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Exchange variations of generalized dual parallel curves and surfaces

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Abstract. Parallel curves (or offset curves) and parallel surfaces (or offset surfaces) have a big importance for CAD/CAM, robotics, cam design and many industrial applications, especially for mathematical modelling of cutting paths milling machines. Any vector space has a corresponding dual vector space that consists of all linear functions on vector space. Dual spaces are used in mathematics such as describing measures, distributions, and Hilbert spaces. Consequently, the dual space is an important concept in functional analysis. This paper proposes a novel definition of generalized and standard dual parallel curves and surfaces. Additionally, we give some properties of generalized dual parallel curves and surfaces using this novel definition. We also express the variation of the generalized dual parallel curves, the first and second variation of area change of the standard dual parallel surfaces and the first variation of area change of the generalized dual parallel surfaces.

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1. Introduction

A dual number is written as,

$$X = x + \varepsilon x_0$$

where, x and x_0 are real numbers and $\varepsilon^2 = 0$, $\varepsilon \neq 0$, [4], [5]. The sum and the product of the two dual numbers $X = x + \varepsilon x_0$ and $Y = y + \varepsilon y_0$ are,

$$\begin{aligned} X + Y &= (x + y) + \varepsilon (x_0 + y_0) \\ X.Y &= xy + \varepsilon (x_0 y + x y_0). \end{aligned}$$

The equality of the two dual numbers X and Y is defined as

$$X = Y \Longleftrightarrow x = y \quad and \quad x_0 = y_0.$$

The parametric equation of the oriented line in E^3 is written as

$$\mathbf{R} = \boldsymbol{\nu} + t\mathbf{r}, \tag{1}$$

where $\mathbf{R}$ and $\boldsymbol{\nu}$ are two points on the line, $\mathbf{r}$ is a unit vector along the line, and t is real parameter. We can define the moment of $\mathbf{r}$ according to the origin as

$$\mathbf{r_0} = \mathbf{R} \times \mathbf{r} = \boldsymbol{\nu} \times \mathbf{r}, \tag{2}$$

or, in other words, $\mathbf{r}$ and $\mathbf{r_0}$ are not independent of the choice of the points on the line, and the relationships $\langle \mathbf{r}, \mathbf{r} \rangle = 1$ and $\langle \mathbf{r}, \mathbf{r_0} \rangle = 0$ are satisfied. Therefore, the two vectors $\mathbf{r}$ and $\mathbf{r_0}$ construct the oriented line.

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If we take the space D^3 of the triples of dual numbers with $X = x_i + \varepsilon x_{0i}$, $(i = 1, 2, 3)$, we can define a unit dual vector as

$$\mathbf{R} = \mathbf{r} + \varepsilon \mathbf{r_0} \tag{3}$$

where $\mathbf{r}$ and $\mathbf{r_0}$ are the real vectors. Hence, each line in E^3 can be represented by a dual vector using equation (3) in D^3 together with $\langle \mathbf{R}, \mathbf{R} \rangle = 1$. Parallel curves and surfaces have great importance in industrial applications, such as tool path generation for numerical control (NC) machining and robot path planning. We can apply parallel operations to curves, surfaces, or entire 3D models. We can also use the generalized parallel curves and surfaces for the generation of a rich variety of shapes. In our previous study [2], we proposed a novel definition of generalized and standard offset curves and surfaces. We also gave some analytic properties of generalized offset curves and surfaces using this new definition. Additionally, we examined the first variation of standard offset curves, and the first and second variation of area change of standard offset surfaces. In this paper, we present a new definition of generalized and standard dual parallel curves and surfaces, and we also give some properties of generalized dual parallel curves and surfaces using this novel definition. Additionally, we express the variation of generalized dual parallel curves, the first and second variations of area change of standard dual parallel surfaces, and the first variation of area change of generalized dual parallel surfaces.

2. The generalized dual parallel curves

Two definitions for generalized parallel curves in real space are given in [2] and [3].

Let

$$\mathbf{R} = \mathbf{R}(t) = (\mathbf{r}(t), \mathbf{r_0}(t)) \tag{4}$$

be a dual curve and its Blaschke trihedron given by $(\mathbf{R}_1 = \mathbf{R}, \mathbf{R}_2, \mathbf{R}_3)$. The derivatives of this trihedron are defined as

$$\begin{gathered} \mathbf{R}_1^{'} = P\mathbf{R}_2 \qquad \mathbf{R}_2^{'} = -P\mathbf{R}_1 + Q\mathbf{R}_3 \quad \mathbf{R}_3^{'} = -Q\mathbf{R}_2, \\ P = \left(\mathbf{R}_1^{'2}\right)^{1/2} \qquad Q = \frac{\left(\mathbf{R}_1, \mathbf{R}_1^{'}, \mathbf{R}_1^{''}\right)}{\mathbf{R}_1^{'2}}, \end{gathered} \tag{5}$$

where $\mathbf{R}_i^{'} = {}^{d\mathbf{R}_i}/_{dt}$, and P and Q are the dual curvature and torsion of the dual curve, respectively [1].

Definition 2.1. We can define the generalized dual parallel curve of the dual curve $\mathbf{R}(t) = \mathbf{r}(t) + \varepsilon \mathbf{r_0}(t)$ with the following equation:

$$\mathbf{R}_*(t) = \mathbf{r}_*(t) + \varepsilon \mathbf{r}_{*\mathbf{0}}(t) = \mathbf{R}(t) + \Lambda \bar{D}_1(t)\, \mathbf{R}_1 + \Lambda \bar{D}_2(t)\, \mathbf{R}_2 \tag{6}$$

where $D_1(t) = \Lambda \bar{D}_1(t)$, $D_2(t) = \Lambda \bar{D}_2(t)$, $D_1(t) = d_1(t) + \varepsilon d_{10}(t)$, $D_2(t) = d_2(t) + \varepsilon d_{20}(t)$, $\bar{D}_1(t) = \bar{d}_1(t) + \varepsilon \bar{d}_{10}(t)$, $\bar{D}_2(t) = \bar{d}_2(t) + \varepsilon \bar{d}_{20}(t)$, and $\Lambda(t) = \lambda(t) + \varepsilon \lambda_0(t)$. Additionally, the parallel distance is $\Lambda \sqrt{\bar{D}_1^2(t) + \bar{D}_2^2(t)}$.

Remark 2.1. If the dual part of the generalized dual parallel curve in (6) equals zero, then we obtain the generalized real parallel curve in [2]. Hence, in this paper we express the generalized dual parallel curves with (6) as the generalized form of the generalized real parallel curves in [2].

The parametric derivatives of $\mathbf{R}_*(t)$ can be written as

$$\begin{aligned}\mathbf{R}_*' &= \mathbf{R}' + \Lambda\left[\bar{D}_1'\mathbf{R}_1 + \bar{D}_1\mathbf{R}_1' + \bar{D}_2'\mathbf{R}_2 + \bar{D}_2\mathbf{R}_2'\right] \\ &= P\mathbf{R}_2 + \Lambda\left[\left(\bar{D}_1' - P\bar{D}_2\right)\mathbf{R}_1 + \left(\bar{D}_2' + P\bar{D}_1\right)\mathbf{R}_2\right].\end{aligned} \tag{7}$$

If we take $\alpha = \bar{D}_1' - P\bar{D}_2$ and $\beta = \bar{D}_2' + P\bar{D}_1$, then we can rewrite equation (7) as

$$\mathbf{R}_*' = P\mathbf{R}_2 + \Lambda\left(\alpha\mathbf{R}_1 + \beta\mathbf{R}_2\right).$$

Thus, we have

$$\begin{aligned}\mathbf{R}_*'' &= P\mathbf{R}_2' + \Lambda\left(\alpha'\mathbf{R}_1 + \alpha\mathbf{R}_1' + \beta'\mathbf{R}_2 + \beta\mathbf{R}_2'\right) \\ &= -P^2\mathbf{R}_1 + \Lambda\left[\left(\alpha' - \beta P\right)\mathbf{R}_1 + \left(\beta' + \alpha P\right)\mathbf{R}_2\right].\end{aligned} \tag{8}$$

Hence, we can find the following equations.

$$\begin{aligned}\mathbf{R}_{*1} &= \mathbf{R}_* \\ \mathbf{R}_{*2} &= \frac{\mathbf{R}_{*1}'}{\sqrt{\mathbf{R}_{*1}'^2}} = \frac{P\mathbf{R}_2+\Lambda(\alpha\mathbf{R}_1+\beta\mathbf{R}_2)}{\sqrt{P^2+\Lambda^2(\alpha^2+\beta^2)+2\Lambda P\beta}}.\end{aligned} \tag{9}$$

To determine the curvature of the generalized parallel dual curve $\mathbf{R}_*(t)$, we can write $\mathbf{R}_*(t)$ as

$$\mathbf{R}_* = \mathbf{R} + D_1\mathbf{R}_1 + D_2\mathbf{R}_2 = \mathbf{R} + \mathbf{Y} = \mathbf{R} + \Lambda\bar{\mathbf{Y}}. \tag{10}$$

Then, using equations (5) and (10) we can define the curvature P_* of the generalized dual parallel curve $\mathbf{R}_*(t)$ as

$$P_* = \frac{\sqrt{\mathbf{R}_*''^2}}{\mathbf{R}_*'^2} = 1 - \Lambda\left(\frac{2}{P}\mathbf{R}_2\bar{\mathbf{Y}}' + \frac{1}{P^2}\mathbf{R}_1\bar{\mathbf{Y}}''\right). \tag{11}$$

Theorem 2.1. *The generalized dual parallel curve $\mathbf{R}_*(t) = \mathbf{R}(t) + \Lambda\bar{D}_1(t)\mathbf{R}_1 + \Lambda\bar{D}_2(t)\mathbf{R}_2$ can be represented as a standard dual parallel curve using the following equation:*

$$\mathbf{R}_* = \mathbf{R}_\blacklozenge + D\mathbf{R}_{\blacklozenge 2}, \tag{12}$$

where $\mathbf{R}_\blacklozenge$ is another dual curve, $D = d + \varepsilon\bar{d}_0$ is a constant value, and $\mathbf{R}_{\blacklozenge 2}$ is the normal vector of $\mathbf{R}_\blacklozenge$.

