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Statistical Voronoi mean and applications to approximation theorems

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Abstract. In this paper, we give statistical Voronoi mean which is a new statistical summability method, is not need to be regular and positive. We prove a Korovkin type approximation theorem via this method that covers many important summability methods scattered in the literature. Also, we demonstrate that our theorem is stronger than proved by earlier authors with an interesting application. Finally, we establish the rate of convergence.

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1. Introduction and preliminaries

The notion of statistical convergence was first introduced by Fast [5] and Steinhaus [14], independently. Now, we begin with this definition:

Let $S \subseteq \mathbb{N}$, the set of natural numbers, and $S_n := \{k \leq n \ : k \in S\}$. Then *the natural density of* S, denoted by $\delta(S)$, is given by $\delta(S) := \lim_n \frac{1}{n} |S_n|$ if the limit exists, where $|S_n|$ denotes the cardinality of the set S_n ([12]).

A sequence $\{s_n\}$ of numbers is statistically convergent to s provided that, for every $\varepsilon > 0$, the set $S_\varepsilon := \{k \leq n : |s_k - s| \geq \varepsilon\}$ has natural density zero, i.e. for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |s_k - s| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim_n s_n = s$ or $s_n \overset{st}{\to} s$ ([5], [14]). Note that if the sequence is convergent then the sequence is statistically convergent to the same number, but a statistically convergent sequence need not to be convergent. Also, it is important to say that, the sequence $\{s_n\}$ is statistically convergent to s iff there exists a subset S of $\mathbb{N}$ such that $\lim_k s_{n_k} = s$ where $\mathbb{N} \backslash S = \{n_k \ : k \in \mathbb{N}\}$ and $\delta(S) = 0$ ([6]).

Korovkin type approximation theory generally deals with convergence of sequences of positive linear operators ([9]). In some Korovkin type theorems, in the case of the lack of convergence, it is effective to use the summability methods. Recently, the idea of statistical $(C,1)$ −summability was introduced in [10], statistical $(\overline{N}, p)$ −summability in [11] and, more general than these methods, the statistical A−summability in [4]. Using the concept of statistical A−summability, Demirci and Karakuş ([3]) have provided a Korovkin-type approximation theorem. More recently,

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many authors have introduced new statistical summability methods and prove Korovkin type approximation theorems by these methods (see for example [1, 8, 13]). As it is known, the non-regular summability methods give interesting results within summability theory. The main motivation of this paper is to introduce new statistical summability method, is not need to be regular and positive and, includes many known summability methods and also, prove a Korovkin type theorem via this new interesting method. Hence we get Korovkin type approximation results that include the earlier ones can be obtained by our new method with proper chooses.

Voronoi mean, is a non-regular generalisation of the Nörlund mean, has been introduced by Bingham and Gashi in [2]. Now, we remind this method:

Definition 1.1. [2] Let the real sequences $\{p_n, q_n, u_n\}$ with $u_n \neq 0$ for $n \geq 0$, be given. The real sequence $\{s_n\}$ has Voronoi mean s, written $s_n \to s\left(\mathcal{V}, p_n, q_n, u_n\right)$, if

$$\frac{1}{u_n}\sum_{k=0}^{n} p_{n-k} q_k s_k \to s \ \ (n \to \infty) . \tag{1}$$

There are many special cases of the Voronoi mean:

Remark 1.1. (i) The Voronoi mean reduced to the generalised Nörlund mean (N, p_n, q_n) if

$$u_n := (p * q)_n := \sum_{k=0}^{n} p_{n-k} q_k .$$

Also, if $q_n = 1$ then we get the Nörlund mean (N, p_n) and for $k > 0$ and $p_n = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}$ then we get the Cesáro mean (C, k).

