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New inequalities for operator monotone functions

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Abstract. In this paper we show that, if $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ with $f(0) = 0$ and

$$0 < \alpha \le A \le \beta < \gamma \le B \le \delta$$

for some positive constants α, β, γ, δ, then

$$0 \le \frac{\gamma - \beta}{\delta - \beta} \left[f(\beta) \beta^{-1} - f(\delta) \delta^{-1} \right] \le f(A) A^{-1} - f(B) B^{-1}$$
$$\le \frac{\delta - \alpha}{\gamma - \alpha} \left[f(\alpha) \alpha^{-1} - f(\gamma) \gamma^{-1} \right].$$

In particular, we obtain the following inequalities for powers of positive operators

$$0 \le \frac{\gamma - \beta}{\delta - \beta} \left(\beta^{r-1} - \delta^{r-1} \right) \le A^{r-1} - B^{r-1} \le \frac{\delta - \alpha}{\gamma - \alpha} \left(\alpha^{r-1} - \gamma^{r-1} \right).$$

for $r \in (0, 1]$.

The logarithmic inequalities

$$0 \le \frac{\gamma - \beta}{\delta - \beta} \ln \left[\frac{(\beta + 1)^{\beta^{-1}}}{(\delta + 1)^{\delta^{-1}}} \right] \le A^{-1} \ln (A + 1) - B^{-1} \ln (B + 1)$$
$$\le \frac{\delta - \alpha}{\gamma - \alpha} \ln \left[\frac{(\alpha + 1)^{\alpha^{-1}}}{(\gamma + 1)^{\gamma^{-1}}} \right],$$

are also valid.

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1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x \rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \ge f(B)$ holds for any $A \ge B > 0$.

In 1934, K. Löwner [6] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1.1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{ts}{t + s} dw(s) \tag{1}$$

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where $b \geq 0$ *and a positive measure* w *on* $[0,\infty)$ *such that*

$$\int_0^\infty \frac{s}{1+s} dw(s) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : [0,\infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0,1]$, [5]. The function ln is also operator monotone on $(0,\infty)$.

Let A and B be strictly positive operators on a Hilbert space H such that $B - A \geq m1_H > 0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$\begin{aligned} f(B) - f(A) &\geq f(\|A\| + m) - f(\|A\|) \\ &\geq f(\|B\|) - f(\|B\| - m) > 0. \end{aligned} \tag{2}$$

If $B > A > 0$, then

$$\begin{aligned} f(B) - f(A) \quad &\geq \quad f\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - f(\|A\|) \\ &\geq \quad f(\|B\|) - f\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0. \end{aligned} \tag{3}$$

The inequality between the first and third term in (3) was obtained earlier by H. Zuo and G. Duan in [8].

By taking $f(t) = t^r$, $r \in (0,1]$ in (3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality

$$\begin{aligned} B^r - A^r &\geq \left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r - \|A\|^r \\ &\geq \|B\|^r - \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r > 0 \end{aligned} \tag{4}$$

provided $B > A > 0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\begin{aligned} \ln B - \ln A &\geq \ln\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - \ln(\|A\|) \\ &\geq \ln(\|B\|) - \ln\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0. \end{aligned} \tag{5}$$

Notice that the inequalities between the first and third terms in (4) and (5) were obtained earlier by M. S. Moslehian and H. Najafi in [7].

Motivated by the above results, in this paper we show that, if $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ with $f(0) = 0$ and

$$0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$$

for some positive constants α, β, γ, δ, then

$$\begin{aligned} 0 &\le \frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}\right] \le f(A)A^{-1}-f(B)B^{-1} \\ &\le \frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}\right]. \end{aligned}$$

In particular, we obtain the following inequalities for powers of positive operators

$$0 \le \frac{\gamma-\beta}{\delta-\beta}\left(\beta^{r-1}-\delta^{r-1}\right) \le A^{r-1}-B^{r-1} \le \frac{\delta-\alpha}{\gamma-\alpha}\left(\alpha^{r-1}-\gamma^{r-1}\right).$$

for $r \in (0,1]$.

The logarithmic inequalities

$$\begin{aligned} 0 &\le \frac{\gamma-\beta}{\delta-\beta}\ln\left[\frac{(\beta+1)^{\beta^{-1}}}{(\delta+1)^{\delta^{-1}}}\right] \le A^{-1}\ln(A+1)-B^{-1}\ln(B+1) \\ &\le \frac{\delta-\alpha}{\gamma-\alpha}\ln\left[\frac{(\alpha+1)^{\alpha^{-1}}}{(\gamma+1)^{\gamma^{-1}}}\right], \end{aligned}$$

are also valid.