Proof of Theorem 2.1. The generalized dual parallel curve can be written as

$$\begin{aligned}\mathbf{R}_* &= \mathbf{R} + D_1\mathbf{R}_1 + D_2\mathbf{R}_2 \\ &= \left(\mathbf{R} + A\mathbf{R}_1 + B\mathbf{R}_2\right) + \left(D_1 - A\right)\mathbf{R}_1 + \left(D_2 - B\right)\mathbf{R}_2 \\ &\triangleq \mathbf{R}_\blacklozenge + D\mathbf{R}_{\blacklozenge 2},\end{aligned} \tag{13}$$

where $D_1 = \Lambda\bar{D}_1$, $D_2 = \Lambda\bar{D}_2$, $A = \Lambda\bar{A}$ and $B = \Lambda\bar{B}$ with the functions of t being $A = a + \varepsilon a_0$, $B = b + \varepsilon b_0$, $\bar{A} = \bar{a} + \varepsilon\bar{a}_0$, and $\bar{B} = \bar{b} + \varepsilon\bar{b}_0$. Moreover,

$$\begin{aligned} D &= \Lambda\sqrt{\left(\bar{D}_1 - \bar{A}\right)^2 + \left(\bar{D}_2 - \bar{B}\right)^2} = \Lambda\bar{D} \\ \mathbf{R}_{\blacklozenge 2} &= \Lambda\left[\frac{\left(\bar{D}_1-\bar{A}\right)}{D}\mathbf{R}_1 + \frac{\left(\bar{D}_2-\bar{B}\right)}{D}\mathbf{R}_2\right]. \end{aligned} \tag{14}$$

□

Remark 2.2. If the dual part of the standard dual parallel curve in (12) equals zero, then we obtain the standard real parallel curve with equation (9) (see [2]). Hence, in this paper we express the standard dual parallel curves with (12) as the generalized form of the standard real parallel curves in equation (9) in [2].

$$\begin{aligned} \mathbf{R}_{\blacklozenge}^{'} &= \mathbf{R}^{'} + \Lambda\left[\bar{A}^{'}\mathbf{R}_1 + \bar{A}\mathbf{R}_1^{'} + \bar{B}^{'}\mathbf{R}_2 + \bar{B}\mathbf{R}_2^{'}\right] \\ &= P\mathbf{R}_2 + \Lambda\left[\left(\bar{A}^{'} - \bar{B}P\right)\mathbf{R}_1 + \left(\bar{B}^{'} + \bar{A}P\right)\mathbf{R}_2\right]. \end{aligned} \tag{15}$$

The following two equations must be satisfied to provide the relationship in (15).
(*i*) $\left\langle D\mathbf{R}_{\blacklozenge 2}, \mathbf{R}_{\blacklozenge}^{'}\right\rangle = 0$, that is,

$$\Lambda^2\left[\left(\bar{D}_1 - \bar{A}\right)\mathbf{R}_1 + \left(\bar{D}_2 - \bar{B}\right)\left(\bar{B}^{'} - \bar{A}P\right)\right] + \Lambda\left(\bar{D}_2 - \bar{B}\right)P = 0. \tag{16}$$

(*ii*) $\Lambda^2\left[\left(\bar{D}_1 - \bar{A}\right)^2 + \left(\bar{D}_2 - \bar{B}\right)^2\right] = const$. That is,

$$\Lambda^2\left[\left(\bar{D}_1 - \bar{A}\right)\left(\bar{D}_1^{'} - \bar{A}^{'}\right) + \left(\bar{D}_2 - \bar{B}\right)\left(\bar{D}_2^{'} - \bar{B}^{'}\right)\right] = 0. \tag{17}$$

Next, to compute $A = \Lambda\bar{A}$ and $B = \Lambda\bar{B}$ using the differential equations (16) and (17) the equations below can be used.

$$\begin{aligned} \Lambda^2\left(\bar{D}_2P - \bar{D}_1^{'}\right)\bar{A} &= \left[\Lambda^2\left(\bar{D}_1P + \bar{D}_2^{'}\right) + \Lambda P\right]B \\ &\quad - \left[\Lambda\bar{D}_2P + \Lambda^2\left(\bar{D}_1\bar{D}_1^{'} + \bar{D}_2\bar{D}_2^{'}\right)\right]. \end{aligned} \tag{18}$$

Additionally, let

$$-\Lambda\bar{D}_2P - \Lambda^2\left(\bar{D}_1\bar{D}_1^{'} + \bar{D}_2\bar{D}_2^{'}\right) = -\left[\Lambda\bar{D}_2P + \Lambda^2\left(\bar{D}_1\alpha + \bar{D}_2\beta\right)\right], \tag{19}$$

where $\alpha = \bar{D}_1^{'} - \bar{D}_2P$ and $\beta = \bar{D}_1P + \bar{D}_2^{'}$. Hence, we can obtain

$$\Lambda\left[\left(\bar{D}_2 - \bar{B}\right)P + \Lambda\alpha\left(\bar{D}_1 - \bar{A}\right) + \Lambda\beta\left(\bar{D}_2 - \bar{B}\right)\right] = 0. \tag{20}$$

If $\alpha \neq 0$ and $\beta \neq 0$, the values of $A = \Lambda\bar{A}$ and $B = \Lambda\bar{B}$ can be determined by using (17) and (20) with the following equation.

$$\begin{aligned} \bar{A} &= \bar{D}_1 - \frac{\beta C}{\sqrt{\Lambda^4\beta^2 + \left(\Lambda^2\alpha + \Lambda P\right)^2}} \\ \bar{B} &= \bar{D}_2 + \frac{\alpha C}{\sqrt{\Lambda^4\beta^2 + \left(\Lambda^2\alpha + \Lambda P\right)^2}}, \end{aligned} \tag{21}$$

where C is an arbitrary constant.

3. The variation of the generalized dual parallel curve

Let the dual curve $\boldsymbol{R}(t)$ be given. We can write the generalized dual parallel curve of this curve as

$$\boldsymbol{R_*} = \boldsymbol{R} + D_1\boldsymbol{R_1} + D_2\boldsymbol{R_2} = \boldsymbol{R} + \boldsymbol{Y}, \tag{22}$$

where

$$D_1 = \Lambda\bar{D}_1,\, D_2 = \Lambda\bar{D}_2,\, and\, \boldsymbol{Y} = \Lambda\bar{\boldsymbol{Y}}. \tag{23}$$

If the condition $\Lambda \to 0$ occurs, then the generalized dual parallel curve $\boldsymbol{R_*}(t)$ approaches to the curve $\boldsymbol{R}(t)$. If we expand the series according to Λ' powers of the dual arc length S_* of the generalized dual parallel curve, we can obtain

$$S_* = S + \delta S + ... \tag{24}$$

Therefore, δS is called the first variation of the dual arc length and is defined as

$$\delta S = \Lambda \lim_{\Lambda\to 0}\frac{S_* - S}{\Lambda}. \tag{25}$$

Since $\mathbf{R}_1^{'} = P\mathbf{R}_2$ and $\mathbf{R}_2^{'} = -P\mathbf{R}_1$, after some computations, we can write the first variation of the dual arc length as

$$\delta S = P\,[D_2]_{S_1}^{S_2} + \int_{S_1}^{S_2} D_1P^2 dS. \tag{26}$$

On the other hand, let us examine the exchange variation of curvatures of these curves. If we write the generalized dual parallel curve of this curve as

$$\boldsymbol{R_*} = \boldsymbol{R} + D_1\boldsymbol{R_1} + D_2\boldsymbol{R_2} = \boldsymbol{R} + \boldsymbol{Y} = \boldsymbol{R} + \Lambda\bar{\boldsymbol{Y}}, \tag{27}$$

since the curvature of the generalized dual parallel curve $\boldsymbol{R_*}(t)$ is described with

$$P_* = \frac{\sqrt{\boldsymbol{R_*''^2}}}{\boldsymbol{R_*'^2}}, \tag{28}$$

we can obtain the curvature variation of these curves. It is

$$P_* = 1 - \Lambda\left(\frac{2}{P}\boldsymbol{R_2}\bar{\boldsymbol{Y}}^{'} + \frac{1}{P^2}\boldsymbol{R_1}\bar{\boldsymbol{Y}}^{''}\right). \tag{29}$$

Therefore, the variation for the equation

$$\delta P = \Lambda \lim_{\Lambda\to 0}\frac{1 - P_*}{\Lambda}, \tag{30}$$

can express the curvature variation between the progenitor dual curve $\boldsymbol{R}(t)$ and the generalized dual parallel curve $\boldsymbol{R_*}(t)$ [1], i.e.,

$$\delta P = \Lambda\left(\frac{2}{P}\boldsymbol{R_2}\bar{\boldsymbol{Y}}^{'} + \frac{1}{P^2}\boldsymbol{R_1}\bar{\boldsymbol{Y}}^{''}\right). \tag{31}$$

4. The generalized dual parallel surfaces

Two definitions for generalized parallel surfaces in real space are given in [2] and [3].

Let a dual surface and its right-handed system be $\mathbf{R}(u,v) = \mathbf{r}(u,v) + \varepsilon \mathbf{r}_0(u,v)$ and $(\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{R})$, respectively, which can then be given as

$$\mathbf{E}_1 = \frac{\mathbf{R}_u(u,v)}{\|\mathbf{R}_u(u,v)\|}, \quad \mathbf{E}_2 = \frac{\mathbf{R}_v(u,v)}{\|\mathbf{R}_v(u,v)\|}, \quad \mathbf{R} = \boldsymbol{R}(u,v), \tag{32}$$

where $\mathbf{E}_1 = \mathbf{e_1} + \varepsilon \mathbf{e_{10}}$, $\mathbf{E}_2 = \mathbf{e_2} + \varepsilon \mathbf{e_{20}}$, $\mathbf{R} = \mathbf{r} + \varepsilon \mathbf{r}_0$, and $\mathbf{R}_u(u,v)$ and $\mathbf{R}_v(u,v)$ are the partial derivatives of $\mathbf{R}(u,v)$ with regards to parameters u and v. A generalized dual parallel surface $\boldsymbol{R}^*(u,v) = \boldsymbol{r}^*(u,v) + \varepsilon \boldsymbol{r}^*_{\mathbf{0}}(u,v)$ can be defined along with the variable parallel direction and parallel distance by using the coordinate system $(\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{R})$ of the dual surface $\mathbf{R}(u,v)$. In this paper, we suppose that the coordinate system $(\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{R})$ of the dual surface $\mathbf{R}(u,v)$ is an orthonormal coordinate system.

Definition 4.1. The generalized dual parallel surface $\boldsymbol{R}^*(u,v)$ for a dual surface $\mathbf{R}(u,v)$, $(u,v) \in [0,1] \times [0,1]$ can be obtained using

$$\boldsymbol{R}^*(u,v) = \boldsymbol{R}(u,v) + D_1(u,v)\mathbf{E}_1 + D_2(u,v)\mathbf{E}_2 + D_3(u,v)\mathbf{R}, \tag{33}$$

where $D_1 = d_1 + \varepsilon d_{10}$, $D_2 = d_2 + \varepsilon d_{20}$ and $D_3 = d_3 + \varepsilon d_{30}$ are the functions of the variables u and v. The parallel direction and distance are then computed using $D_1\mathbf{E}_1$, $D_2\mathbf{E}_2$, and $D_3\mathbf{R}$.

We can express the generalized dual parallel surface, using the equation in [1] as

$$\boldsymbol{R}^*(u,v) = \boldsymbol{R}(u,v) + \delta \boldsymbol{R}(u,v), \tag{34}$$

where

$$\delta \boldsymbol{R} = D_1\mathbf{E}_1 + D_2\mathbf{E}_2 + D_3\mathbf{R} \tag{35}$$

and

$$D_1 = \Lambda \bar{D}_1, \quad D_2 = \Lambda \bar{D}_2, \textit{ and } \quad D_3 = \Lambda \bar{D}_3 \ , \tag{36}$$

in which, $\bar{D}_1(t) = \bar{d}_1(t) + \varepsilon \bar{d}_{10}(t)$, $\bar{D}_2(t) = \bar{d}_2(t) + \varepsilon \bar{d}_{20}(t)$, $\bar{D}_3(t) = \bar{d}_3(t) + \varepsilon \bar{d}_{30}(t)$, and $\Lambda(t) = \lambda(t) + \varepsilon \lambda_0(t)$.

The generalized dual parallel surface $\mathbf{R}^*(u,v)$ approaches to the progenitor dual surface $\mathbf{R}(u,v)$ for $\Lambda \to 0$.

Remark 4.1. If the dual part of the generalized dual parallel surface in (33) equals zero, then we obtain the generalized real parallel surface (see [2]). Hence, in this paper, we express the generalized dual parallel surfaces with (33) as the generalized form of the generalized real parallel surfaces in [2].

