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The limit cycle of the unforced Rayleigh system

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Abstract. We prove the existence and the uniqueness of the limit cycle for the unforced Rayleigh system [1], [2].

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1. Introduction

The equation

$$\varepsilon\ddot{x} + \frac{\dot{x}^3}{3} - \dot{x} + ax = 0, \qquad \text{with} \qquad a, \varepsilon \in \mathbb{R}, \tag{1}$$

known as the Rayleigh equation, is an example of autonomous second-order differential equation which has an unique limit cycle [1], [2].

Consider the Cauchy problem $x(0) = x_0, y(0) = y_0$ for the nonlinear system associated to (1),

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\frac{a}{\varepsilon}x - \frac{1}{\varepsilon}\left(\frac{\dot{y}^3}{3} - \dot{y}\right), \end{cases} \tag{2}$$

where $a, \varepsilon \in \mathbb{R}, \varepsilon \neq 0$ are parameters, $x, y : \mathbb{R} \to \mathbb{R}, x = x(t), y = y(t)$ are the unknown functions, t is the independent variable and the dot over quantities stands for the derivative with respect to t. Since the vector field defined by (2) is smooth, the solution of the Cauchy problem for (2) exists, is unique and smooth. It defines a two dimensional dynamical system depending on the parameters a and ε. In [3] we have proved the existence and the uniqueness of the limit cycle of this dynamical system. In the following we present an alternative proof for those in [3].

We consider the case $\varepsilon \cdot a > 0$ (i.e. $\varepsilon > 0, a > 0$) because, in this case, the equilibrium $(0, 0) \in \mathbb{R}^2$ of the system (2) is node or focus with index 1 and then, the system can have periodic solutions [5]. The definitions and properties from Sections 2 and 3 serve to prove the main result in Section 4.

2. Properties of the positive orbits

We rewrite the system (2) as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \frac{1}{\varepsilon}\left(h(y) - a \cdot x\right), \end{cases} \tag{3}$$

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where $h:\mathbb{R}\to\mathbb{R}, h(y)=y-\frac{y^3}{3}$ and we consider the curves

$$\begin{aligned} v^+ &= \{(x,y)\mid x>0,\ y=0\}, \\ g^- &= \{(x,y)\mid a\cdot x=h(y),\ y<0\}, \\ v^- &= \{(x,y)\mid x<0,\ y=0\}, \\ g^+ &= \{(x,y)\mid a\cdot x=h(y),\ y>0\}, \end{aligned}$$

which delimit the regions (Figure 1),

$$\begin{aligned} A &= \{(x,y)\mid a\cdot x>h(y),\ y<0\}, \\ B &= \{(x,y)\mid a\cdot x<h(y),\ y<0\}, \\ C &= \{(x,y)\mid a\cdot x<h(y),\ y>0\}, \\ D &= \{(x,y)\mid a\cdot x>h(y),\ y>0\}. \end{aligned} \tag{4}$$

The next proposition characterizes the positive orbits of the system (3).

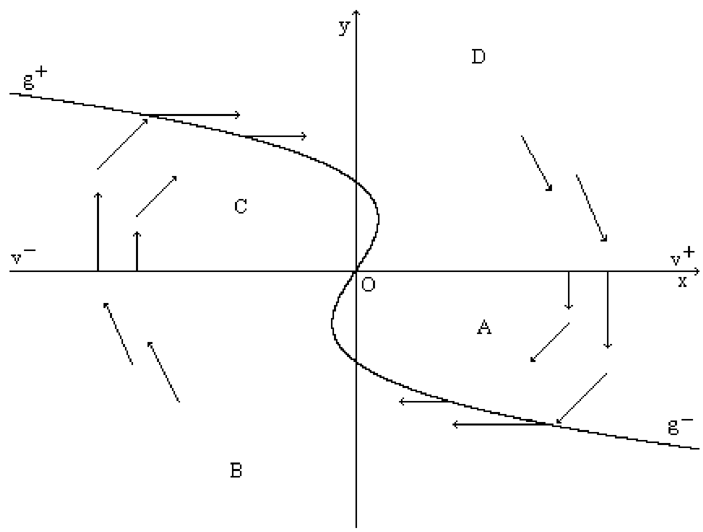

FIGURE 1. The regions A, B, C, D and the positive orbits of the system (2).