2. Main results

We have the following identity of interest in itself:

Lemma 2.1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$ with the representation (1). Then for all A, $B>0$ we have the identity*

$$\begin{aligned} &f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right) \\ &= \int_0^\infty \lambda\left(\int_0^1 \left((1-t)B+tA+\lambda 1_H\right)^{-1}(B-A)\right. \\ &\left.\times\left((1-t)B+tA+\lambda 1_H\right)^{-1}dt\right)dw(\lambda). \end{aligned} \tag{6}$$

Proof. Since f is operator monotone on $[0,\infty)$, then there exists $b\ge 0$ and w is a positive measure satisfying

$$\int_0^\infty \frac{\lambda}{1+\lambda}dw(\lambda)<\infty$$

such that [1, p. 144-145]

$$f(t)=f(0)+bt+\int_0^\infty \frac{\lambda t}{t+\lambda}dw(\lambda). \tag{7}$$

We have for $t>0$ that

$$g(t):=\frac{f(t)-f(0)}{t}-b=\int_0^\infty \frac{\lambda}{t+\lambda}dw(\lambda).$$

Therefore for all A, $B>0$

$$g(B)-g(A)=\int_0^\infty \lambda\left[(B+\lambda 1_H)^{-1}-(A+\lambda 1_H)^{-1}\right]dw(\lambda). \tag{8}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotonic on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0} \left[\frac{f(T + tS) - f(T)}{t} \right] = T^{-1} S T^{-1} \tag{9}$$

for T, $S > 0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0, 1]$ by

$$f_{C,D}(t) := f((1 - t) C + tD), \ t \in [0, 1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t)) \, dt = \int_0^1 \nabla f_{(1-t)C+tD}(D - C) \, dt. \tag{10}$$

If we write this equality for the function $f(t) = -t^{-1}$ and C, $D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1 - t) C + tD)^{-1} (D - C) ((1 - t) C + tD)^{-1} \, dt. \tag{11}$$

Now, if we replace in (11) $C = B + \lambda 1_H$ and $D = A + \lambda 1_H$ for $\lambda > 0$, then we get

$$\begin{aligned} &(B + \lambda 1_H)^{-1} - (A + \lambda 1_H)^{-1} \\ &= \int_0^1 ((1 - t) B + tA + \lambda 1_H)^{-1} (A - B) ((1 - t) B + tA + \lambda 1_H)^{-1} \, dt. \end{aligned} \tag{12}$$

Therefore, by (8),

$$\begin{aligned} g(B) - g(A) &= \int_0^\infty \lambda \left(\int_0^1 ((1 - t) B + tA + \lambda 1_H)^{-1} (A - B) \right. \\ &\quad \left. \times ((1 - t) B + tA + \lambda 1_H)^{-1} \, dt \right) dw(\lambda) \\ &= -\int_0^\infty \lambda \left(\int_0^1 ((1 - t) B + tA + \lambda 1_H)^{-1} (B - A) \right. \\ &\quad \left. \times ((1 - t) B + tA + \lambda 1_H)^{-1} \, dt \right) dw(\lambda), \end{aligned} \tag{13}$$

namely

$$\begin{aligned} &[f(A) - f(0)] A^{-1} - [f(B) - f(0)] B^{-1} \\ &= \int_0^\infty \lambda \left(\int_0^1 ((1 - t) B + tA + \lambda 1_H)^{-1} (B - A) \right. \\ &\quad \left. \times ((1 - t) B + tA + \lambda 1_H)^{-1} \, dt \right) dw(\lambda), \end{aligned}$$

which is equivalent to (6). □

Our main result is as follows:

Theorem 2.2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. If*

$$0<\alpha\le A\le\beta<\gamma\le B\le\delta \tag{14}$$

for some positive constants α, β, γ, δ, then

$$\begin{aligned}0&\le\frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}-f(0)\left(\beta^{-1}-\delta^{-1}\right)\right]\\&\le f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}-f(0)\left(\alpha^{-1}-\gamma^{-1}\right)\right].\end{aligned} \tag{15}$$

If $f(0)=0$, then we have the simpler inequality

$$\begin{aligned}0&\le\frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}\right]\le f(A)A^{-1}-f(B)B^{-1}\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}\right].\end{aligned} \tag{16}$$