Now, we can determine the first and second fundamental quantities and the Gauss and mean curvatures of the generalized dual parallel surface $\boldsymbol{R}^*(u,v)$. Since, $\boldsymbol{R}^*(u,v) = \boldsymbol{r}^*(u,v) + \varepsilon \boldsymbol{r}^*_{\mathbf{0}}(u,v)$, $\boldsymbol{r}^*(u,v) = \boldsymbol{r}(u,v) + d_1\mathbf{e_1} + d_2\mathbf{e_2} + d_3\mathbf{r}$, and $\boldsymbol{r}^*_{\mathbf{0}}(u,v) = \boldsymbol{r_0}(u,v) + \bar{d}_1\mathbf{e_1} + \bar{d}_2\mathbf{e_2} + \bar{d}_3\mathbf{r}$, the first and second fundamental quantities are

$$\begin{aligned}
&E = e + \varepsilon e_0, && F = f + \varepsilon f_0, && G = g + \varepsilon g_0 \\
&e = (\boldsymbol{r_u})^2 = A_1^2, && f = \boldsymbol{r_u}.\boldsymbol{r_v}, && g = (\boldsymbol{r_v})^2 = A_2^2 \\
&e_0 = 2\boldsymbol{r_u}.\boldsymbol{r_{0v}} = 2A_1A_{10}, && f_0 = r_u.\boldsymbol{r_{0v}} + \boldsymbol{r_v}.\boldsymbol{r_{0u}}, && \\
&g_0 = 2\boldsymbol{r_v}.\boldsymbol{r_{0v}} = 2A_2A_{20}. && &&
\end{aligned} \tag{37}$$

Next, we can obtain partial derivatives using equations (34), (35) and (36):

$$
\begin{aligned}
\boldsymbol{R}_{\boldsymbol{u}}^{*} = & \boldsymbol{r}_{\boldsymbol{u}} + \Lambda\left[\left(\bar{d}_{1u}+\bar{d}_{2}\frac{A_{1v}}{A_{2}}+\bar{d}_{3}A_{1}\right)\mathbf{e_1}+\left(-\bar{d}_{1}\frac{A_{1v}}{A_{2}}+\bar{d}_{2u}\right)\mathbf{e_2}+\left(\bar{d}_{1}\frac{A_{1}}{R_{1}}+\bar{d}_{3u}\right)\mathbf{r}\right] \\
& + \varepsilon\Lambda\left[\left(\bar{d}_{10u}+\bar{d}_{1}\frac{A_{1v}}{A_{2}}-\bar{d}_{2u}+\bar{d}_{20}\frac{A_{1v}}{A_{2}}+\bar{d}_{30}A_{1}\right)\mathbf{e_1}\right. \\
& \left.+\left(\bar{d}_{1u}-\bar{d}_{10}\frac{A_{1v}}{A_{2}}+\bar{d}_{20u}-\bar{d}_{2}\frac{A_{1v}}{A_{2}}\right)\mathbf{e_2}+\left(\bar{d}_{10}\frac{A_{1}}{R_{1}}+\bar{d}_{2}\frac{A_{1}}{R_{1}}+\bar{d}_{30u}\right)\mathbf{r}\right]
\end{aligned}
$$

$$
\triangleq \quad A_{1}\mathbf{e_1}+\Lambda\bar{B}_{1}\mathbf{e_1}+\Lambda\bar{B}_{2}\mathbf{e_2}+\Lambda\bar{B}_{3}\mathbf{r}+\varepsilon\Lambda\left(\bar{U}_{1}\mathbf{e_1}+\bar{U}_{2}\mathbf{e_2}+\bar{U}_{3}\mathbf{r}\right) \tag{38}
$$

and

$$
\begin{aligned}
\boldsymbol{R}_{\boldsymbol{v}}^{*} = & \boldsymbol{r}_{\boldsymbol{v}} + \Lambda\left[\left(\bar{d}_{2v}+\bar{d}_{1}\frac{A_{2u}}{A_{1}}+\bar{d}_{3}A_{2}\right)\mathbf{e_2}+\left(\bar{d}_{1v}-\bar{d}_{2}\frac{A_{2u}}{A_{1}}\right)\mathbf{e_1}+\left(\bar{d}_{2}\frac{A_{2}}{R_{2}}+\bar{d}_{3v}\right)\mathbf{r}\right] \\
& + \varepsilon\Lambda\left[\left(\bar{d}_{10v}-\bar{d}_{1}\frac{A_{2u}}{A_{1}}-\bar{d}_{2v}-\bar{d}_{2}\frac{A_{2u}}{A_{1}}-\bar{d}_{20}\frac{A_{2u}}{A_{1}}\right)\boldsymbol{e_1}\right. \\
& +\left(\bar{d}_{1v}+\bar{d}_{10}\frac{A_{2u}}{A_{1}}+\bar{d}_{20v}+\bar{d}_{2}\frac{A_{2u}}{A_{1}}+\bar{d}_{30}A_{2}\right)\boldsymbol{e_2} \\
& \left.+\left(\bar{d}_{1}\frac{A_{2}}{R_{2}}+\bar{d}_{2}\frac{A_{2}}{R_{2}}+\bar{d}_{20}\frac{A_{2}}{R_{2}}+\bar{d}_{30v}\right)\mathbf{r}\right]
\end{aligned}
$$

$$
\triangleq \quad A_{2}\mathbf{e_2}+\Lambda\bar{C}_{1}\mathbf{e_1}+\Lambda\bar{C}_{2}\mathbf{e_2}+\Lambda\bar{C}_{3}\mathbf{r}+\varepsilon\Lambda\left(\bar{V}_{1}\mathbf{e_1}+\bar{V}_{2}\mathbf{e_2}+\bar{V}_{3}\mathbf{r}\right) \tag{39}
$$

where $\boldsymbol{r}_{\boldsymbol{u}}$, $\bar{d}_{1u}$, $\bar{d}_{2u}$, A_{2u} and $\boldsymbol{r}_{\boldsymbol{v}}$, $\bar{d}_{1v}$, $\bar{d}_{2v}$, A_{2v} are the corresponding partial derivatives of $\boldsymbol{r}$, $\bar{d}_1$, $\bar{d}_2$, A_2 with respect to the parameter u and v, respectively. Also, R_1 and R_2 are called the radii of the principal curvatures. Therefore, we arrive at the following equations using (38) and (39):

$$
\begin{aligned}
\boldsymbol{R}_{\boldsymbol{uu}}^{*} = & A_{1u}\mathbf{e_1}+A_{1}\mathbf{e_{1u}}+\Lambda\left[\bar{B}_{1u}\mathbf{e_1}+\bar{B}_{1}\mathbf{e_{1u}}+\bar{B}_{2u}\mathbf{e_2}+\bar{B}_{2}\mathbf{e_{2u}}+\bar{B}_{3u}\mathbf{r}+\bar{B}_{3}\mathbf{r_u}\right] \\
& +\varepsilon\Lambda\left[\bar{U}_{1u}\mathbf{e_1}+\bar{U}_{1}\mathbf{e_{1u}}+\bar{U}_{2u}\mathbf{e_2}+\bar{U}_{2}\mathbf{e_{2u}}+\bar{U}_{3u}\boldsymbol{r}+\bar{U}_{3}\mathbf{r_u}\right] \\
= & A_{1u}\mathbf{e_1}-\frac{A_{1}A_{1v}}{A_{2}}\mathbf{e_2}+\frac{A_{1^2}}{R_{1}}\mathbf{r} \\
& +\Lambda\left[\left(\bar{B}_{1u}+\frac{\bar{B}_{2}A_{1v}}{A_{2}}+\bar{B}_{3}A_{1}\right)\boldsymbol{e_1}+\left(\bar{B}_{2u}-\frac{\bar{B}_{1}A_{1v}}{A_{2}}\right)\boldsymbol{e_2}+\left(\bar{B}_{3u}+\frac{\bar{B}_{1}A_{1}}{R_{1}}\right)\boldsymbol{r}\right] \\
& +\varepsilon\Lambda\left[\left(\bar{U}_{1u}+\frac{\bar{U}_{2}A_{1v}}{A_{2}}+\bar{U}_{3}A_{1}\right)\mathbf{e_1}+\left(\bar{U}_{2u}-\frac{\bar{U}_{1}A_{1v}}{A_{2}}\right)\mathbf{e_2}+\left(\bar{U}_{3u}+\frac{\bar{U}_{1}A_{1}}{R_{1}}\right)\mathbf{r}\right],
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\boldsymbol{R}_{\boldsymbol{uv}}^{*} = & A_{1v}\mathbf{e_1}+A_{2u}\mathbf{e_2} \\
& +\Lambda\left[\left(\bar{B}_{1v}-\frac{\bar{B}_{2}A_{2u}}{A_{1}}\right)\mathbf{e_1}+\left(\bar{\mathrm{B}}_{2\mathrm{u}}+\frac{\bar{B}_{1}A_{2u}}{A_{1}}+\bar{\mathrm{B}}_{3}A_{2}\right)\mathbf{e_2}+\left(\bar{\mathrm{B}}_{3\mathrm{v}}+\frac{\bar{B}_{2}A_{2}}{R_{2}}\right)\mathbf{r}\right] \\
& +\varepsilon\Lambda\left[\left(\bar{U}_{1v}-\frac{\bar{U}_{2}A_{2u}}{A_{1}}\right)\mathbf{e_1}+\left(\bar{U}_{2v}+\frac{\bar{U}_{1}A_{2u}}{A_{1}}+\bar{U}_{3}A_{2}\right)\mathbf{e_2}+\left(\bar{U}_{3v}+\frac{\bar{U}_{2}A_{2}}{R_{2}}\right)\mathbf{r}\right],
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
\boldsymbol{R}^{*}_{\boldsymbol{vv}} = & -\frac{A_2A_{2u}}{A_1}\mathbf{e_1} + A_{2v}\mathbf{e_2} + \frac{A_2^2}{R_2}\mathbf{r} \\
& + \Lambda\left[\left(\bar{C}_{1v} - \frac{\bar{C}_2A_{2u}}{A_1}\right)\mathbf{e_1} + \left(\bar{C}_{2v} + \frac{\bar{C}_1A_{2u}}{A_1} + \bar{C}_3A_2\right)\mathbf{e_2} + \left(\bar{C}_{3v} + \frac{\bar{C}_2A_2}{R_2}\right)\mathbf{r}\right] \\
& + \left(\bar{\mathrm{B}}_{2\mathrm{u}} + \frac{\bar{B}_1A_{2u}}{A_1} + \bar{\mathrm{B}}_3A_2\right)\mathbf{e_2} + \left(\bar{\mathrm{B}}_{3\mathrm{v}} + \frac{\bar{B}_2A_2}{R_2}\right)\mathbf{r} \\
& + \varepsilon\Lambda\left[\left(\bar{V}_{1v} - \frac{\bar{V}_2A_{2u}}{A_1}\right)\mathbf{e_1} + \left(\bar{V}_{2v} + \frac{\bar{V}_1A_{2u}}{A_1} + \bar{V}_3A_2\right)\mathbf{e_2} + \left(\bar{V}_{3v} + \frac{\bar{V}_2A_2}{R_2}\right)\mathbf{r}\right].
\end{aligned}
\tag{42}
$$

