \le \beta$, we have

$$\frac{1}{h_r^\beta} \sum_{k\in I_r} f(|\zeta_k - \zeta|) \le p\frac{1}{h_r^\alpha} \sum_{k\in I_r} g(|\zeta_k - \zeta|).$$

Taking the limits on both sides as $r \to \infty$, we obtain that $(\zeta_k) \in N_\theta^\alpha(g)$ implies $(\zeta_k) \in N_\theta^\beta(f)$. □

Remark 3.1. The following illustration demonstrates that at least for some $\alpha, \beta \in (0, 1]$ such that $\alpha \le \beta$ and certain different modulus functions f and g, the inclusion $N_\theta^\alpha(g) \subset N_\theta^\beta(f)$ is strict.

Example 3.1. Let the lacunary sequence $\theta = (k_r)$ be given and choose $\alpha = \beta = 1$ and also define (ζ_k) as ζ_k to be $\left[\sqrt{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $\zeta_k = 0$ otherwise, where $[t]$ denotes an integral part of the real number t. Now if we take the modulus functions $f(u) = \frac{u}{u+1}$ and $g(u) = u$, then $\sup_{u\in(0,\infty)} \frac{f(u)}{g(u)} = 1 < \infty$ and thus $N_\theta^\alpha(g) \subset N_\theta^\beta(f)$ by Theorem 3.1. By using the $f(0) = 0$ equality, we have

$$\frac{1}{h_r^\beta} \sum_{k\in I_r} f(|\zeta_k|) = \frac{1}{h_r}\left[\sqrt{h_r}\right] f\left(\left[\sqrt{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r\left(\left[\sqrt{h_r}\right]+1\right)}.$$

Taking the limits as $r \to \infty$, we get that $N_\theta^\beta(f) - \lim \zeta_k = 0$. So that $(x_k) \in N_\theta^\beta(f)$. But since

$$\frac{1}{h_r^\alpha} \sum_{k\in I_r} g(|\zeta_k|) = \frac{1}{h_r}\left[\sqrt{h_r}\right] g\left(\left[\sqrt{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r}$$

and $\frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r} \to 1$ as $r \to \infty$, we get $(\zeta_k) \notin N_\theta^\alpha(g)$. Hence $(\zeta_k) \in N_\theta^\beta(f) - N_\theta^\alpha(g)$ and the inclusion $N_\theta^\alpha(g) \subset N_\theta^\beta(f)$ is being strict.

The outcome below of inclusions is obtained from Theorem 3.1.

Corollary 3.2. *Suppose f and g are modulus functions, $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$, and suppose $\theta = (k_r)$ is a lacunary sequence*

(i) If $\sup_{u \in (0, \infty)} \frac{f(u)}{g(u)} < \infty$, then $N_\theta^\alpha(g) \subset N_\theta^\alpha(f)$.

(ii) If $\sup_{u \in (0, \infty)} \frac{f(u)}{g(u)} < \infty$, then $N_\theta(g) \subset N_\theta(f)$.

(iii) $N_\theta^\alpha(f) \subset N_\theta^\beta(f)$.

(iv) $N_\theta^\alpha \subset N_\theta^\beta$.

Theorem 3.3. *Suppose f and g are modulus functions, $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$, and suppose $\theta = (k_r)$ is a lacunary sequence. If*

$$\inf_{u \in (0, \infty)} \frac{f(u)}{g(u)} > 0,$$

then $N_\theta^\alpha(f) \subset N_\theta^\beta(g)$ and the inclusion is strict.

Proof. Let $q = \inf_{u \in (0, \infty)} \frac{f(u)}{g(u)} > 0$. So that $\frac{f(u)}{g(u)} \geq q$ and $qg(u) \leq f(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$. Now if (ζ_k) is strongly $N_\theta^\alpha(f)$-summable to $\zeta \in \mathbb{C}$, then

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} g(|\zeta_k - \zeta|) \leq \frac{1}{h_r^\alpha} \sum_{k \in I_r} \frac{1}{q} f(|\zeta_k - \zeta|).$$

Since $\alpha \leq \beta$, we have

$$\frac{1}{h_r^\beta} \sum_{k \in I_r} g(|\zeta_k - \zeta|) \leq \frac{1}{h_r^\alpha} \sum_{k \in I_r} \frac{1}{q} f(|\zeta_k - \zeta|).$$

Taking the limits on both sides as $r \to \infty$, we obtain that $(\zeta_k) \in N_\theta^\alpha(f)$ implies $(\zeta_k) \in N_\theta^\beta(g)$.

