Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 30(2), 2003, Pages 48–53
ISSN: 1223-6934

MV-algebra of fractions relative to an $\wedge$-closed system

Dumitru Buşneag and Dana Piciu

Abstract. The aim of this paper is to introduce the notion of MV-algebra of fractions relative to an $\wedge$-closed system.

2000 Mathematics Subject Classification. 06D35, 03G25.
Key words and phrases. MV-algebra, MV-algebra of fractions, $\wedge$-closed system.

1. Definitions and preliminaries

Definition 1.1. *([4], [5]) An MV**-algebra** is an algebra $(A, +, ^*, 0)$ of type $(2, 1, 0)$ satisfying the following equations:*
(a_1) $x + (y + z) = (x + y) + z,$
(a_2) $x + y = y + x,$
(a_3) $x + 0 = x,$
(a_4) $x^{**} = x,$
(a_5) $x + 0^* = 0^*,$
(a_6) $(x^* + y)^* + y = (y^* + x)^* + x.$

MV-algebras were originally introduced by Chang in [4] in order to give an algebraic counterpart of the Lukasiewicz many valued logic (MV = many valued). Note that axioms a_1-a_3 state that $(A, +, 0)$ is an abelian monoid; following tradition, we denote an MV-algebra $(A, +, ^*, 0)$ by its universe A.

Remark 1.1. *If in a_6 we put $y = 0$ we obtain $x^{**} = 0^{**} + x$, so, if $0^{**} = 0$ then $x^{**} = x$ for every $x \in A$. Hence, the axiom a_4 is equivalent with (a_4') $0^{**} = 0$.*

Examples:

E_1) A singleton $\{0\}$ is a trivial example of an MV-algebra; an MV-algebra is said *nontrivial* provided its universe has more that one element.

E_2) Let $(G, \oplus, -, 0, \leq)$ an l-group. For each $u \in G$, $u > 0$, let

$$[0, u] = \{x \in G : 0 \leq x \leq u\}$$

and for each $x, y \in [0, u]$, let $x+y = u \wedge (x \oplus y)$ and $x^* = u - x$. Then $([0, u], +, ^*, 0)$ is an MV-algebra. In particular, if consider the real unit interval $[0, 1]$ and for all $x, y \in [0, 1]$ we define $x + y = \min\{1, x + y\}$ and $x^* = 1 - x$, then $([0, 1], +, ^*, 0)$ is an MV-algebra.

E_3) If $(A, \vee, \wedge, ^*, 0, 1)$ is a Boolean lattice, then $(A, \vee, ^*, 0)$ is an MV-algebra.

E_4) The rational numbers in $[0, 1]$, and, for each integer $n \geq 2$, the n-element set $L_n = \left\{0, \frac{1}{(n-1)}, ..., \frac{(n-2)}{(n-1)}, 1\right\}$ yield examples of subalgebras of $[0, 1]$.

E_5) Given an MV-algebra A and a set X, the set A^X of all functions $f : X \longrightarrow A$ becomes an MV-algebra if the operations $+$, and * and the element 0 are defined

Received: 3 June 2003.

pointwise. The continues functions from $[0,1]$ into $[0,1]$ form a subalgebra of the MV-algebra $[0,1]^{[0,1]}$.

In the rest of this paper, by A we denote an MV-algebra.

On A we define the constant 1 and the operations ,,$\cdot$" and ,,$-$" as follows: $1 = 0^*$, $x \cdot y = (x^* + y^*)^*$ and $x - y = x \cdot y^* = (x^* + y)^*$ (we consider the ,,$*$" operation more binding that any other operation, and the ,,$\cdot$" more binding that $+$ and $-$).

Lemma 1.1. *([2]-[7]) For $x, y \in A$, the following conditions are equivalent:*
(i) $x^* + y = 1$,
(ii) $x \cdot y^* = 0$,
(iii) $y = x + (y - x)$,
(iv) *There is an element $z \in A$ such that $x + z = y$.*

For any two elements $x, y \in A$ let us agree to write $x \leq y$ iff x and y satisfy the equivalent conditions $(i) - (iv)$ in the above lemma. So, $\leq$ is a partial order relation on A (which is called the *natural order* on A).