Proof. From (14) we have

$$0<\gamma-\beta\le B-A\le\delta-\alpha,$$

which implies that

$$\begin{aligned}0&\le(\gamma-\beta)\left((1-t)B+tA+\lambda\right)^{-2}\\&\le\left((1-t)B+tA+\lambda\right)^{-1}(B-A)\left((1-t)B+tA+\lambda\right)^{-1}\\&\le(\delta-\alpha)\left((1-t)B+tA+\lambda\right)^{-2}\end{aligned}$$

for all $t\in[0,1]$ and $\lambda\ge0$.

By integration over $t\in[0,1]$ we deduce

$$\begin{aligned}0&\le(\gamma-\beta)\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt\\&\le\int_0^1\left((1-t)B+tA+\lambda\right)^{-1}(B-A)\left((1-t)B+tA+\lambda\right)^{-1}dt\\&\le(\delta-\alpha)\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt\end{aligned}$$

for all $\lambda\ge0$.

If we multiply this inequality by λ and integrate over the measure $w(\lambda)$, we get

$$\begin{aligned}0&\le(\gamma-\beta)\int_0^\infty\lambda\left(\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt\right)dw(\lambda)\\&\le\int_0^\infty\lambda\left(\int_0^1\left((1-t)B+tA+\lambda\right)^{-1}(B-A)\left((1-t)B+tA+\lambda\right)^{-1}dt\right)dw(\lambda)\\&\le(\delta-\alpha)\int_0^\infty\lambda\left(\int_0^1\left((1-t)B+tA+\lambda\right)^{-2}dt\right)dw(\lambda),\end{aligned}$$

and, by (6) we derive the inequality of interest

$$\begin{aligned}
0 &\le (\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}dt\right)dw(\lambda) \\
&\le f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right) \\
&\le (\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}dt\right)dw(\lambda).
\end{aligned} \tag{17}$$

From (14) we derive that

$$(1-t)B+tA+\lambda \le (1-t)\delta+t\beta+\lambda,$$

which implies that

$$((1-t)B+tA+\lambda)^{-1} \ge ((1-t)\delta+t\beta+\lambda)^{-1}$$

and

$$((1-t)B+tA+\lambda)^{-2} \ge ((1-t)\delta+t\beta+\lambda)^{-2}$$

for all $t\in[0,1]$ and $\lambda\ge 0$.

Also

$$(1-t)B+tA+\lambda \ge (1-t)\gamma+t\alpha+\lambda,$$

which implies that

$$((1-t)B+tA+\lambda)^{-1} \le ((1-t)\gamma+t\alpha+\lambda)^{-1}$$

and

$$((1-t)B+tA+\lambda)^{-2} \le ((1-t)\gamma+t\alpha+\lambda)^{-2}$$

for all $t\in[0,1]$ and $\lambda\ge 0$.

Therefore

$$\begin{aligned}
&(\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)\delta+t\beta+\lambda)^{-2}dt\right)dw(\lambda) \\
&\qquad\le (\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}dt\right)dw(\lambda)
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&(\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}dt\right)dw(\lambda) \\
&\qquad\le (\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)\gamma+t\alpha+\lambda)^{-2}dt\right)dw(\lambda).
\end{aligned} \tag{19}$$

Since

$$\begin{aligned}
&(\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)\delta+t\beta+\lambda)^{-2}dt\right)dw(\lambda) \\
&= \frac{\gamma-\beta}{\delta-\beta}\int_0^\infty \lambda\left(\int_0^1 ((1-t)\delta+t\beta+\lambda)^{-1}(\delta-\beta)((1-t)\delta+t\beta+\lambda)^{-1}dt\right)dw(\lambda) \\
&= \frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}-f(0)\left(\beta^{-1}-\delta^{-1}\right)\right] \text{ (by (6))}
\end{aligned}$$

and

$$\begin{aligned}
&(\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)\alpha+t\gamma+\lambda)^{-2}dt\right)dw(\lambda)\\
&=\frac{\delta-\alpha}{\gamma-\alpha}\int_0^\infty \lambda\left(\int_0^1 ((1-t)\gamma+t\alpha+\lambda)^{-1}(\gamma-\alpha)((1-t)\gamma+t\alpha+\lambda)^{-1}dt\right)dw(\lambda)\\
&=\frac{\delta-\alpha}{\gamma-\alpha}\left[f(A)\alpha^{-1}-f(\gamma)\gamma^{-1}-f(0)\left(\alpha^{-1}-\gamma^{-1}\right)\right]\ \text{(by (6))},
\end{aligned}$$