For the strict inclusion, the sequence of Example 3.1 with modulus functions $g(u) = \frac{u}{u+1}$ and $f(u) = u$ serve the purpose in the case $\alpha = \beta = 1$. □

The outcome below of inclusions is a result of Theorem 3.2.

Corollary 3.4. *Suppose f and g are modulus functions, $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$, and suppose $\theta = (k_r)$ is a lacunary sequence.*

(i) If $\inf_{u \in (0, \infty)} \frac{f(u)}{g(u)} > 0$, then $N_\theta^\alpha(f) \subset N_\theta^\alpha(g)$.

(ii) If $\inf_{u \in (0, \infty)} \frac{f(u)}{g(u)} > 0$, then $N_\theta(f) \subset N_\theta(g)$

(iii) $N_\theta^\alpha(f) \subset N_\theta^\beta(f)$.

(iv) $N_\theta^\alpha \subset N_\theta^\beta$.

The following result is obtained from Theorem 3.1 and Theorem 3.2.

Corollary 3.5. *Suppose f and g are modulus functions, $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$ and $\theta = (k_r)$ is a lacunary sequence. If*

$$0 < \inf_{u\in(0,\infty)} \frac{f(u)}{g(u)} \leq \sup_{u\in(0,\infty)} \frac{f(u)}{g(u)} < \infty,$$

then $N_\theta^\alpha(f) = N_\theta^\alpha(g)$.

Corollary 3.6. *Suppose f is any modulus function, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\sup\limits_{u\in(0,\infty)} \frac{f(u)}{u} < \infty$, then $N_\theta^\alpha \subset N_\theta^\beta(g)$ for any $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$.*

Since $\sup\limits_{u\in(0,\infty)} \frac{f(u)}{u} < \infty$, taking $g(u) = u$ in Theorem 3.1, the proof follows directly.
The following result is obtained by taking $\beta = \alpha$ in the above corollary.

Corollary 3.7. *Suppose f is any modulus function, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\sup\limits_{u\in(0,\infty)} \frac{f(u)}{u} < \infty$, then $N_\theta^\alpha \subset N_\theta^\alpha(f)$ for any $\alpha \in (0, 1]$.*

Corollary 3.8. *Suppose f is any modulus function, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\inf\limits_{u\in(0,\infty)} \frac{f(u)}{u} > 0$, then $N_\theta^\alpha(f) \subset N_\theta^\beta$ for any $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$.*

Since $\inf\limits_{u\in(0,\infty)} \frac{f(u)}{u} > 0$, taking $g(u) = u$ in Theorem 3.2 the proof follows directly.
The following result is obtained by taking $\beta = \alpha$ in the above corollary.

Corollary 3.9. *Suppose f is any modulus function, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\inf\limits_{u\in(0,\infty)} \frac{f(u)}{u} > 0$, then $N_\theta^\alpha(f) \subset N_\theta^\alpha$ for any $\alpha \in (0, 1]$.*

The following result is obtained from Corollary 3.5 and Corollary 3.7.

Corollary 3.10. *Suppose f is any modulus function, and suppose $\theta = (k_r)$ is a lacunary sequence. If $0 < \inf\limits_{u\in(0,\infty)} \frac{f(u)}{u} \leq \sup\limits_{u\in(0,\infty)} \frac{f(u)}{u} < \infty$, then $N_\theta^\alpha(f) = N_\theta^\alpha$ for any $\alpha \in (0, 1]$.*

Theorem 3.11. *Suppose f and g are any unbounded modulus functions, $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\inf\limits_{u\in(0,\infty)} \frac{f(u)}{g(u)} > 0$ and $\lim\limits_{u\to\infty} \frac{g(u)}{u} > 0$, then every strongly $N_\theta^\alpha(f)$-summable sequence is $S_\theta^\beta(g)$-statistically convergent.*