Proposition 2.1. [5] *Any positive orbit of the system (3), with exception of those through $(0,0)$, is clockwise and intersects the curves v^+, g^-, v^-, g^+, in this order, passing through the regions A, B, C, D.*

Proof. Let γ_+ be a positive orbit, $\gamma_+\neq\mathbf{0}$ and let be $(x_0,y_0)=(x(0),y(0))$ his initial point. Function of the position of the point (x_0,y_0), relative at the regions (4), we distinguish eight cases (Figure 1).

Case 1. $(x_0,\ y_0)\in v^+$. From the initial conditions it results $y(0)=0$, $x(0)>0$ and $\dot{y}(0)<0$. For small t, $\dot{y}(t)<0$, then y decreases. Since $y(0)=0$, the trajectory through $(x_0,\ y_0)$ enters A.

Case 2. $(x_0,\ y_0)\in A$. From the inequality $a\cdot x(0)>h(y(0))$ we have $\dot{y}(0)<0$ and then, for small t, the function y decreases, i.e. $y(t)<y(0)<0$. From the last inequality we have $\dot{x}(t)<y(0)<0$ then, for small t the function x decreases. We have $x(t) = x(0) + y(s) \cdot t$, for $s \in (0,t)$ and from $y(s) \leq y_0$ we have $x(t) \leq x_0 + y_0 \cdot t$. Since the functions x and y decrease, the trajectory through $(x_0,\ y_0)$ intersects the curve g^-, and the coordinate x of the intersection point is less than the coordinate x of the intersection point of the straight line $\Delta(x_0, y_0)$ and the curve g^-, where $\Delta = \Delta(x_0, y_0)$ is

$$\Delta : \begin{cases} x = x_0 + y_0 \cdot \tau, \\ y = \tau, \end{cases} \tag{5}$$

and $\tau \in \mathbb{R}$.

Case 3. $(x_0,\ y_0) \in g^-$. From the initial conditions we have $y(0) < 0$, $\dot{y}(0) = 0$ and $\dot{x}(0) < 0$. For small t, $\dot{x}(t) < 0, \dot{y}(t) \approx 0$ then, x decreases and $y(t) \approx y(0)$. The trajectory through $(x_0,\ y_0)$ enters B.

Case 4. $(x_0, y_0) \in B$. From the initial conditions we have $y(0) < 0$, $\dot{y}(0) > 0$ and $\dot{x}(0) < 0$. For small t, $\dot{y}(t) > 0$ and $\dot{x}(t) < 0$ then, y increases and x decreases. Since $y(0) < 0$ we have $y(t_1) = 0$, or equivalently $(x(t_1),\ y(t_1)) \in v^-$ and $x(t_1) < x_0$ for $t_1 > 0$. Then, $x(t_1) \geq x_0 + y_0 \cdot t_1$ and the x coordinate of the intersection point of the trajectory through (x_0, y_0) and v^- is bigger than the x coordinate of the intersection point between the stright line (5) and v^-.

Similarly we prove the cases 5-8. □

3. The dynamics of the points on the Ox axis

The existence of the limit cycle for (3) is proved in the Proposition 4.1. In order to prove this proposition we define three maps and we prove the Proposition 3.1.

Let be the set $v^* = v^+ \cup v^- \cup \{(0,0)\}$. For each point $(b,0) \in v^*$ we associate the value $b \in \mathbb{R}$. Then, we have the natural order $(b,0) < (c,0) \Leftrightarrow b < c$ in the set v^*. Next, we define three maps.