(ii) The Voronoi mean reduced to the Euler method E_p of order $p \in (0, 1)$ if

$$p_n = \frac{(1-p)^n}{n!}, \ q_n = \frac{p^n}{n!} \text{ and } u_n = (p * q)_n .$$

(iii) The Voronoi mean reduced to the weighted mean or the discontinuous Riesz mean $\left(\overline{N}, q_n\right)$ if

$$p_n = 1 \text{ and } u_n = (1 * q)_n .$$

Also, if $q_n = 1$ and $q_n = \frac{1}{n+1}$ then we get the Cesáro mean $(C, 1)$ and the logarithmic mean l, respectively.

(iv) The Voronoi mean reduced to the Jajte mean– the summability method for the law of large numbers (LLN) if

$$p_n = 1 \text{ and } \sum_{k=0}^{n} \frac{q_k}{u_n} \text{ not necessarily converging to 1 as } n \to \infty.$$

(v) The Voronoi mean reduced to the Chow–Lai mean– the summability method for the LLN if

$$q_n = 1, \ u_n \to \infty \text{ and } \sum_{n=0}^{\infty} p_n^2 < \infty$$

(see for details [2]).

As it is well known a summability method is regular if it sums a convergent sequence to its limit. The necessary and sufficient conditions for the $(\mathcal{V}, p_n, q_n, u_n)$ mean to be regular are:

(a) $\sum\limits_{k=0}^{n} |p_{n-k}q_k| < K |u_n|$, with K independent of n,

(b) $\frac{p_{n-k}q_k}{u_n} \to 0$ as $n \to \infty$ for each $k \geq 0$,

(c) $\sum\limits_{k=0}^{n} \frac{p_{n-k}q_k}{u_n} \to 1$ as $n \to \infty$.

Now, we give our new statistical summability method via Voronoi mean:

Definition 1.2. Let the real sequences $\{p_n, q_n, u_n\}$ with $u_n \neq 0$ for $n \geq 0$, be given. We say that $\{s_n\}$ is statistically summable to s by the Voronoi mean, written $s_n \overset{st}{\to} s\,(\mathcal{V}, p_n, q_n, u_n)$, if

$$st - \lim_n \frac{1}{u_n}\sum_{k=0}^{n} p_{n-k}q_k s_k = s.$$

Example 1.1. Let us consider the real sequences $\{p_n, q_n, u_n\}$ with $u_n = n^2 + 1$, $p_n = -1$, $q_n = \frac{1}{n+1}$ and a sequence $\{s_n\}$ as

$$s_n = \begin{cases} n+1, & n \text{ is odd}, \\ 0, & n \text{ is even}. \end{cases} \tag{2}$$

So, $\frac{1}{u_n}\sum\limits_{k=0}^{n} p_{n-k}q_k s_k = \begin{cases} -\frac{n+1}{n^2+1}, & n \text{ is odd}, \\ 0, & n \text{ is even}. \end{cases}$ Then, clearly, $\{s_n\}$ has Voronoi mean $s = 0$, hence $\{s_n\}$ is statistically summable to $s = 0$ by Voronoi mean, i.e.

$$s_n \overset{st}{\to} s = 0\left(\mathcal{V}, -1, \frac{1}{n+1}, n^2+1\right).$$

However, $\{s_n\}$ neither convergent (ordinary) nor statistical convergent to $s = 0$.

In the following result, we characterize the sequences that statistical summable by Voronoi mean through the sebsequences has Voronoi mean.

Theorem 1.1. *The sequence $\{s_n\}$ is statistically summable to s by the Voronoi mean iff there exists a set $N = \{n_1 < n_2 < n_3 < ... < n_k < ...\} \subseteq \mathbb{N}$ such that $\delta(N) = 1$ and $\{s_{n_k}\}$ has Voronoi mean s.*

Proof. Assume that there exists a set $N := \{n_1 < n_2 < n_3 < ... < n_k < ...\} \subseteq \mathbb{N}$ such that $\delta(N) := 1$ and $\{s_{n_k}\}$ has Voronoi mean s. Then there is a positive integer K such that for $k > K$,

$$\left|\frac{1}{u_{n_k}}\sum_{k=0}^{n_k} p_{n_k-k}q_k s_k - s\right| < \varepsilon.$$