Proof. Suppose that $q = \inf\limits_{u\in(0,\infty)} \frac{f(u)}{g(u)} > 0$. Then $\frac{f(u)}{g(u)} \geq q$ and so $qg(u) \leq f(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$. Now if (ζ_k) is strongly $N_\theta^\alpha(f)$-summable to $\zeta \in \mathbb{C}$ and

$\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$, then

$$\begin{aligned} \frac{1}{h_r^\alpha} \sum_{k \in I_r} f(|\zeta_k - \zeta|) &\geq q \frac{1}{h_r^\alpha} \sum_{k \in I_r} g(|\zeta_k - \zeta|) \\ &\geq q \frac{1}{h_r^\beta} \sum_{k \in I_r} g(|\zeta_k - \zeta|) \\ &= q \frac{1}{h_r^\beta} \sum_{\substack{k \in I_r \\ |\zeta_k - l| \geq \varepsilon}} g(|\zeta_k - \zeta|) + q \frac{1}{h_r^\beta} \sum_{\substack{k \in I_r \\ |\zeta_k - l| < \varepsilon}} g(|\zeta_k - \zeta|) \\ &\geq q \frac{1}{h_r^\beta} \sum_{\substack{k \in I_r \\ |\zeta_k - l| \geq \varepsilon}} g(|\zeta_k - \zeta|) \\ &\geq q \frac{1}{h_r^\beta} |\{k \in I_r : |\zeta_k - \zeta| \geq \varepsilon\}| \, g(\varepsilon). \end{aligned}$$

Since $|\{k \in I_r : |\zeta_k - \zeta| \geq \varepsilon\}|$ is a positive integer, we get

$$\begin{aligned} \frac{1}{h_r^\alpha} \sum_{k \in I_r} f(|\zeta_k - \zeta|) &\geq \frac{1}{h_r^\beta} g(|\{k \in I_r : |\zeta_k - \zeta| \geq \varepsilon\}|) \frac{g(\varepsilon)}{g(1)} q \\ &= \frac{g(|\{k \in I_r : |\zeta_k - \zeta| \geq \varepsilon\}|)}{g\left(h_r^\beta\right)} \frac{g\left(h_r^\beta\right)}{h_r^\beta} \frac{g(\varepsilon)}{g(1)} q. \end{aligned}$$

Taking the limits on both sides as$r \to \infty$, we obtain that $(\zeta_k) \in N_\theta^\alpha(f)$ implies $(\zeta_k) \in S_\theta^\beta(g)$ since $\lim_{u \to \infty} \frac{g(u)}{u} > 0$. This fulfills the proof. □

Remark 3.2. In general, contrary to the above theorem could not be possible. This fact could be seen in the illustration below.

Example 3.2. Let θ be given and select the sequence (ζ_k) as in Example 3.1 and also consider the modulus functions $g(u) = f(u) = u$. So that $\inf_{u \in (0, \infty)} \frac{f(u)}{g(u)} > 0$ and $\lim_{u \to \infty} \frac{g(u)}{u} > 0$. Now if we take $0 < \alpha \leq \frac{1}{2} < \beta \leq 1$, then for any $\varepsilon > 0$, we have

$$\lim_{r \to \infty} \frac{1}{g\left(h_r^\beta\right)} g(|\{k \in I_r : |\zeta_k| \geq \varepsilon\}|) = \lim_{r \to \infty} \frac{\left[\sqrt{h_r}\right]}{h_r^\beta} = 0.$$

So, $(\zeta_k) \in S_\theta^\beta(g)$. But since

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} f(|\zeta_k|) = \lim_{r \to \infty} \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r^\alpha} = \infty,$$

so that $(\zeta_k) \notin N_\theta^\alpha(f)$.

The following result is obtained by taking $g(u) = f(u)$ in Theorem 3.3.

Corollary 3.12. *Suppose f is an unbounded modulus, $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\lim_{u \to \infty} \frac{f(u)}{u} > 0$, then every strongly $N_\theta^\alpha(f)$-summable sequence is $S_\theta^\beta(f)$-statistically convergent.*

The following result is obtained by taking $\beta = \alpha$ in Theorem 3.3.