Definition 3.1. [5] *Let $p \in v^+$ where $p \equiv (x_0, y_0)$, $y_0 = 0$ and let be $\varphi_t(p) = (x(t), y(t))$, the solution curve corresponding to the Cauchy problem $x(0) = x_0, y(0) = y_0$ for (3). We note $t_1 = t_1(p)$ the smallest $t > 0$, so that $\varphi_{t_1}(p) \in v^+$. We define $\sigma : v^+ \to v^+$ by*

$$\sigma(p) = \varphi_{t_1}(p).$$

The maps $p \to t_1(p)$ and $p \to \sigma(p)$ are continuous [6] and there are Poincaré maps relative to the set v^+.

Remark 3.1. [5] *The map σ keeps the order of the points in the set v^+, i.e. for any two points $p,\ q \in v^+$ in relation $p \leq q$, we have $\sigma(p) \leq \sigma(q)$.*

Proof. For two points $p, q \in v^+$ in relation $p \leq q$, we suppose $\sigma(p) > \sigma(q)$. Then, there are $\widetilde{t} > 0$, $0 < \widetilde{t} < t_1$, so that $\varphi_{\widetilde{t}}(p) = \varphi_{\widetilde{t}}(q)$, and the solution of (3) with initial condition $\widetilde{p} = \varphi_{\widetilde{t}}(p)$ is not unique. Then, the map σ keeps the order of the points in the set v^+. □

Definition 3.2. [5] *Let be $p \in v^+$. We note $t_2 = t_2(p)$ the smallest $t > 0$ so that $\varphi_{t_2}(p) \in v^-$. We define the map $\alpha : v^+ \to v^-$, by*

$$\alpha\ (p) = \varphi_{t_2}(p).$$

The map α is continuous and reverses the order of the points, i.e. for $p \leq q$, p, $q \in v^+$, we have $\alpha(p) \geq \alpha(q)$.

Definition 3.3. [5] *Let be the points $p \in v^+$, $p \equiv (x_1, y_1)$, $y_1 = 0$ and $q \in v^-$, $q \equiv (x_2, y_2), y_2 = 0$. We suppose $q = \alpha(p)$. We define the continuous map $\delta : v^+ \to \mathbb{R}$, by*

$$\delta(p) = \frac{a}{2\,\varepsilon}\left(x_2^2 - x_1^2\right).$$

For the point $p \in \mathbb{R}^2$ we denote by $\|p\|$ the euclidian norm for the position vector of p.

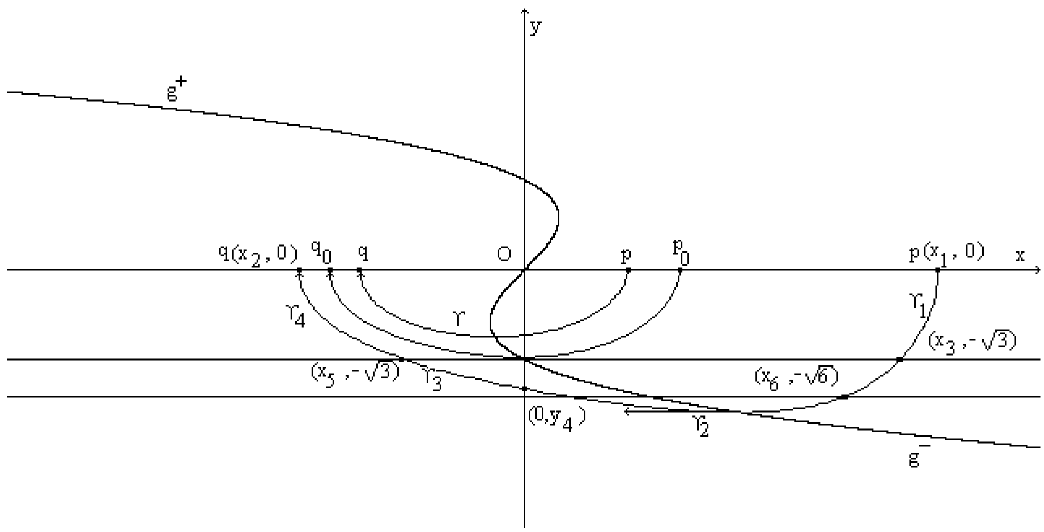

FIGURE 2. The Poincaré map σ.