Theorem 1.1. *([2]-[7]) If $x, y, z \in A$ then the following hold:*
$c_1)$ $1^* = 0$,
$c_2)$ $x + y = (x^* \cdot y^*)^*$,
$c_3)$ $x + 1 = 1$,
$c_4)$ $(x - y) + y = (y - x) + x$,
$c_5)$ $x + x^* = 1, x \cdot x^* = 0$,
$c_6)$ $x - 0 = x$, $0 - x = 0$, $x - x = 0$, $1 - x = x^*$, $x - 1 = 0$,
$c_7)$ $x + x = x$ *iff* $x \cdot x = x$,
$c_8)$ $x \leq y$ *iff* $y^* \leq x^*$,
$c_9)$ *If* $x \leq y$, *then* $x + z \leq y + z$ *and* $x \cdot z \leq y \cdot z$,
$c_{10})$ *If* $x \leq y$, *then* $x - z \leq y - z$ *and* $z - y \leq z - x$,
$c_{11})$ $x - y \leq x, x - y \leq y^*$,
$c_{12})$ $(x + y) - x \leq y$,
$c_{13})$ $x \cdot z \leq y$ *iff* $z \leq x^* + y$,
$c_{14})$ $x + y + x \cdot y = x + y$.

Remark 1.2. *([2]-[7]) On A, the natural order determines a bounded distributive lattice structure. Specifically, the join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by:*

$$x \vee y = (x - y) + y = (y - x) + x = x \cdot y^* + y = y \cdot x^* + x,$$
$$x \wedge y = (x^* \vee y^*)^* = x \cdot (x^* + y) = y \cdot (y^* + x).$$

Clearly, $x \cdot y \leq x \wedge y \leq x, y \leq x \vee y \leq x + y$.

We shall denote this distributive lattice with 0 and 1 by $L(A)$ (see [4]-[5]). For any MV-algebra A we shall write $B(A)$ as an abbreviation of set of all complemented elements of $L(A)$; elements of $B(A)$ are called the *boolean* elements of A.

Theorem 1.2. *([4]-[5]) For every element x in an MV-algebra A, the following conditions are equivalent:*
(i) $x \in B(A)$,
(ii) $x \vee x^* = 1$,
(iii) $x \wedge x^* = 0$,
(iv) $x + x = x$,
(v) $x \cdot x = x$,
(vi) $x + y = x \vee y$, *for all* $y \in A$,

(vii) $x \cdot y = x \wedge y$, *for all* $y \in A$.

Corollary 1.1. *(*[4]-[5]*)*

(i) $B(A)$ *is subalgebra of the* MV*-algebra* A. *A subalgebra* B *of* A *is a boolean algebra iff* $B \subseteq B(A)$,

(ii) *An* MV*-algebra* A *is a boolean algebra iff the operation* $+$ *is idempotent, i.e., the equation* $x + x = x$ *is satisfied by* A.

Theorem 1.3. *(*[2]-[6]*) If* $x, y, z, (x_i)_{i \in I}$ *are elements of* A, *then the following hold:*

c_{15}) $x + y = (x \vee y) + (x \wedge y)$,

c_{16}) $x \cdot y = (x \vee y) \cdot (x \wedge y)$,

c_{17}) $x + \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x + x_i)$,

c_{18}) $x + \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (x + x_i)$,

c_{19}) $x \cdot \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x \cdot x_i)$,

c_{20}) $x \cdot \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (x \cdot x_i)$,

c_{21}) $x \wedge \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x \wedge x_i)$,

c_{22}) $x \vee \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (x \vee x_i)$ *(if all suprema and infima exist),*

Lemma 1.2. *If* a, b, x *are elements of* A, *then:*

c_{23}) $[(a \wedge x) + (b \wedge x)] \wedge x = (a + b) \wedge x$,

c_{24}) $a^* \wedge x \geq x \cdot (a \wedge x)^*$.

Proof. c_{23}). By c_{18} we have $[(a \wedge x) + (b \wedge x)] \wedge x = ((a \wedge x) + b) \wedge ((a \wedge x) + x) \wedge x = ((a \wedge x) + b) \wedge x = (a + b) \wedge (x + b) \wedge x = (a + b) \wedge x$.

c_{24}). We have $x \cdot (a \wedge x)^* = x \cdot (a^* \vee x^*) \overset{c_{19}}{=} (x \cdot a^*) \vee (x \cdot x^*) \overset{c_5}{=} (x \cdot a^*) \vee 0 = x \cdot a^* \leq a^* \wedge x$. □

Corollary 1.2. *If* $a \in B(A)$, *then:*

c_{25}) $a^* \wedge x = x \cdot (a \wedge x)^*$ *for all* $x \in A$,

c_{26}) $a \wedge (x + y) = (a \wedge x) + (a \wedge y)$,

c_{27}) $a \vee (x + y) = (a \vee x) + (a \vee y)$.