then (18) and (19) become

$$\begin{aligned}
&\frac{\gamma-\beta}{\delta-\beta}\left[f(\beta)\beta^{-1}-f(\delta)\delta^{-1}-f(0)\left(\beta^{-1}-\delta^{-1}\right)\right]\\
&\qquad\le(\gamma-\beta)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}dt\right)dw(\lambda)
\end{aligned}\tag{20}$$

and

$$\begin{aligned}
&(\delta-\alpha)\int_0^\infty \lambda\left(\int_0^1 ((1-t)B+tA+\lambda)^{-2}dt\right)dw(\lambda)\\
&\qquad\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\alpha)\alpha^{-1}-f(\gamma)\gamma^{-1}-f(0)\left(\alpha^{-1}-\gamma^{-1}\right)\right].
\end{aligned}\tag{21}$$

Finally, on making use of (17), (20) and (22), we derive (15). □

Its is well known that, if $P\ge 0$, then

$$|\langle Px,y\rangle|^2\le\langle Px,x\rangle\langle Py,y\rangle$$

for all x, $y\in H$.

Therefore, if $T>0$, then

$$\begin{aligned}
0&\le\langle x,x\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2\\
&\le\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle
\end{aligned}$$

for all $x\in H$.

If $x\in H$, $\|x\|=1$, then

$$1\le\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\le\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1}\le T.\tag{22}$$

Corollary 2.3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. If A, $B>0$ and $\|A\|\left\|B^{-1}\right\|<1$, then*

$$\begin{aligned}
0&\le\frac{1-\|A\|\left\|B^{-1}\right\|}{(\|B\|-\|A\|)\left\|B^{-1}\right\|}\left[f(\|A\|)\|A\|^{-1}-f(\|B\|)\|B\|^{-1}-f(0)\left(\|A\|^{-1}-\|B\|^{-1}\right)\right]\\
&\le f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\\
&\le\frac{\left(\|B\|\left\|A^{-1}\right\|-1\right)\left\|B^{-1}\right\|}{\left\|A^{-1}\right\|-\left\|B^{-1}\right\|}\left[f\left(\left\|A^{-1}\right\|^{-1}\right)\left\|A^{-1}\right\|-f\left(\left\|B^{-1}\right\|^{-1}\right)\left\|B^{-1}\right\|\right.\\
&\quad\left.-f(0)\left(\left\|A^{-1}\right\|-\left\|B^{-1}\right\|\right)\right].
\end{aligned}\tag{23}$$

If $f(0)=0$, then we have the simpler inequality

$$
\begin{aligned}
0 &\le \frac{1-\|A\|\left\|B^{-1}\right\|}{(\|B\|-\|A\|)\left\|B^{-1}\right\|}\left[f(\|A\|)\|A\|^{-1}-f(\|B\|)\|B\|^{-1}\right] \\
&\le f(A)A^{-1}-f(B)B^{-1} \\
&\le \frac{\left(\|B\|\left\|A^{-1}\right\|-1\right)\left\|B^{-1}\right\|}{\left\|A^{-1}\right\|-\left\|B^{-1}\right\|}\left[f\left(\left\|A^{-1}\right\|^{-1}\right)\left\|A^{-1}\right\|-f\left(\left\|B^{-1}\right\|^{-1}\right)\left\|B^{-1}\right\|\right]. \quad (24)
\end{aligned}
$$

Proof. Since $\|A\|\left\|B^{-1}\right\|<1$, then

$$
0<\left\|A^{-1}\right\|^{-1}\le A\le\|A\|<\left\|B^{-1}\right\|^{-1}\le B\le\|B\|.
$$