Therefore, the first fundamental quantities of the generalized dual parallel surface $\boldsymbol{R}^{*}(u,v)$ can be expressed with

$$
\begin{aligned}
e^* = \quad & \mathbf{r}_{\boldsymbol{u}}^2 = A_1^2 + 2\Lambda A_1\bar{B}_1 + \Lambda^2\left(\bar{B}_1^2 + \bar{B}_2^2 + \bar{B}_3^2\right) \\
e_0^* = \quad & 2\boldsymbol{r_u}\boldsymbol{r_{0u}} = 2\left[\Lambda A_1\bar{U}_1 + \Lambda^2\left(\bar{B}_1\bar{U}_1 + \bar{B}_2\bar{U}_2 + \bar{B}_3\bar{U}_3\right)\right] \\
f^* = \quad & \boldsymbol{r_u}\boldsymbol{r_v} = \Lambda\left(A_1\bar{C}_1 + A_2\bar{B}_2\right) + \Lambda^2\left(\bar{C}_1\bar{B}_1 + \bar{C}_2\bar{B}_2 + \bar{C}_3\bar{B}_3\right) \\
f_0^* = \quad & \boldsymbol{r_u}\boldsymbol{r_{0v}} + \boldsymbol{r_v}\boldsymbol{r_{0u}} = \Lambda A_1\bar{V}_1 + \Lambda^2\left(\bar{V}_1\bar{B}_1 + \bar{V}_2\bar{B}_2 + \bar{V}_3\bar{B}_3\right) \\
& + \Lambda A_2\bar{U}_2 + \Lambda^2\left(\bar{C}_1\bar{U}_1 + \bar{C}_2\bar{U}_2 + \bar{C}_3\bar{U}_3\right) \\
g^* = \quad & \mathbf{r}_{\boldsymbol{v}}^2 = A_2^2 + 2\Lambda A_2\bar{C}_2 + \Lambda^2\left(\bar{C}_1^2 + \bar{C}_2^2 + \bar{C}_3^2\right) \\
g_0^* = \quad & 2\boldsymbol{r_v}\boldsymbol{r_{0v}} = 2\left[\Lambda A_2\bar{V}_2 + \Lambda^2\left(\bar{C}_1\bar{V}_1 + \bar{C}_2\bar{V}_2 + \bar{C}_3\bar{V}_3\right)\right].
\end{aligned}
\tag{43}
$$

Consequently, we can determine the components of the coordinate system $(\boldsymbol{E}_{\boldsymbol{1}}^{*}, \boldsymbol{E}_{\boldsymbol{2}}^{*}, \boldsymbol{R}^{*})$ of the dual surface $\boldsymbol{R}^{*}(u,v)$ as

$$
\begin{aligned}
\boldsymbol{E}_{\boldsymbol{1}}^{*} &= \frac{\boldsymbol{R}_{\boldsymbol{u}}^{*}}{\left\|\boldsymbol{R}_{\boldsymbol{u}}^{*}\right\|} = \frac{A_1\mathbf{e_1} + \Lambda\left(\bar{B}_1\mathbf{e_1} + \bar{B}_2\mathbf{e_2} + \bar{B}_3\mathbf{r}\right) + \varepsilon\Lambda\left(\bar{U}_1\mathbf{e_1} + \bar{U}_2\mathbf{e_2} + \bar{U}_3\mathbf{r}\right)}{\sqrt{E^*}} \\
\boldsymbol{E}_{\boldsymbol{2}}^{*} &= \frac{\boldsymbol{R}_{\boldsymbol{v}}^{*}}{\left\|\boldsymbol{R}_{\boldsymbol{v}}^{*}\right\|} = \frac{A_2\mathbf{e_2} + \Lambda\left(\bar{C}_1\mathbf{e_1} + \bar{C}_2\mathbf{e_2} + \bar{C}_3\mathbf{r}\right) + \varepsilon\Lambda\left(\bar{V}_1\mathbf{e_1} + \bar{V}_2\mathbf{e_2} + \bar{V}_3\mathbf{r}\right)}{\sqrt{G*}} \\
\boldsymbol{R}^{*} &= \frac{\boldsymbol{R} + D_1\mathbf{E}_1 + D_2\mathbf{E}_2 + D_3\mathbf{R}}{\left\|\boldsymbol{R} + D_1\mathbf{E}_1 + D_2\mathbf{E}_2 + D_3\mathbf{R}\right\|}.
\end{aligned}
\tag{44}
$$

Now, let's examine the first and the second fundamental form quantities of the generalized dual parallel surface $\boldsymbol{R}^{*}(u,v)$. First, we can write the below equations using the equations in [1] as

$$
1 = -\frac{1}{4w^4}\begin{vmatrix} e & e_u & e_v \\ f & f_u & f_v \\ g & g_u & g_v \end{vmatrix} - \frac{1}{2w}\left\{\frac{\partial}{\partial v}\frac{e_{v-}f_u}{w} - \frac{\partial}{\partial u}\frac{f_v - g_u}{w}\right\}, \quad w^2 = eg - f^2 \tag{45}
$$

and

$$0=\frac{2h}{w^4}\begin{vmatrix} e & e_u & e_v\\ f & f_u & f_v\\ g & g_u & g_v\end{vmatrix}-\frac{1}{4w^4}\left\{\begin{vmatrix} e_0 & e_u & e_v\\ f_0 & f_u & f_v\\ g_0 & g_u & g_v\end{vmatrix}+\begin{vmatrix} e & e_{0u} & e_v\\ f & f_{0u} & f_v\\ g & g_{0u} & g_v\end{vmatrix}+\begin{vmatrix} e & e_u & e_{0v}\\ f & f_u & f_{0v}\\ g & g_u & g_{0v}\end{vmatrix}\right\}$$

$$+\frac{h}{w}\left\{\frac{\partial}{\partial v}\frac{e_v-f_u}{w}-\frac{\partial}{\partial u}\frac{f_v-g_u}{w}\right\}-\frac{1}{2w}\left\{\frac{\partial}{\partial v}\frac{e_{0v}-f_{0u}}{w}-\frac{\partial}{\partial u}\frac{f_{0v}-g_{0u}}{w}\right\}$$
$$+\frac{1}{w}\left\{\frac{\partial}{\partial v}h\frac{e_v-f_u}{w}-\frac{\partial}{\partial u}h\frac{f_v-g_u}{w}\right\}. \tag{46}$$

Using equation (37) and

$$\begin{aligned} E &= -L = e+\varepsilon e_0\\ F &= -M = f+\varepsilon f_0\\ G &= -N = g+\varepsilon g_0, \end{aligned}$$

it is shown that the first and the second fundamental forms and the equations (45) and (46) specify the generalized dual parallel surface $\boldsymbol{R^*}(u,v)$, where L, M and N are components of the second fundamental form.

On the other hand, the Gauss curvature (K) and the mean curvature (H) of the generalized dual parallel surface $\boldsymbol{R^*}(u,v)$ can be expressed with

$$\begin{aligned} K^* &= \tfrac{1}{4}\frac{e_0^*g_0^*-f_0^{*2}}{e^*g^*-f^{*2}}\\ &= \frac{1}{4}\left\{\frac{4\left[\Lambda A_1\bar{U}_1+\Lambda^2K\right]\left[\Lambda A_2\bar{V}_2+\Lambda^2L\right]-\left[\Lambda\left(A_1\bar{V}_1+A_2\bar{U}_2\right)+\Lambda^2(M+N)\right]^2}{\left[A_1^2+2\Lambda A_1\bar{B}_1+\Lambda^2\bar{B}_1^2\right]\left[A_2^2+2\Lambda A_2\bar{C}_2+\Lambda^2\bar{C}_2^2\right]-\left[\Lambda\left(A_1\bar{C}_1+A_2\bar{B}_2\right)+\Lambda^2\left(\bar{C}_1\bar{B}_1+\bar{C}_2\bar{B}_2+\bar{C}_3\bar{B}_3\right)\right]}\right\} \end{aligned} \tag{47}$$

and

$$\begin{aligned} H^* &= \tfrac{1}{4}\frac{e^*g_0^*-2f^*f_0^*+g^*e_0^*}{e^*g^*-f^{*2}}\\ &= \frac{1}{4}\frac{2\left[A_1^2+2\Lambda A_1\bar{B}_1+\Lambda^2\bar{B}_1^2\right]\left[\Lambda A_2\bar{V}_2+\Lambda^2L\right]-2\left[\lambda\left(A_1\bar{C}_1+A_2\bar{B}_2\right)+\Lambda^2\left(\bar{C}_1\bar{B}_1+\bar{C}_2\bar{B}_2+\bar{C}_3\bar{B}_3\right)\right]}{\left[A_1^2+2\Lambda A_1\bar{B}_1+\Lambda^2\bar{B}_1^2\right]\left[A_2^2+2\Lambda A_2\bar{C}_2+\Lambda^2\bar{C}_2^2\right]-\left[\Lambda\left(A_1\bar{C}_1+A_2\bar{B}_2\right)+\Lambda^2\left(\bar{C}_1\bar{B}_1+\bar{C}_2\bar{B}_2+\bar{C}_3\bar{B}_3\right)\right]^2}\\ &\quad-\frac{\left[\Lambda A_1\bar{V}_1+\Lambda^2M+\Lambda A_2\bar{U}_2+\Lambda^2N\right]+2\left[A_2^2+2\Lambda A_2\bar{C}_2+\Lambda^2\bar{C}_2^2\right]\left[\Lambda A_1\bar{U}_1+\Lambda^2K\right]}{\left[A_1^2+2\Lambda A_1\bar{B}_1+\Lambda^2\bar{B}_1^2\right]\left[A_2^2+2\Lambda A_2\bar{C}_2+\Lambda^2\bar{C}_2^2\right]-\left[\Lambda\left(A_1\bar{C}_1+A_2\bar{B}_2\right)+\Lambda^2\left(\bar{C}_1\bar{B}_1+\bar{C}_2\bar{B}_2+\bar{C}_3\bar{B}_3\right)\right]^2}. \end{aligned} \tag{48}$$

Theorem 4.1. *The generalized dual parallel surface* $\boldsymbol{R^*}(u,v)=\boldsymbol{R}(u,v)+D_1(u,v)\mathbf{E}_1+D_2(u,v)\mathbf{E}_2+D_3(u,v)\mathbf{R}$ *can be defined as a standard dual parallel surface with*

$$\mathbf{R}^*=\mathbf{R_1}+\mathrm{D}\mathbf{R_1}=(1+\mathrm{D})\mathbf{R_1}, \tag{49}$$

where $\boldsymbol{R_1}(u,v)$ *is a new dual surface,* D *is a constant, and* $D=\Lambda\bar{D}$ *.*

Proof of Theorem 4.1. The generalized dual parallel surface can be written with the following equation.

$$\begin{aligned} \boldsymbol{R^*} &= R+D_1\boldsymbol{E_1}+D_2\boldsymbol{E_2}+D_3\boldsymbol{R}\\ &= (\boldsymbol{R}+Z_1\boldsymbol{E_1}+Z_2\boldsymbol{E_2}+Z_3\boldsymbol{R})+(D_{1-}Z_1)\boldsymbol{E_1}+(D_2-Z_2)\boldsymbol{E_2}+(D_3-Z_3)\boldsymbol{R}\\ &\triangleq\ \boldsymbol{R_1}+\Lambda\bar{D}\boldsymbol{R_1}, \end{aligned} \tag{50}$$

where $Z_1 = z_1 + \varepsilon z_{10}$, $Z_2 = z_2 + \varepsilon z_{20}$ and $Z_3 = z_3 + \varepsilon z_{30}$ are the functions of u and v, and

$$\begin{aligned} D_1 = \Lambda\bar{D}_1, D_2 = \Lambda\bar{D}_2, \qquad & D_3 = \Lambda\bar{D}_3 \\ Z_1 = \Lambda\bar{Z}_1, Z_2 = \Lambda\bar{Z}_2, \qquad & Z_3 = \Lambda\bar{Z}_3. \end{aligned}$$