Proof. c_{25}). See the proof of c_{24}.

c_{26}). We have: $(a \wedge x) + (a \wedge y) = [(a \wedge x) + a] \wedge [(a \wedge x) + y] = [(a \wedge x) \vee a] \wedge [(a + y) \wedge (x + y)] = a \wedge (a + y) \wedge (x + y) = a \wedge (x + y)$.

c_{27}). We have $(a \vee x) + (a \vee y) = (a + x) + (a + y) = (a + a) + (x + y) = a + (x + y) = a \vee (x + y)$. □

Definition 1.2. *(*[2]-[7]*) Let* A *and* B *be* $MV-$*algebras. A function* $f : A \to B$ *is a* ***morphism of*** $MV-$***algebras*** *iff it satisfies the following conditions, for every* $x, y \in A :$

(a_7) $f(0) = 0$,

(a_8) $f(x + y) = f(x) + f(y)$,

(a_9) $f(x^*) = (f(x))^*$.

Remark 1.3. *It follows that:*

$$f(1) = 1,$$

$$f(x \cdot y) = f(x) \cdot f(y),$$
$$f(x \vee y) = f(x) \vee f(y),$$
$$f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

2. MV-algebra of fractions relative to an $\wedge-$closed system

Definition 2.1. *A nonempty subset* $S \subseteq A$ *is called* $\wedge-$***closed system*** *in* A *if* $1 \in S$ *and* $x, y \in S$ *implies* $x \wedge y \in S$.

We denote by $S(A)$ the set of all $\wedge-$ closed systems of A (clearly $\{1\}$, $A \in S(A)$).
For $S \in S(A)$, on the MV-algebra A we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

Lemma 2.1. θ_S *is a congruence on* A.

Proof. The reflexivity (since $1 \in S \cap B(A)$) and the symmetry of θ_S are immediately. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $y \wedge f = z \wedge f$. If denote $g = e \wedge f \in S \cap B(A)$, then $g \wedge x = (e \wedge f) \wedge x = (e \wedge x) \wedge f = (y \wedge e) \wedge f = (y \wedge f) \wedge e = (z \wedge f) \wedge e = z \wedge (f \wedge e) = z \wedge g$, hence $(x, z) \in \theta_S$. To prove the compatibility of θ_S with the operations $+$ and * , let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$.

Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$.

By c_{26} we obtain: $(x + z) \wedge g = (x \wedge g) + (z \wedge g) = (x \wedge e \wedge f) + (z \wedge f \wedge e) = (y \wedge e \wedge f) + (t \wedge f \wedge e) = (y \wedge g) + (t \wedge g) = (y + t) \wedge g$, hence $(x + z, y + t) \in \theta_S$.

From $x \wedge e = y \wedge e$ we deduce $x^* + e^* = y^* + e^*$, so $e \cdot (e^* + x^*) = e \cdot (e^* + y^*)$, hence $x^* \wedge e = y^* \wedge e$, that is $(x^*, y^*) \in \theta_S$. □

For x we denote by x/S the equivalence class of x relative to θ_S and by

$$A[S] = A/\theta_S.$$

By $p_S : A \to A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, in $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$,

$$x/S + y/S = (x + y)/S$$
$$(x/S)^* = x^*/S$$

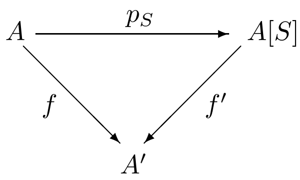
So, p_S is an onto morphism of MV-algebras.

Remark 2.1. *Since for every* $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ *we deduce that* $s/S = 1/S = \mathbf{1}$, *hence* $p_S(S \cap B(A)) = \{\mathbf{1}\}$.

Proposition 2.1. *If* $a \in A$, *then* $a/S \in B(A[S])$ *iff there exists* $e \in S \cap B(A)$ *such that* $e \wedge a \in B(A)$. *So, if* $e \in B(A)$, *then* $e/S \in B(A[S])$.

Proof. For $a \in A$, we have $a/S \in B(A[S]) \Leftrightarrow a/S + a/S = a/S \Leftrightarrow (a + a)/S = a/S \Leftrightarrow$ there exists $e \in S \cap B(A)$ such that $(a + a) \wedge e = a \wedge e \overset{c_{26}}{\Leftrightarrow} (a \wedge e) + (a \wedge e) = a \wedge e \Leftrightarrow a \wedge e \in B(A)$. If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 \wedge e = e \in B(A)$ we deduce that $e/S \in B(A[S])$. □

Theorem 2.1. *If* A' *is an* MV*-algebra and* $f : A \to A'$ *is an morphism of* MV*-algebras such that* $f(S \cap B(A)) = \{\mathbf{1}\}$, *then there exists an unique morphism of* MV*-algebras* $f' : A[S] \to A'$ *such that the diagram*



is commutative (i.e. $f' \circ p_S = f$*).*

Proof. If $x, y \in A$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$. Since f is morphism of MV-algebras, we obtain that $f(x \wedge e) = f(y \wedge e) \Leftrightarrow f(x) \wedge f(e) = f(y) \wedge f(e) \Leftrightarrow f(x) \wedge \mathbf{1} = f(y) \wedge \mathbf{1} \Leftrightarrow f(x) = f(y)$.