By employing the inequality (15) for $\alpha=\left\|A^{-1}\right\|^{-1}$, $\beta=\|A\|$, $\gamma=\left\|B^{-1}\right\|^{-1}$ and $\delta=\|B\|$ we get

$$
\begin{aligned}
0 &\le \frac{\left\|B^{-1}\right\|^{-1}-\|A\|}{\|B\|-\|A\|}\left[f(\|A\|)\|A\|^{-1}-f(\|B\|)\|B\|^{-1}-f(0)\left(\|A\|^{-1}-\|B\|^{-1}\right)\right] \\
&\le f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right) \\
&\le \frac{\|B\|-\left\|A^{-1}\right\|^{-1}}{\left\|B^{-1}\right\|^{-1}-\left\|A^{-1}\right\|^{-1}} \\
&\times\left[f\left(\left\|A^{-1}\right\|^{-1}\right)\left\|A^{-1}\right\|-f\left(\left\|B^{-1}\right\|^{-1}\right)\left\|B^{-1}\right\|-f(0)\left(\left\|A^{-1}\right\|-\left\|B^{-1}\right\|\right)\right],
\end{aligned}
$$

which is equivalent to (24). □

3. Some Examples

If $0<\alpha\le A\le\beta<\gamma\le B\le\delta$ for some positive constants α, β, γ, δ, then (by 16)

$$
0\le\frac{\gamma-\beta}{\delta-\beta}\left(\beta^{r-1}-\delta^{r-1}\right)\le A^{r-1}-B^{r-1}\le\frac{\delta-\alpha}{\gamma-\alpha}\left(\alpha^{r-1}-\gamma^{r-1}\right). \quad (25)
$$

for $r\in(0,1]$.

If A, $B>0$ and $\|A\|\left\|B^{-1}\right\|<1$, then by (24) we get

$$
\begin{aligned}
0 &\le \frac{1-\|A\|\left\|B^{-1}\right\|}{(\|B\|-\|A\|)\left\|B^{-1}\right\|}\left[\left(\|A\|^{r-1}-\|B\|^{r-1}\right)\right] \quad (26) \\
&\le A^{r-1}-B^{r-1} \\
&\le \frac{\left(\|B\|\left\|A^{-1}\right\|-1\right)\left\|B^{-1}\right\|}{\left\|A^{-1}\right\|-\left\|B^{-1}\right\|}\left(\left\|A^{-1}\right\|^{1-r}-\left\|B^{-1}\right\|^{1-r}\right).
\end{aligned}
$$

for $r\in(0,1]$.

The function $f(t)=\ln(t+1)$ is operator monotone on $[0,\infty)$ and $f(0)=0$. If we write the inequality (16) for this function, we derive

$$
\begin{aligned}
0 &\le \frac{\gamma-\beta}{\delta-\beta}\ln\left[\frac{(\beta+1)^{\beta^{-1}}}{(\delta+1)^{\delta^{-1}}}\right]\le A^{-1}\ln(A+1)-B^{-1}\ln(B+1) \\
&\le \frac{\delta-\alpha}{\gamma-\alpha}\ln\left[\frac{(\alpha+1)^{\alpha^{-1}}}{(\gamma+1)^{\gamma^{-1}}}\right], \quad (27)
\end{aligned}
$$

provided that $0<\alpha\leq A\leq\beta<\gamma\leq B\leq\delta$ for some positive constants α, β, γ, δ.

If A, $B>0$ and $\|A\|\left\|B^{-1}\right\|<1$, then by (24) we get

$$\begin{aligned}
0 &\leq \frac{1-\|A\|\left\|B^{-1}\right\|}{(\|B\|-\|A\|)\left\|B^{-1}\right\|}\left[\|A\|^{-1}\ln(\|A\|+1)-\|B\|^{-1}\ln(\|B\|+1)\right] \\
&\leq A^{-1}\ln(A+1)-B^{-1}\ln(B+1) \\
&\leq \frac{\left(\|B\|\left\|A^{-1}\right\|-1\right)\left\|B^{-1}\right\|}{\left\|A^{-1}\right\|-\left\|B^{-1}\right\|}\left[\left\|A^{-1}\right\|\ln\left(\left\|A^{-1}\right\|^{-1}+1\right)-\left\|B^{-1}\right\|\ln\left(\left\|B^{-1}\right\|^{-1}+1\right)\right].
\end{aligned} \tag{28}$$

For other examples of operator monotone functions see also [2], [4] and the references therein.

References

[1] R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics 169, Springer-Verlag, New York, 1997. xii+347

[2] J. I. Fujii and Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998), 301–306.

[3] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008), 972–980.

[4] T. Furuta, Precise lower bound of $f(A)-f(B)$ for $A>B>0$ and non-constant operator monotone function f on $[0,\infty)$, *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[5] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951), 415–438.

[6] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934), 177–216.

[7] M. S. Moslehian and H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.

[8] H. Zuo and G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

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