Moreover,

$$D = \Lambda\sqrt{\left(\overline{D}_1 - \bar{Z}_1\right)^2 + \left(\overline{D}_2 - \bar{Z}_2\right)^2 + \left(\overline{D}_3 - \bar{Z}_3\right)^2} = \Lambda\overline{D}$$
$$\boldsymbol{R_1} = \Lambda\left[\frac{\left(\overline{D}_1 - \bar{Z}_1\right)}{D}\boldsymbol{E_1} + \frac{\left(\overline{D}_2 - \bar{Z}_2\right)}{D}\boldsymbol{E_2} + \frac{\left(\overline{D}_3 - \bar{Z}_3\right)}{D}\boldsymbol{R}\right]. \tag{51}$$

Three functions, $\bar{Z}_1$, $\bar{Z}_2$, and $\bar{Z}_3$, determine the generalized dual parallel surface $\boldsymbol{R^*}(u,v) = \boldsymbol{R}(u,v) + D_1(u,v)\mathbf{E}_1 + D_2(u,v)\mathbf{E}_2 + D_3(u,v)\mathbf{R}$, and, using the equality below,

$$\boldsymbol{R} + \Lambda\left(\bar{Z}_1\boldsymbol{E_1} + \bar{Z}_2\boldsymbol{E_2} + \bar{Z}_3\boldsymbol{R}\right) = \Lambda\left[\frac{\left(\overline{D}_1 - \bar{Z}_1\right)}{D}\boldsymbol{E_1} + \frac{\left(\overline{D}_2 - \bar{Z}_2\right)}{D}\boldsymbol{E_2} + \frac{\left(\overline{D}_3 - \bar{Z}_3\right)}{D}\boldsymbol{R}\right]$$

we can find $\bar{Z}_1$, $\bar{Z}_2$, and $\bar{Z}_3$ with the following equations:

$$\begin{aligned} \bar{Z}_1 &= \frac{\overline{D}_1}{1+D} \\ \bar{Z}_2 &= \frac{\overline{D}_2}{1+D} \\ \bar{Z}_3 &= \frac{\Lambda\overline{D}_3 - D}{\Lambda(1+D)} \end{aligned} \tag{52}$$

□

Remark 4.2. If the dual part of the standard dual parallel surface in (49) equals zero, then we obtain the special forms of the surface $\boldsymbol{r_1}(u,v)$ and the standard parallel surface $\boldsymbol{r^*}$ in [2] as sphere surfaces. Therefore, in this paper, we present the standard dual parallel surfaces with (49) as the generalized form of the special form of the standard real parallel surfaces in [2].

5. The first variation of area changing on standard dual parallel surfaces

Let us determine the area change after obtaining the standard dual parallel surface. Let $\boldsymbol{R_1}(u,v)$ be the original dual surface. We can express the standard dual parallel surface of $\boldsymbol{R_1}(u,v)$ using equation (49) with

$$\mathbf{R}^* = \mathbf{R_1} + \mathrm{D}\mathbf{R_1} = (1+\mathrm{D})\mathbf{R_1}. \tag{53}$$

We can find the terms of the first fundamental form of the standard dual parallel surface $\boldsymbol{R^*}(u,v)$ by discarding the second degree terms compared to Λ as

$$E^* = \boldsymbol{R^*_u}^2, \quad F^* = \boldsymbol{R^*_u}\boldsymbol{R^*_v} \textit{ and } \quad G^* = \boldsymbol{R^*_v}^2 \ .$$

Then, we can obtain the following equations.

$$\begin{array}{llll} E^* & = E+2DE & = (1+2D)E \\ F^* & = F+2DF & = (1+2D)F \\ G^* & = G+2DG & = (1+2D)G. \end{array} \tag{54}$$

On the other hand, using the equation below:

$$W^{*^2} = E^*G^* - F^{*2},$$

we find the relationship

$$W^{*^2} = (1+2D)^2EG - (1+2nD)F^2 \tag{55}$$

$$= (1+2D)^2(EG-F^2) \tag{56}$$

and then we arrive at

$$W^* = W(1+2D). \tag{57}$$

We can determine the area change of the standard dual parallel surface $\boldsymbol{R}^*\,(u,v)$ with the following equation.

$$\iint W^*dudv = \iint Wdudv + 2\iint DWdudv. \tag{58}$$

Since, $Wdudv = do$, by using δD instead of D we can express the first variation of area change as

$$\delta O = 2\iint \delta D do. \tag{59}$$

6. The first variation of area changing of the generalized dual parallel surfaces by utilizing the gauss formula

Let us suppose that the lines of curvature in $\mathbf{R}(u,v)$ are the parameter curves. Then, we can write the equations below.

$$\begin{cases} \boldsymbol{R_{uu}} = & \frac{E_u}{2E}\boldsymbol{R_u} - \frac{E_v}{2G}\boldsymbol{R_v} - E\boldsymbol{R} \\ \boldsymbol{R_{uv}} = & \frac{E_v}{2E}\boldsymbol{R_u} - \frac{G_u}{2G}\boldsymbol{R_v} \\ \boldsymbol{R_{vv}} = & -\frac{G_u}{2E}\boldsymbol{R_u} + \frac{G_v}{2G}\boldsymbol{R_v} - G\boldsymbol{R}. \end{cases}$$

Using equations (34), (35), and (36) we can find the following equations:

$$\begin{cases} \boldsymbol{R_u^*} = & \left(1 + D_{1u} + D_3 + \frac{D_1E_u + D_2E_v}{2E}\right)\boldsymbol{R_u} + (*)\mathbf{R}_v + (*)\mathbf{R} \\ \boldsymbol{R_v^*} = & (*)\boldsymbol{R_u} + \left(1 + D_{2v} + D_3 + \frac{D_1G_u + D_2G_v}{2G}\right)\boldsymbol{R_v} + (*)\boldsymbol{R}, \end{cases}$$

where $(*)$ indicates the first degree terms according to Λ. If we only take the linear terms according to Λ, we get

$$\begin{aligned} \boldsymbol{R_u^*} \times \boldsymbol{R_v^*} = & \Bigg(1 + D_{2v} + D_3 + \frac{D_1G_u + D_2G_v}{2G} \\ & + D_{1u} + D_3 + \frac{D_1E_u + D_2E_v}{2E}\Bigg)(\boldsymbol{R_u} \times \boldsymbol{R_v}) \\ & + (*)(\boldsymbol{R_v} \times \boldsymbol{R}) + (*)(\boldsymbol{R} \times \boldsymbol{R_u}). \end{aligned} \tag{60}$$

On the other hand, since we know that the lines of curvature are the parameter curves the vectors $\boldsymbol{R_u}$, $\boldsymbol{R_v}$ and $\boldsymbol{R}$ and also the vectors $\boldsymbol{R_u}\times\boldsymbol{R_v}$, $\boldsymbol{R_v}\times\boldsymbol{R}$ and $\boldsymbol{R}\times\boldsymbol{R_u}$

are perpendicular to each other in twos. Next, we can find the equation below by discarding the second-degree terms according to Λ:

$$\begin{aligned}(\boldsymbol{R}_{\boldsymbol{u}}^{*} \times \boldsymbol{R}_{\boldsymbol{v}}^{*})^2 &= \phi^2(\boldsymbol{R}_{\boldsymbol{u}} \times \boldsymbol{R}_{\boldsymbol{v}})^2 \\ &= \phi^2({\boldsymbol{R}_{\boldsymbol{u}}}^2 {\boldsymbol{R}_{\boldsymbol{v}}}^2) \\ &= \phi^2\left[(\boldsymbol{r}_{\boldsymbol{u}} + \varepsilon \boldsymbol{r}_{\boldsymbol{0u}})^2(\boldsymbol{r}_{\boldsymbol{v}} + \varepsilon \boldsymbol{r}_{\boldsymbol{0v}})^2\right] \\ &= \phi^2\left[eg + \varepsilon(eg_0 + e_0 g)\right] \\ &= \phi^2 EG,\end{aligned}$$

where the amount ϕ indicates the coefficient of the $\boldsymbol{R}_{\boldsymbol{u}} \times \boldsymbol{R}_{\boldsymbol{v}}$ in (60). Thus, we can determine the area of the generalized dual parallel surface $\boldsymbol{R}^{*}(u,v)$ with

$$\begin{aligned}\overline{O} &= \int \sqrt{(\boldsymbol{R}_{\boldsymbol{u}}^{*} \times \boldsymbol{R}_{\boldsymbol{v}}^{*})}\,dudv \\ &= \int \phi\sqrt{EG}\,dudv \\ &= O + \delta O,\end{aligned} \tag{61}$$

where

$$\delta O = \iint \left\{\frac{\partial}{\partial u}(D_1\sqrt{EG}) + \frac{\partial}{\partial v}(D_2\sqrt{EG})\right\} dudv + 2\iint D_3\sqrt{EG}\,dudv. \tag{62}$$

When we take into consideration the equation $(\boldsymbol{R}_{\boldsymbol{u}}, \boldsymbol{R}_{\boldsymbol{v}}, \boldsymbol{R}) = \sqrt{EG}$, and using equations (34) and (35), the following equations can be obtained.

$$\begin{cases} D_1\sqrt{EG} = (\delta\boldsymbol{R}, \boldsymbol{R}_{\boldsymbol{v}}, \boldsymbol{R}) \\ D_2\sqrt{EG} = (\boldsymbol{R}_{\boldsymbol{u}}, \delta\boldsymbol{R}, \boldsymbol{R}). \end{cases} \tag{63}$$

If the equations in (63) are substituted into (62), and the equation $d\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{u}} d_u + \boldsymbol{R}_{\boldsymbol{v}} d_v$ is taken into consideration, we have the following equation:

$$\oint (\delta\boldsymbol{R}, d\boldsymbol{R}, \boldsymbol{R}).$$

Hence, we can determine the first variation of the area between the generalized dual parallel surface and progenitor dual surface with the following equation.

$$\delta O = \oint (\delta\boldsymbol{R}, d\boldsymbol{R}, \boldsymbol{R}) + 2\int D_3 do$$

or

$$\delta O = \oint (\delta\boldsymbol{R}, d\boldsymbol{R}, \boldsymbol{R}) + 2\iint (\delta\boldsymbol{R}, \boldsymbol{R}_{\boldsymbol{u}}, \boldsymbol{R}_{\boldsymbol{v}})\,dudv. \tag{64}$$

7. The second variation of the area change between the standard dual parallel and the original dual surfaces

Let us take the standard dual parallel surface $\mathbf{R}^* = \mathbf{R_1} + \mathrm{D}\mathbf{R_1} = (1+\mathrm{D})\mathbf{R_1}$ in (49) the second-degree terms according to Λ into consideration. Using the equations below:

$$\begin{aligned}\boldsymbol{R}_{\boldsymbol{u}}^{*} &= \boldsymbol{R}_{\boldsymbol{1u}} + D\boldsymbol{R}_{\boldsymbol{1u}} + D_u\boldsymbol{R}_{\boldsymbol{1}} \\ \boldsymbol{R}_{\boldsymbol{v}}^{*} &= \boldsymbol{R}_{\boldsymbol{1v}} + D\boldsymbol{R}_{\boldsymbol{1v}} + D_v\boldsymbol{R}_{\boldsymbol{1}}\end{aligned}$$

we can write following equations.