From this observation we deduce that the map $f' : A[S] \to A'$ defined for $x \in A$ by $f'(x/S) = f(x)$ is correctly defined. Clearly, f' is an morphism of MV-algebras. The unicity of f' follows from the fact that p_S is a onto map. □

Remark 2.2. *Theorem 2.1 allows us to call* $A[S]$ *the* MV***-algebra of fractions relative to the*** $\wedge-$***closed system*** S*.*

Examples

1. If $S = \{1\}$ or S is such that $1 \in S$ and $S \cap (B(A) \backslash \{1\}) = \varnothing$, then for $x, y \in A, (x, y) \in \theta_S \Longleftrightarrow x \wedge 1 = y \wedge 1 \Longleftrightarrow x = y$, hence in this case $A[S] = A$.

2. If S is an $\wedge-$closed system such that $0 \in S$ (for example $S = A$ or $S = B(A)$), then for every $x, y \in A$, $(x, y) \in \theta_S$ (since $x \wedge 0 = y \wedge 0$ and $0 \in S \cap B(A)$), hence in this case $A[S] = \mathbf{0}$.

3. If $\mathcal{P}$ is a prime ideal of A (that is $\mathcal{P} \neq A$ and if $x \wedge y \in \mathcal{P}$ implies $x \in \mathcal{P}$ or $y \in \mathcal{P}$), then $S = A \backslash \mathcal{P}$ is an $\wedge-$closed system. We denote $A[S]$ by $A_{\mathcal{P}}$. The set $M = \{x/S : x \in \mathcal{P}\}$ is a maximal ideal of $A_{\mathcal{P}}$. Indeed, if $x, y \in \mathcal{P}$, then $x/S + y/S = (x + y)/S \in M$ (since $x + y \in \mathcal{P}$). If $x, y \in A$ such that $x \in \mathcal{P}$ and $y/S \leq x/S$ then there exists $e \in S \cap B(A)$ such that $y \wedge e \leq x \wedge e$. Since $x \in \mathcal{P}$, then $y \wedge e \in \mathcal{P}$, hence $y \in \mathcal{P}$ (since $e \notin \mathcal{P}$), so $y/S \in M$. To prove the maximality of M let I an ideal of $A_{\mathcal{P}}$ such that $M \subseteq I$ and $M \neq I$. Then there exists $x/S \in I$ such that $x/S \notin M$, (that is $x \notin \mathcal{P} \Longleftrightarrow x \in S$), hence $x/S = \mathbf{1}$, so $I = A_{\mathcal{P}}$. Moreover, M is the only maximal ideal of $A_{\mathcal{P}}$ (since if we have another maximal ideal M' of $A_{\mathcal{P}}$, then $M' \nsubseteq M$ hence there exists $x/S \in M'$ such that $x/S \notin M$, so $x/S = \mathbf{1}$ and $M' = A_{\mathcal{P}}$, a contradiction!). In other words $A_{\mathcal{P}}$ is a local MV -algebra. The process of passing from A to $A_{\mathcal{P}}$ is called *localization* at $\mathcal{P}$.

References

[1] D. Buşneag, Hilbert algebra of fractions relative to an $\underline{\vee}-$ closed system, *Analele Universităţii din Craiova, Seria Matematica-Fizica-Chimie*, **XVI**, 34-38 (1998).

[2] D. Buşneag, D. Piciu, Meet-irreducible ideals in an MV-algebra, *Analele Universităţii din Craiova, Seria Matematica-Informatica*, **XXVIII**, 110-119 (2001).

[3] D. Buşneag, D. Piciu, On the lattice of ideals of an MV-algebra, *Scientiae Mathematicae Japonicae*, **56**(2), 367-372 (2002), :e6, 221-226.

[4] C.C. Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.*, **88**, 467-490 (1958).

[5] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, *Algebraic foundation of many -valued Reasoning*, Dordrecht, Kluwer Academic Publishers, 2000.

[6] G. Georgescu, A. Iorgulescu, Pseudo-MV algebras, *Multi. Val. Logic*, **6**, 95-135 (2001).

[7] P. Hájek, *Metamathematics of fuzzy logic*, Dordrecht, Kluwer Academic Publishers, 1998.

(Dumitru Buşneag) Department of Mathematics, University of Craiova,
Al. I. Cuza street, 13, Craiova RO-200585, Romania, Tel/Fax: 40-251412673
E-mail address: busneag@central.ucv.ro

(Dana Piciu) Department of Mathematics, University of Craiova,
Al. I. Cuza street, 13, Craiova RO-200585, Romania, Tel/Fax: 40-251412673
E-mail address: danap@central.ucv.ro