Proposition 3.1. [5] *There are $r > 0$ so that: (i) $\delta(p) > 0$ for $0 < \|p\| < r$; (ii) $\delta(p)$ decreases at $-\infty$ when $\|p\| \geq r$ and $\|p\| \to \infty$.*

Proof. Let $p_0 \in v^+$and let be $t(p_0)$ the smallest $t > 0$ so that $\varphi_{t(p_0)}(p_0) = \left(0, -\sqrt{3}\right)$, where $\left(0, -\sqrt{3}\right)$ is the intersection point between the curve g^- and the semiaxis $\{(x, y)\,|x = 0,\ y < 0\}$ (Figure 2). The point p_0 is unique. Indeed, if exists another point $p_1 \in v^+$, so that the trajectory through p_1 passes through the point $\left(0, -\sqrt{3}\right)$, the dynamical system (3) has two different trajectories which intersect, impossible. We denote $r = \|p_0\|$.

(i) Let be $p \in v^+$ and $q \in v^-$, so that $\alpha(p) = q$, where $p \equiv (x_1, y_1)$, $q \equiv (x_2,\ y_2)$, $y_1 = 0$, $y_2 = 0$ (Figure 2) and let be the continuous and differentiable function $V : \mathbb{R}^2 \to \mathbb{R}$,

$$V(x,\ y) = \frac{1}{2}\left(\frac{a}{\varepsilon}\ x^2 + y^2\right).$$

Since $y_1 = 0$ and $y_2 = 0$, $\delta(p)$ we write

$$\delta\ (p) = \frac{a}{2\,\varepsilon}\ \left(x_2^2 - x_1^2\right) + \frac{1}{2}\left(y_2^2 - y_1^2\right) = V\ (x_2, y_2) - V\ (x_1, y_1),$$

and

$$\delta\ (p) = \int_0^{t_1} \frac{d}{dt}\ V\ \left(x\ (t)\,,\ y\ (t)\right) dt \tag{6}$$

where $t_1 = t_1(p)$ is the smallest $t > 0$ with $q = \alpha(p) = \varphi_{t_1}(p)$. Because

$$\tfrac{d}{dt}V\left(x\ (t)\,,\ y\ (t)\right) = \tfrac{a}{\varepsilon}x\,(t)\,\dot{x}\ (t) + y\,(t)\,\dot{y}\ (t) = \tfrac{a}{\varepsilon}x\,(t)\,y\,(t) + y\,(t)\cdot\tfrac{1}{\varepsilon}\left[h\left(y\ (t)\right) - ax\ (t)\right] = \tfrac{1}{\varepsilon}y\,(t)^2\left(1 - \tfrac{y\ (t)\ ^2}{3}\right),$$

we write (6) as

$$\delta\ (p) = \frac{1}{\varepsilon}\int_\gamma y^2\ \left(1 - \frac{y^2}{3}\right) dt, \tag{7}$$

where γ is the part of the solution curve for (3) between the points p and q (Figure 2).

Let be $p \in v^+$ with the property $\|p\| < r$, or equivalently, $p < p_0$. From Proposition 2.1 it results that for $t \in [0, t_1]$ we have $-\sqrt{3} \le y\,(t) \le 0$. Then, $\delta\ (p) = \int_0^{t_1} y\ (t)^2\left(1 - \frac{y\ (t)^2}{3}\right) dt > 0$ because the integrand is positive. This immediately proves (i).