$$\begin{aligned} E^* = &\quad \boldsymbol{R_u^*}^2 = (1+2D)E + D^2\boldsymbol{R_{1u}}^2 + D_u^2 \\ F^* = &\quad \boldsymbol{R_u^*}.\boldsymbol{R_v^*} = (1+2n)F + D^2\boldsymbol{R_{1u}}\boldsymbol{R_{1v}} + D_uD_v \\ G^* = &\quad \boldsymbol{R_v^*}^2 = (1+2n)G + D^2\boldsymbol{R_{1v}}^2 + D_v^2 \end{aligned}$$

or

$$\begin{aligned} E^* = &\quad \boldsymbol{R_u^*}^2 = E + 2DE - D^2(2H+K)E + D_u^2 \\ F^* = &\quad \boldsymbol{R_u^*}.\boldsymbol{R_v^*} = F + 2DF - D^2(2H+K)F + D_uD_v \\ G^* = &\quad \boldsymbol{R_v^*}^2 = G + 2DG - D^2(2H+K)G + D_v^2. \end{aligned}$$

Let us determine $W^{*2} = E^*G^* - F^{*2}$ using

$$\nabla D = \frac{ED_v^2 - 2FD_uD_v + GD_u^2}{W^2}.$$

Then we have

$$\begin{aligned} W^{*2} &\quad = E^*G^* - F^{*2} \\ &\quad = (EG - F^2)\left[\left[(1+2D) - D^2(2H+K)\right]^2\right. \\ &\quad \left.+\nabla D\left[(1+2D) - D^2(2H+K)\right]\right]. \end{aligned} \tag{65}$$

8. Examples

8.1. Example 1. Let the progenitor dual spherical curve $\mathbf{R}(t)$ be given as

$$\mathbf{R}(t) = \left(\sin(2t), \sin(t)\cos(2t), \cos(t)^2\right).$$

We can show the progenitor real spherical curve on the inner sphere (real sphere) and dual spherical curve, which it will be able to correspond to the ruled surface, on the outer sphere (dual sphere), in Figure (1). Similarly, the relationship between the progenitor real spherical curve and its generalized parallel spherical curves on the inner sphere (real sphere), and the progenitor dual spherical curve and its generalized dual parallel spherical curves, which they will be able to correspond to the ruled surfaces, on the outer sphere (dual sphere) can be seen in Figure (2).

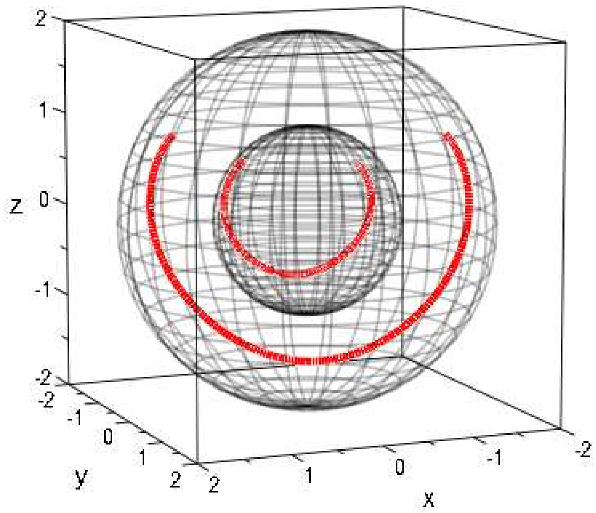


Figure 1. The progenitor real and dual spherical curves.

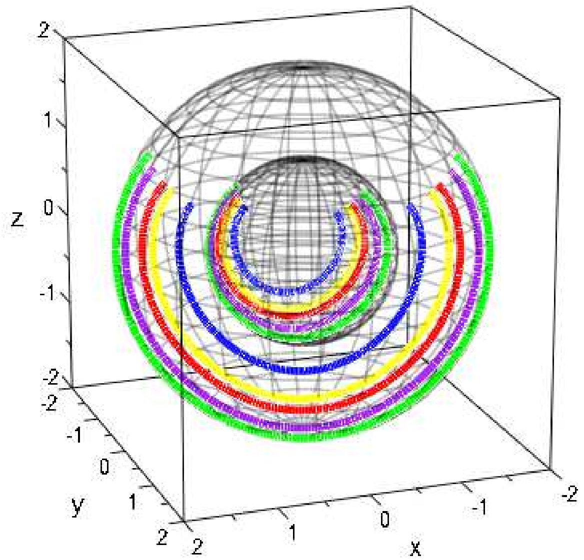


Figure 2. The progenitor real and dual spherical curves (red), and their generalized real and dual parallel spherical curves for different values.

8.2. Example 2. Next, let the progenitor dual spherical curve $\mathbf{R}(t)$ be given as

$$\mathbf{R}(t) = (\cos(t), \sin(t), \cos(t)\sin(t))\,.$$

We can see the progenitor real spherical curve on the inner sphere (real sphere) and dual spherical curve, which it will be able to correspond to the ruled surface, on the outer sphere (dual sphere), in Figure (3). Additionally, the relationship between the progenitor real spherical curve and its generalized parallel spherical curves on the inner sphere (real sphere), and the progenitor dual spherical curve and its generalized dual parallel spherical curves , which they will be able to correspond to the ruled surfaces, on the outer sphere (dual sphere), can be shown in Figure (4).

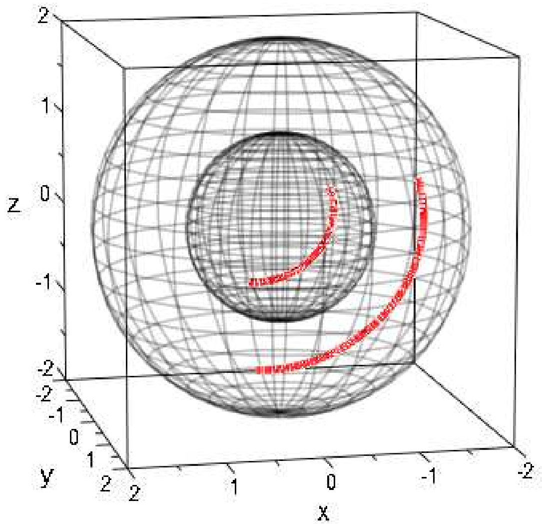


Figure 3. The progenitor real and dual spherical curves.

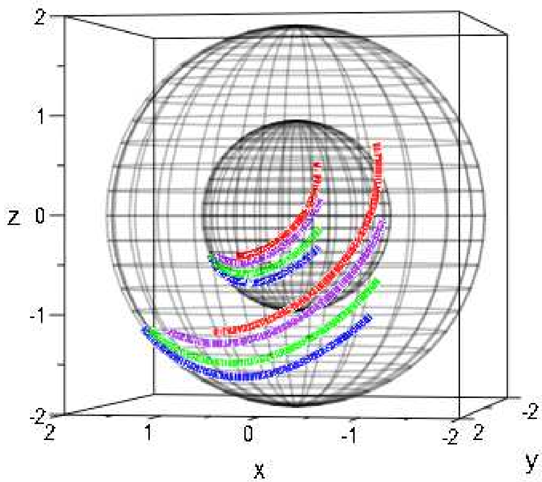
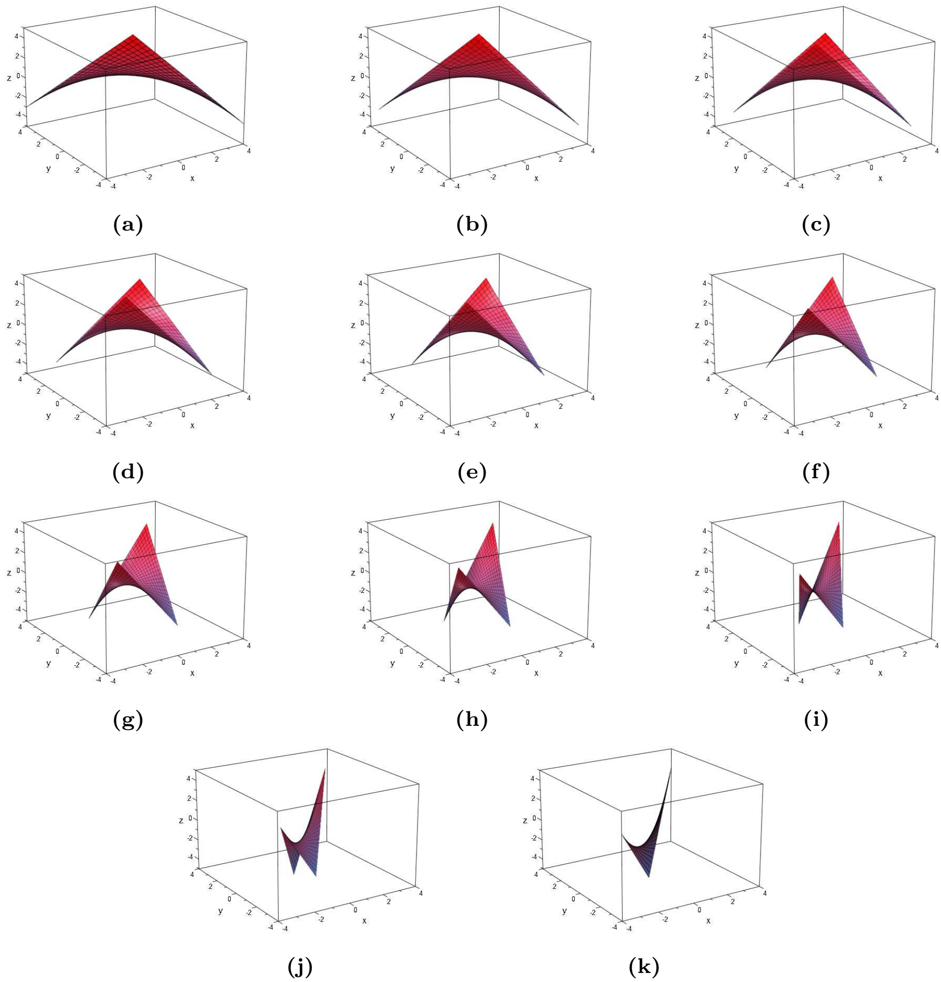


Figure 4. The progenitor real and dual spherical curves (red), and their generalized real and dual parallel spherical curves for different values.

8.3. Example 3. Let the progenitor dual surface $\boldsymbol{R}(u,v)$ be given as

$$\boldsymbol{R}(u,v) = \frac{1}{\sqrt{u^2+v^2+1}}\left[(u,\, v,\, -1) + \varepsilon\left(-u-uv^2,\, v+vu^2,\, v^2-u^2\right)\right].$$

We can show the progenitor dual surface $((a)-(k))$ for different parameter values in real space in Figure(5), the real and dual spherical parameter curves that correspond to the dual parameter curves of the progenitor dual surface in Figure(6), and the generalized dual parallel surfaces for different values of Λ in real space in Figure (7).



(a) **(b)** **(c)** **(d)** **(e)** **(f)** **(g)** **(h)** **(i)** **(j)** **(k)**

Figure 5. The progenitor dual surface $((a)-(k))$ for different parameter values in real space.