Corollary 3.13. *Suppose f and g are unbounded modulus functions, $\alpha \in (0, 1]$, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\inf_{u \in (0, \infty)} \frac{f(u)}{g(u)} > 0$ and $\lim_{u \to \infty} \frac{g(u)}{u} > 0$, then every strongly $N_\theta^\alpha(f)$-summable sequence is $S_\theta^\alpha(g)$-statistically convergent.*

The following result is obtained by taking $g(u) = u$ in Corollary 3.10, which is also Theorem 2.9 of [29], for the case $p = 1$.

Corollary 3.14. *Suppose f is unbounded modulus function, $\alpha \in (0, 1]$, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\inf\limits_{u\in(0,\infty)} \frac{f(u)}{u} > 0$, then every strongly $N_\theta^\alpha(f)$-summable sequence is S_θ^α-statistically convergent.*

We obtain the result below by taking $\alpha = 1$ in Corollary 3.11.

Corollary 3.15. *Suppose f is an unbounded modulus function, and suppose $\theta = (k_r)$ is a lacunary sequence. If $\inf\limits_{u\in(0,\infty)} \frac{f(u)}{u} > 0$, then every strongly $N_\theta(f)$-summable sequence is S_θ-statistically convergent.*

The following result is obtained by taking $f(u) = u$ in Corollary 3.12, which is also the first part of Theorem 1 of [17].

Corollary 3.16. *$N_\theta \subset S_\theta$ for any lacunary sequence $\theta = (k_r)$.*

Theorem 3.17. *Suppose f and g are any unbounded modulus functions, $0 < \alpha \le \beta \le 1$, and suppose $\theta = (k_r)$ and $\vartheta = (w_r)$ are lacunary sequences such that $I_r \subset I'_r$ for each $r \in \mathbb{N}$. If $\lim\limits_{r\to\infty} \frac{v_r}{h_r^\beta} = 1$ and $\sup\limits_{u\in(0,\infty)} \frac{g(u)}{u} < \infty$, then every bounded and $S_\theta^\alpha(f)$-convergent sequence is strongly $N_\vartheta^\beta(g)$-summable, i.e.,*

$$\ell_\infty \cap S_\theta^\alpha(f) \subset N_\vartheta^\beta(g).$$

Proof. Assuming f and g are unbounded modulus functions, $I_r = (k_{r-1}, k_r]$, $I' = (w_{r-1}, w_r]$, $h_r = k_r - k_{r-1}$, $v_r = w_r - w_{r-1}$ and $0 < \alpha \le \beta \le 1$. Let $(\zeta_k) \in \ell_\infty \cap S_\theta^\alpha(f)$ and $S_\theta^\alpha(f) - \lim \zeta_k = \zeta \in \mathbb{C}$. In order to verify that $(\zeta_k) \in N_\vartheta^\beta(g)$, we shall first prove that $S_\theta^\alpha(f) \subset S_\theta^\alpha$. Since f is a modulus and $S_\theta^\alpha(f) - \lim \zeta_k = \zeta$, for each $p \in \mathbb{N}$ there exists $r_0 \in \mathbb{N}$ such that, if $r > r_0$, we get

$$f(|\{k \in I_r : |\zeta_k - \zeta| \ge \varepsilon\}|) \le \frac{1}{p} f(h_r^\alpha) \le \frac{1}{p} p f\left(\frac{h_r^\alpha}{p}\right) = f\left(\frac{h_r^\alpha}{p}\right)$$

for any $\varepsilon > 0$. So,

$$\frac{1}{h_r^\alpha} |k \in I_r : |\zeta_k - \zeta| \ge \varepsilon| \le \frac{1}{p}.$$

It follows that $S_\theta^\alpha(f) \subset S_\theta^\alpha$ and so that $\ell_\infty \cap S_\theta^\alpha(f) \subset \ell_\infty \cap S_\theta^\alpha$. Since $\lim\limits_{r\to\infty} \frac{v_r}{h_r^\beta} = 1$, we have $\ell_\infty \cap S_\theta^\alpha \subset N_\vartheta^\beta$ by the second part of Theorem 2.14 of [12]. So that $N_\vartheta^\beta \subset N_\vartheta^\beta(g)$ by Corollary 3.5 since $\sup\limits_{u\in(0,\infty)} \frac{g(u)}{u} < \infty$. Therefore, $\ell_\infty \cap S_\theta^\alpha(f) \subset N_\vartheta^\beta(g)$. □

Remark 3.3. The illustration below demonstrates that for some $\alpha, \beta \in (0, 1]$ such that $\alpha \le \beta$ and some special modulus functions f and g, the inclusion $\ell_\infty \cap S_\theta^\alpha(f) \subset N_\vartheta^\beta(g)$ is strict.