(ii) Let $p \in v^+$ with the property $\|p\| \ge r$, then $p \ge p_0$. On the curve γ we consider the points $\left(x_3, -\sqrt{3}\right)$, $(0, y_4)$, $\left(x_5, -\sqrt{3}\right)$, where the curve intersects the stright line $\left\{(x, y)\,\middle|\, x \in \mathbb{R}, y = -\sqrt{3}\right\}$ and the semiaxis $\{(x, y)\,|x = 0,\ y < 0\}$. We denote γ_i, $i = 1, ..., 4$, the parts of the curve γ between the points p and $\left(x_3, -\sqrt{3}\right)$, $\left(x_3, -\sqrt{3}\right)$ and $(0,\ y_4)$, $(0,\ y_4)$ and $\left(x_5, -\sqrt{3}\right)$ and respectively $\left(x_5, -\sqrt{3}\right)$ and q (Figure 2). Let $\delta_i(p)$, $i = 1, ..., 4$, defined by

$$\delta_i\,(p) = \frac{1}{\varepsilon}\int_{\gamma_i} y^2\left(1 - \frac{y^2}{3}\right) dt.$$

Then, we write $\delta(p)$,

$$\delta\,(p) = \sum_{i\,=\,1}^{4} \delta_i\,(p)\,.$$

Since on the curve γ the coordinate y is function of x, we have

$$\delta_1\,(p) = \frac{1}{\varepsilon}\int_{x_1}^{x_3} y\,(x)\left(1 - \frac{y\,(x)^2}{3}\right) dx = \frac{1}{\varepsilon}\int_{x_3}^{x_1} y\,(x)\left(\frac{y\,(x)^2}{3} - 1\right) dx. \tag{8}$$

When $x \in [x_3, x_1]$ we have $-\sqrt{3} \le y\,(x) \le 0$, then $\delta_1(p) > 0$ (because the integrand is positive), and

$$\delta_1\,(p) = \frac{1}{\varepsilon}\int_{x_3}^{x_1}\left|y\,(x)\left(1 - \frac{y\,(x)^2}{3}\right)\right| dx \le \frac{2}{3\varepsilon}\int_{x_3}^{x_1} dx = \frac{\sqrt{3}}{\varepsilon}\,(x_1 - x_3)\,.$$

It follows

$$0 < \delta_1\,(p) \le \frac{\sqrt{3}}{\varepsilon}\ \,(x_1 - x_3)\,. \tag{9}$$

For $\delta_2(p)$ we have

$$\delta_2(p) = -\frac{1}{\varepsilon}\int_0^{x_3} y(x)\left(1 - \frac{y(x)^2}{3}\right)dx = \frac{1}{\varepsilon}\int_0^{x_3} y(x)\left(\frac{y(x)^2}{3} - 1\right)dx < 0, \quad (10)$$

because the last integrand in (10) is negative. For $x \in [0, x_3]$ and $p, \|p\| \to \infty$ the trajectory through p intersects the stright line $\left\{(x,y) \,\middle|\, x \in \mathbb{R}, y = -\sqrt{6}\right\}$ at the point $(x_6, -\sqrt{6})$. Then, we have

$$\begin{aligned} -\delta_2(p) &= \tfrac{1}{\varepsilon}\textstyle\int_0^{x_3} y(x)\left(1 - \frac{y(x)^2}{3}\right)dx = \\ &= \tfrac{1}{\varepsilon}\textstyle\int_0^{x_6} y(x)\left(1 - \frac{y(x)^2}{3}\right)dx + \tfrac{1}{\varepsilon}\int_{x_6}^{x_3} y(x)\left(1 - \frac{y(x)^2}{3}\right)dx > \frac{\sqrt{6}}{\varepsilon}x_6 \end{aligned}$$