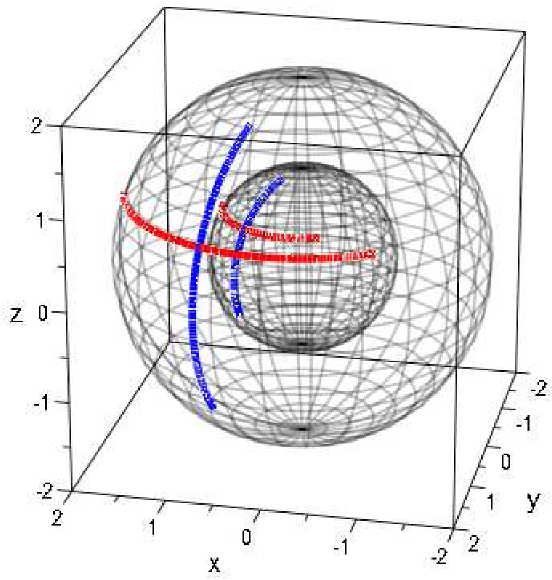


Figure 6. The real and dual spherical parameter curves that correspond to the dual parameter curves of the progenitor dual surface.

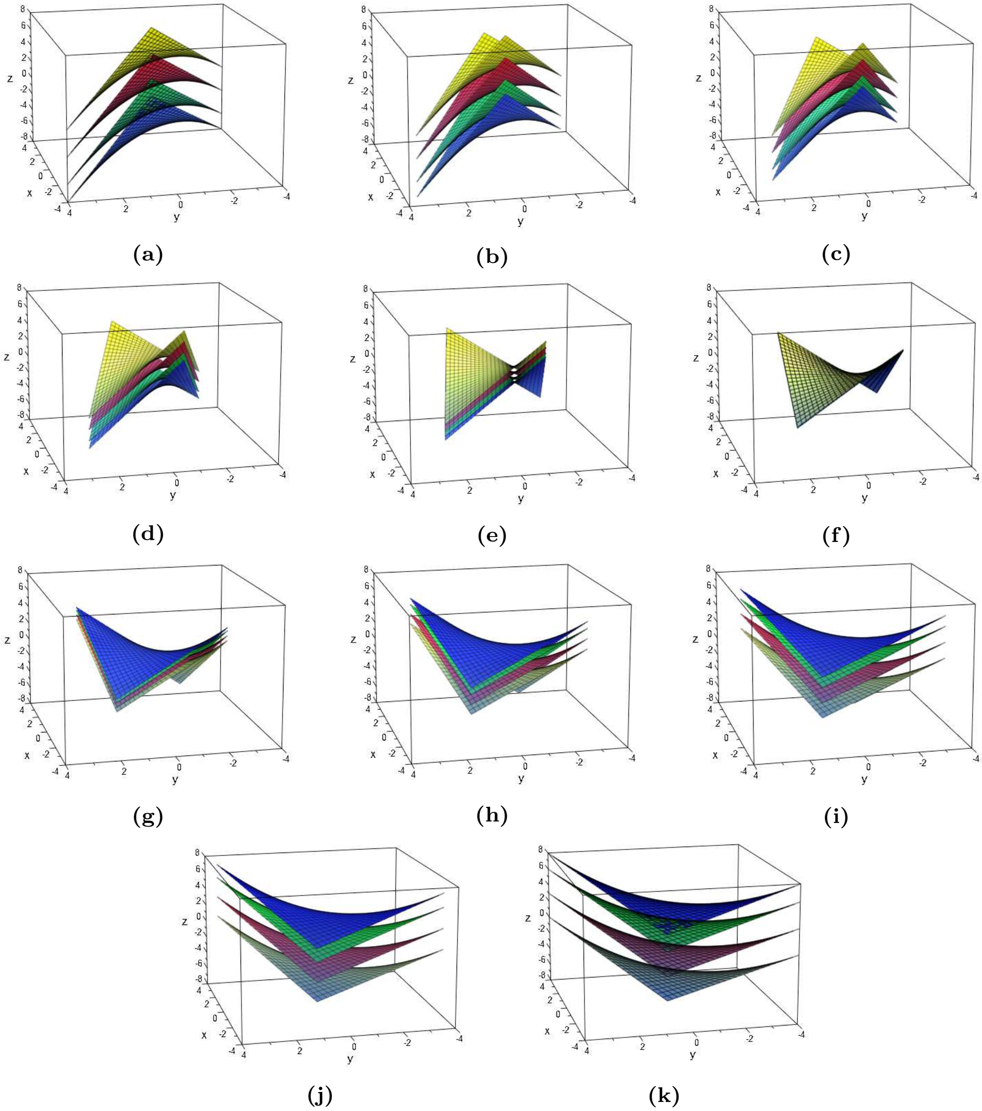


Figure 7. The progenitor dual surface (red) in Figure (5), and its generalized dual parallel surfaces for different values of Λ in real space.

8.4. Example 4. Now, let the progenitor dual surface $\boldsymbol{R}(u,v)$ be given as

$$\begin{aligned}\boldsymbol{R}(u,v) &= (\sin(u)\sin(v),\, \cos(u)\sin(v),\, \cos(v)) \\ &\quad + \varepsilon\left(\sin(u)\cos^2(v),\, -\cos(u)\cos^2(v),\, \cos(2u)\cos(v)\sin(v)\right).\end{aligned}$$

We can see the progenitor dual surface $((a)-(k))$ for different parameter values in real space in Figure(8), the real and dual spherical parameter curves that correspond to the dual parameter curves of the progenitor dual surface in Figure(9), and the different generalized dual parallel surfaces for different values of Λ in real space in Figure (10).

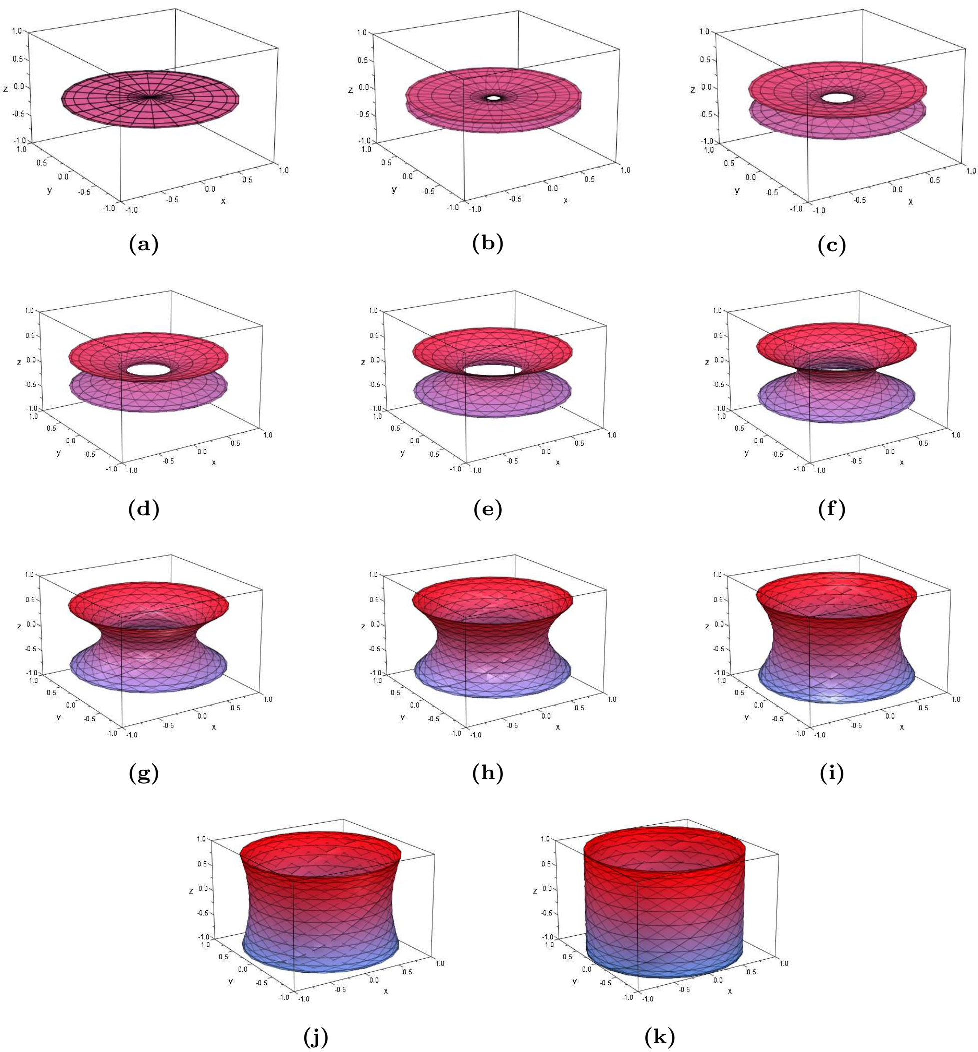


Figure 8. The progenitor dual surface $((a)-(k))$ for different parameter values in real space.

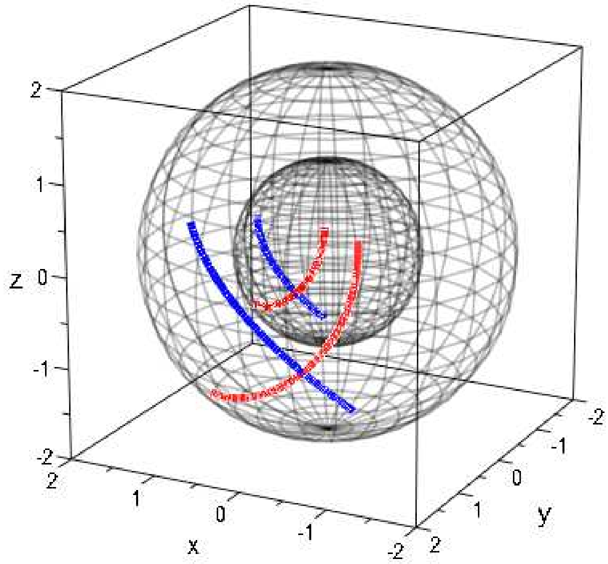


Figure 9. The real and dual spherical parameter curves that correspond to the dual parameter curves of the progenitor dual surface.