Example 3.3. As an example, let the lacunary sequence $\theta = (k_r)$ be provided and $\vartheta = \theta$. Define the sequence (ζ_k) as ζ_k to be $\left[\sqrt[3]{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $\zeta_k = 0$ otherwise. Also, consider the modulus functions $f(u) = g(u) = u$. Now if we take $0 < \alpha \leq \frac{1}{2}$ and $\beta = 1$, then $\lim_{r\to\infty} \frac{v_r}{h_r^\beta} = 1$ and $\sup_{u\in(0,\infty)} \frac{g(u)}{u} = 1 < \infty$. Since $\vartheta = \theta$, then for any $r \in \mathbb{N}$, we have

$$\frac{1}{v_r^\beta} \sum_{k\in I'_r} g(|\zeta_k|) = \frac{1}{v_r^\beta} \sum_{k\in I'_r} g([\sqrt[3]{v_r}]) = \frac{\left[\sqrt{v_r}\right]\left[\sqrt[3]{v_r}\right]}{v_r}.$$

Since $\frac{[\sqrt{v_r}][\sqrt[3]{v_r}]}{v_r} \to 0$ as $r \to \infty$, then $(\zeta_k) \in N_\vartheta^\beta(g)$. But for every $\varepsilon > 0$, we may write

$$\frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |\zeta_k| \geq \varepsilon\}|) = \frac{f\left(\left[\sqrt{h_r}\right]\right)}{f(h_r^\alpha)} = \frac{\left[\sqrt{h_r}\right]}{h_r^\alpha}.$$

So, $(\zeta_k) \notin S_\theta^\alpha(f)$ since $\frac{[\sqrt{h_r}]}{h_r^\alpha} \to \infty$ as $r \to \infty$ for $0 < \alpha < \frac{1}{2}$ and $\frac{[\sqrt{h_r}]}{h_r^\alpha} \to 1$ as $r \to \infty$ for $\alpha = \frac{1}{2}$. Therefore, the inclusn $\ell_\infty \cap S_\theta^\alpha(f) \subset N_\vartheta^\beta(g)$ is strict.

The outcome below of inclusions is a result of Theorem 3.4.

Corollary 3.18. *Suppose f is any unbounded modulus function, $\theta = (k_r)$ and $\vartheta = (w_r)$ are lacunary sequences such that $I_r \subset I'_r$ for each $r \in \mathbb{N}$, and suppose* $0 < \alpha \leq \beta \leq 1$. *If $\lim_{r\to\infty} \frac{v_r}{h_r^\beta} = 1$ and $\sup_{u\in(0,\infty)} \frac{f(u)}{u} < \infty$, then*

(i) $\ell_\infty \cap S_\theta^\alpha(f) \subset N_\vartheta^\beta(f)$.
(ii) $\ell_\infty \cap S_\theta^\alpha(f) \subset N_\vartheta^\alpha(f)$.
(iii) $\ell_\infty \cap S_\theta^\alpha \subset N_\vartheta^\alpha(f)$.

4. Conclusion

In this paper, we have introduced the relations between the sets of strongly lacunary summable sequences and lacunary statistically convergent sequences of complex numbers by using modulus functions f and g under certain conditions and $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$. In the case $\beta = \alpha = 1$, we obtain the relations between $N_\theta(f)$ and $S_\theta(g)$, and in the case $g(u) = f(u)$ and $\beta = \alpha$, we obtain the relations between $N_\theta^\alpha(f)$ and $S_\theta^\alpha(f)$, and also in the case $g(u) = f(u) = u$ and $\alpha = \beta = 1$, we obtain the relations between N_θ and S_θ.

In addition, this research paper will be a good source for the studies to be carried out in related subjects in the next stages and for the scientists who will be researching in related fields. It is strongly possible to obtain some further results in this field. Indeed, choosing some different modulus functions f and g and constants $\alpha, \beta \in (0, 1]$, a wide range of spaces can be obtained for sophisticated applications in the related fields.

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