and

$$\delta_2(p) < -\frac{\sqrt{6}}{\varepsilon}x_6 < 0. \quad (11)$$

When $\|p\| \to \infty$, or equivalently $x_1 \to \infty$, then $x_3, x_6 \to \infty$ but the difference $x_1 - x_3$ stays finite. Indeed, the slope in $(x_3, -\sqrt{3})$ at the curve γ is $\frac{ax_3}{\varepsilon\sqrt{3}}$, then $x_1 - x_3 = \frac{h_1\varepsilon\sqrt{3}}{ax_3}$, where $h_1 < \sqrt{3}$ is the difference of the x-coordinates for the points $(x_1, 0)$ şi $(x_3, -\sqrt{3})$. It follows that $x_3 \to \infty$ and $x_1 - x_3 \to 0$. Then, from (9) and (11), we deduce that $\delta_1(p) + \delta_2(p)$ decrease at $-\infty$. Similarly, $\delta_3(p) + \delta_4(p)$ decreases at $-\infty$ and $\delta(p)$ decreases at $-\infty$. □

4. The limit cycle

In Proposition 4.1 we prove that the existence and the uniqueness of the limit cycle for (3) are equivalent with the existence and the uniqueness of the fixed point for the Poincaré map σ.

Proposition 4.1. [5] *(i) A point $p \in v^+$ is a fixed point for σ if and only if p is situated on a limit cycle for (3); (ii) The map σ has an unique fixed point; (iii) The limit cycle corresponding to the fixed point for σ is attractive.*

Proof. (i) Let be $p \in v^+$ situated on a limit cycle for (3). Then, exists $t_1 > 0$ the smallest positive t with $\varphi_{t_1}(p) = p$, then $\sigma(p) = p$. Let $p \in v^+$, $\sigma(p) \neq p$ and we suppose $\sigma(p) > p$. From the monotonicity of σ we have $\sigma^2(p) > \sigma(p) > p$ and, by induction, $\sigma^n(p) > p, n \in \mathbb{N}$. Similarly, we deduce $\sigma^n(p) < p$ from $\sigma(p) < p$. It results that, for any $t > 0$, we have $\varphi_t(p) \neq p$ when $\sigma(p) \neq p$, then p is not situated on a limit cycle for (3). The first statement of the proposition is proved.

(ii) The function δ is continuous. From Proposition 3.1, for $p \in v^+$ and $\|p\| \to \infty$ the values $\delta(p)$ decrease from positive to $-\infty$ then, there are $p_0 \in v^+$, $\delta(p_0) = 0$. We prove that p_0 is a fixed point for σ.

The vector field

$$G(x,y) = \left(y, \frac{1}{\varepsilon}\left(y - \frac{y^3}{3} - ax\right)\right),$$

given by the right side of the equations (3), has the property

$$G(-x,-y) = -G(x,y) \quad (12)$$

then, if $t \to (x(t), y(t))$ is the solution curve for (3) then $t \to (-x(t), -y(t))$ is also a solution curve for (3). Let be $\alpha(p_0)$ the image of the point p_0 by the map α. Since the points p_0 and $\alpha(p_0)$, have the x-coordinate, 0, from $\delta(p_0) = 0$ and the expression of δ we deduce $\|\alpha(p_0)\| = \|p_0\|$. There are t_1 the smallest $t > 0$ so that

$$\varphi_{t_1}(p_0) = -p_0.$$

From (12) we have

$$\varphi_{t_1}(-p_0) = -(-p_0) = p_0.$$

At time $2 \cdot t_1$ we have

$$\varphi_{2\cdot t_1}(p_0) = p_0,$$

i.e.

$$\sigma(p_0) = p_0,$$

then p_0 is a fixed point for σ.