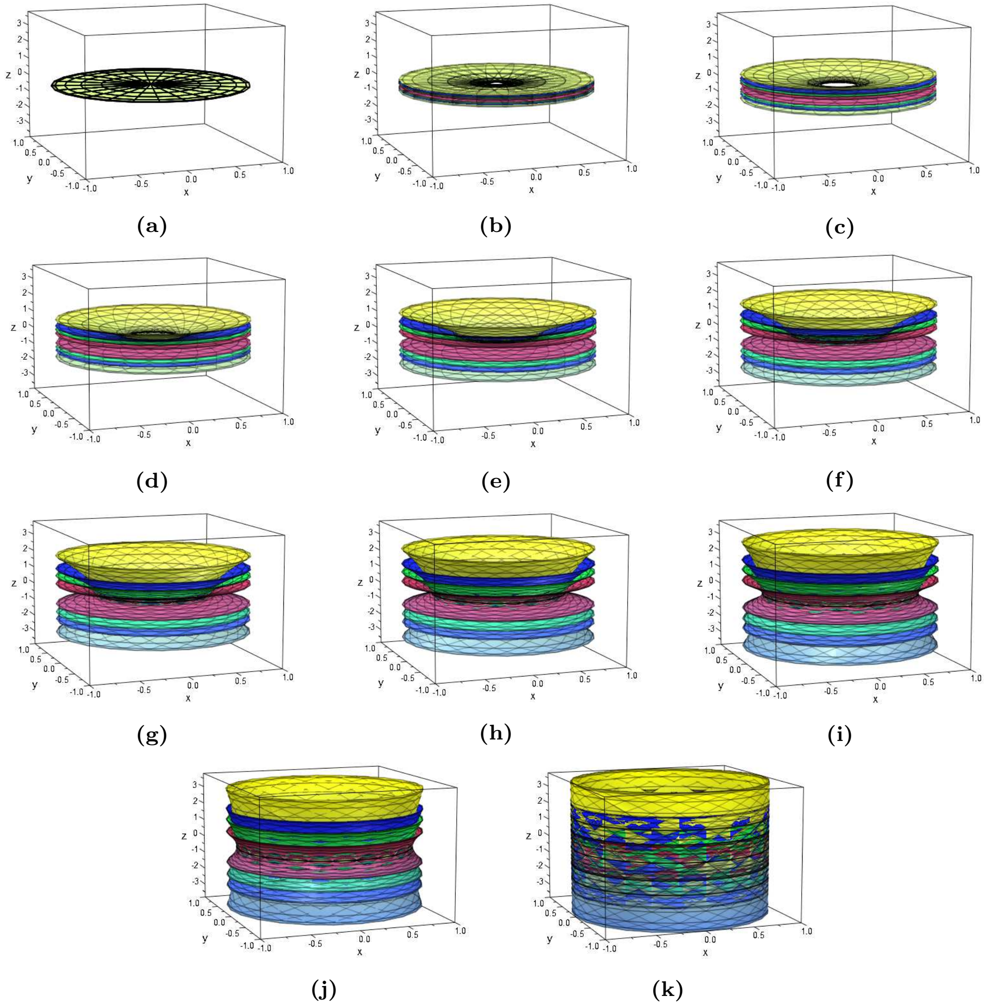


Figure 10. The progenitor dual surface (red) in Figure (8), and its generalized dual parallel surfaces for different values of Λ in real space.

8.5. Example 5: In this example, we apply parallel curves (or offset curves) to path planning for a wheeled mobile robot. For this purpose, assume that, there are four geometric shapes considered as obstacles in the environment scenario in Figure 11. These obstacles can be objects, machines, storage areas or components in a factory in real life.

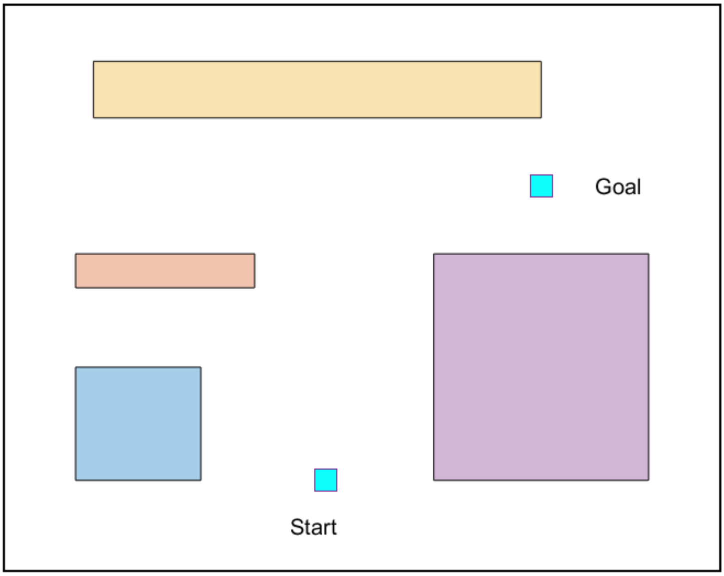


Figure 11. Environment scenario construction

Since Bézier curves have very important properties such that they always pass through starting and end points, and also lie within the convex hull, these curves are widely used for path planning. Therefore, we can take the cubic trigonometric Bézier curve for the predefined path into consideration. First, the definition of cubic trigonometric Bézier curve is given below.

A cubic trigonometric Bezier curve with two shape parameters can be written using four control points as

$$\boldsymbol{r}\left(t\right)=\sum_{i=0}^{3}\boldsymbol{P}_i b_{i,3}\left(t\right), \quad t\in\left[0,1\right], \quad \lambda_1,\lambda_2\in\left[-2,1\right], \tag{66}$$

where

$$\begin{cases} b_{0,3}\left(t\right) & =\left(1-\sin\left(\frac{\pi}{2}t\right)\right)^2\left(1-\zeta_1\sin\left(\frac{\pi}{2}t\right)\right)\\ b_{1,3}\left(t\right) & =\sin\left(\frac{\pi}{2}t\right)\left(1-\sin\left(\frac{\pi}{2}t\right)\right)\left(2+\zeta_1-\zeta_1\sin\left(\frac{\pi}{2}t\right)\right)\\ b_{2,3}\left(t\right) & =\cos\left(\frac{\pi}{2}t\right)\left(1-\cos\left(\frac{\pi}{2}t\right)\right)\left(2+\zeta_2-\zeta_2\cos\left(\frac{\pi}{2}t\right)\right)\\ b_{3,3}\left(t\right) & =\left(1-\cos\left(\frac{\pi}{2}t\right)\right)^2\left(1-\zeta_2\cos\left(\frac{\pi}{2}t\right)\right). \end{cases} \tag{67}$$

The properties of the cubic trigonometric Bezier curve in (66) and its basis functions (67) can be obtained in [6].

Assume that the start and goal points in Figure 11 correspond to the control points $\boldsymbol{P}_0$ and $\boldsymbol{P}_3$, respectively. Moreover, geometric shapes are considered as obstacles. The other two control points can be obtained using these obstacles. First, parallels of the obstacles are obtained as shown in Figure 12.

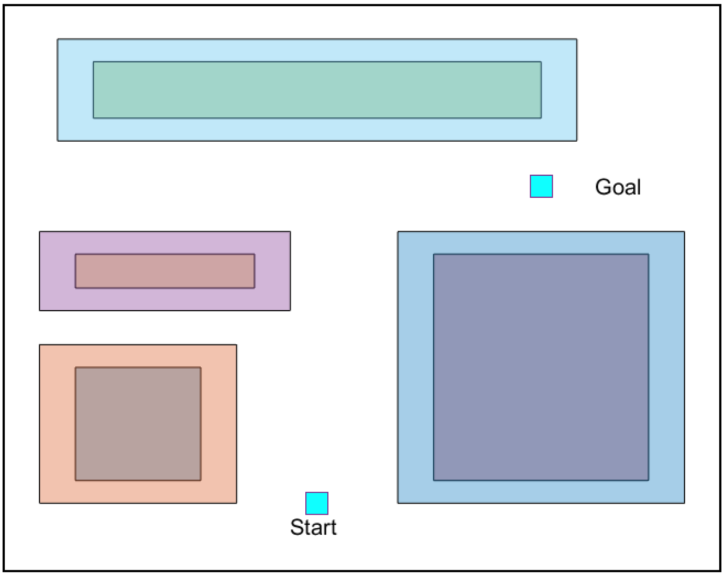


Figure 12. The parallels of the obstacles.

Next, we can determine the predefined skeleton path using the parallels of the obstacles that is given in Figure 13.

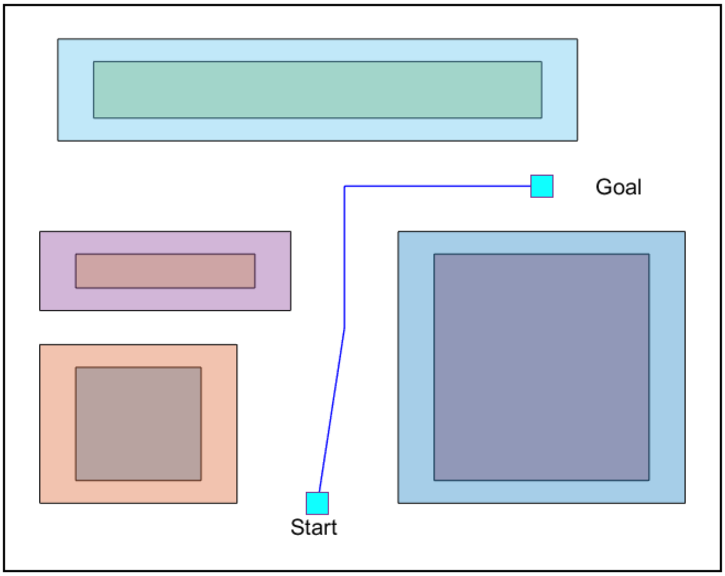


Figure 13. The predefined skeleton path

Our purpose is to obtain a predefined path from a given start point to a goal point by staying close to the predefined skeleton path and without passing through the obstacles. In this paper, the predefined skeleton path is taken into account as convex hull of the predefined cubic trigonometric Bézier path $\boldsymbol{r}(t)$. The predefined cubic trigonometric Bézier path can be altered using shape parameters λ_1 and λ_2, even though there is only one predefined skeleton curve. It is noted that, the offset distance can be taken according to wheeled mobile robot's size. The predefined cubic trigonometric Bézier paths according to several shape parameters are presented in Figure 14.

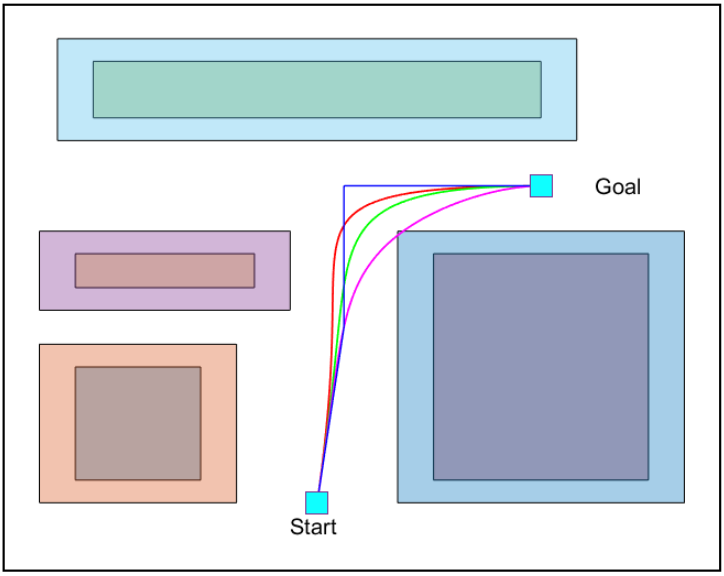


Figure 14. The predefined cubic trigonometric Bézier paths: the red one for $\lambda_1 = -1$ and $\lambda_2 = -2$; the green one for $\lambda_1 = 0$ and $\lambda_2 = -1$ and the magenta one for $\lambda_1 = 1$ and $\lambda_2 = 1$.

The utility of the offsets of the obstacles can be realized in Figure 14. The predefined cubic trigonometric Bézier paths are not able to pass through the obstacles, since the corner points of the predefined skeleton path are determined using the offsets of the obstacles, and all the predefined cubic trigonometric Bezier paths lie within the convex hull (the predefined skeleton path).

9. Conclusion

In this paper, we proposed a novel definition of generalized and standard dual parallel curves and surfaces. We also expressed some properties of generalized dual parallel curves and surfaces using this definition. Moreover, the variation of the generalized dual parallel curves, the first and second variation of area change of the standard dual parallel, and the first variation of area change of the generalized dual parallel surfaces are given. Additionally, when remarks are considered, we presented the generalized form of the generalized and standard real parallel curves and surfaces. We believe that our study can be useful for designing parallel curves and surfaces that are used widely in many industrial applications.

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