We prove that p_0 is the unique fixed point for σ. We define the map $\beta : v^- \to v^+$ by

$$\beta(p) = \varphi_{t_3}(p), p \in v^-,$$

where t_3 is the smallest $t > 0$ with $\varphi_{t_3}(p) \in v^+$. We observe that β reverses the points order, i e. if $p < q$ then $\beta(p) > \beta(q)$, where $p, q \in v^-$. We have $\sigma = \beta \circ \alpha$ and from the symmetry of the vector field G we deduce

$$\beta(p) = -\alpha(-p), p \in v^-. \tag{13}$$

We suppose that p_0 is not the unique fixed point for σ. Let $p_1 \in v^+, p_1 \equiv (x_1, y_1), x_1 > 0, y_1 = 0$, with $\sigma(p_1) = p_1$ and $p_1 \neq p_0$.

We suppose $p_1 > p_0$, or equivalently $\delta(p_1) < 0$, since the map δ is monotonous (Proposition 3.1). Let $q \in v^-, q \equiv (x_2, y_2), x_2 < 0, y_2 = 0$ with $q = \alpha(p_1)$. Then, $\delta(p_1)$ reads,

$$\delta(p_1) = \frac{a}{2\varepsilon}\left(x_2^2 - x_1^2 + y_2^2 - y_1^2\right) = \frac{a}{2\varepsilon}\left(\|\alpha(p_1)\|^2 - \|p_1\|^2\right). \tag{14}$$

From $\delta(p_1) < 0$ and (14) we have $\|\alpha(p_1)\| < \|p_1\|$ and $\alpha(p) > -p$. The map β is monotonous then,

$$\beta(\alpha(p_1)) < \beta(-p_1),$$

i.e.

$$\sigma(p_1) < \beta(-p_1). \tag{15}$$

We replace p_1 with $-p_1$ in (13). Since p_1 is a fixed point for σ, from (15) we deduce

$$p_1 < -\alpha(p_1). \tag{16}$$

From (16) and (14) follows that $\delta(p_1) > 0$ contradicting the above supposition. Similarly, the supposition $p_1 < p_0$, equivalently with $\delta(p_1) > 0$, leads to a contradiction. Then, p_0 is the unique fixed point for σ.

(iii) From the statement (i) the dynamical system associated with (3) has an unique limit cycle which passes through p_0. According with Proposition 2.1, any trajectory of (3) intersects v^+ then, in order to prove that the limit cycle is attractive, is enough to prove that for any $p \in v^+$, the point p_0 is the limit of the sequence $\sigma^n(p), n \in \mathbb{N}$, or equivalently, the trajectory $\varphi_t(p_0)$ attracts all trajectories of (3).

Let $p \in v^+$. We suppose $p > p_0$, or equivalently $\delta(p) < 0$. Since p_0 is the fixed point of σ and σ is monotonous we have

$$\sigma(p) > p_0. \tag{17}$$

From $\delta(p) < 0$ and (14) we deduce $\alpha(p) > -p$. From the last inequality and expression of β we deduce $\sigma(p) < -\alpha(p)$. But, $-\alpha(p) < p$, i.e.

$$\sigma(p) < p. \tag{18}$$

From (17) and (18) we deduce

$$q_0 < \sigma(p) < p$$

and, for $n \in \mathbb{N}$, we have

$$q_0 < \sigma^{n+1}(p) < \sigma^n(p).$$

Let $q_1 \in v^+$, $q_1 \geq p_0$ the limit point of the sequence $\sigma^n(p), n \in \mathbb{N}$ i.e.

$$q_1 = \lim_{n\to\infty} \sigma^n(p). \tag{19}$$

Since σ is continuous, from (19) we have

$$q_1 = \sigma\left(\lim_{n\to\infty} \sigma^n(p)\right) = \sigma(q_1),$$

or, equivalently, q_1 is a fixed point for σ. However, σ has an unique fixed point, then $q_1 = p_0$. The same thing is true if $p < p_0$, i.e.

$$q_0 > \sigma^{n+1}(p) > \sigma^n(p).$$

Similarly, the point q_0 is the limit of the sequence $\sigma^n(p)$. The proof is complete. □

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