Put $N_\varepsilon(\mathcal{V}) := \left\{k \in \mathbb{N} : \left|\frac{1}{u_{n_k}}\sum\limits_{k=0}^{n_k} p_{n_k-k}q_k s_k - s\right| \geq \varepsilon\right\}$ and $N' := \{n_{K+1}, n_{K+2}, ...\}$. Then $\delta(N') = 1$ and $N_\varepsilon(\mathcal{V}) \subseteq \mathbb{N} - N'$ which implies that $\delta(N_\varepsilon(\mathcal{V})) = 0$. Consequently, $\{s_n\}$ is statistically summable to s by the Voronoi mean.

Conversely, let $\{s_n\}$ is statistically summable to s by the Voronoi mean. For $r = 1, 2, 3, ...$, put $N_r(\mathcal{V}) := \left\{j \in \mathbb{N} : \left|\frac{1}{u_{n_j}}\sum\limits_{k=0}^{n_j} p_{n_j-k}q_k s_k - s\right| \geq \frac{1}{r}\right\}$ and $M_r(\mathcal{V}) := \left\{j \in \mathbb{N} : \left|\frac{1}{u_{n_j}}\sum\limits_{k=0}^{n_j} p_{n_j-k}q_k s_k - s\right| < \frac{1}{r}\right\}$. Then $\delta(N_r(\mathcal{V})) = 0$ and

$$M_1(\mathcal{V}) \supset M_2(\mathcal{V}) \supset ... \supset M_r(\mathcal{V}) \supset M_{r+1}(\mathcal{V}) \supset ... \tag{3}$$

and

$$\delta(M_r(\mathcal{V})) = 1, r = 1, 2, 3, ... \tag{4}$$

Now, we have to show that for $j \in M_r(\mathcal{V})$, $\{s_{n_j}\}$ has Voronoi mean s. Suppose that $\{s_{n_j}\}$ does not have Voronoi mean s. Hence, there is $\varepsilon > 0$ such that $\left|\frac{1}{u_{n_j}} \sum_{k=0}^{n_j} p_{n_j-k} q_k s_k - s\right| \geq \varepsilon$ for infinitely many terms. Let

$M_\varepsilon(\mathcal{V}) := \left\{ j \in \mathbb{N} : \left|\frac{1}{u_{n_j}} \sum_{k=0}^{n_j} p_{n_j-k} q_k s_k - s\right| < \varepsilon \right\}$ and $\varepsilon > \frac{1}{r}$ $(r = 1, 2, 3, ...)$. Then $\delta(M_\varepsilon(\mathcal{V})) = 0$, and by (3), $M_r(\mathcal{V}) \subset M_\varepsilon(\mathcal{V})$. Therefore $\delta(M_r(\mathcal{V})) = 0$, that contradicts (4) and hence $\{s_{n_j}\}$ has Voronoi mean s.

This completes the proof. □

2. Korovkin type approximation via statistical Voronoi mean

In this section we prove our main Korovkin type approximation theorem with the help of the statistical Voronoi mean.

Let $C(X)$ be the space of all continuous real valued functions on a compact subset X of the real numbers and $\|.\|$ denotes the usual supremum norm in $C(X)$. Throughout the paper we use the test functions $e_i(x) = x^i$ $(i = 0, 1, 2)$.

Theorem 2.1. *Let the real sequences $\{p_n, q_n, u_n\}$ with $u_n \neq 0$ for $n \geq 0$, be given and $\sum_{k=0}^{n} |p_{n-k} q_k| < K |u_n|$, with K independent of n. Assume that $\{L_n\}$ is a sequence of positive linear operators acting from $C(X)$ into itself, satisfying the following conditions:*

$$st - \lim_n \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k(e_i) - e_i \right\| = 0 \quad (i = 0, 1, 2). \tag{5}$$

Then, for all $f \in C(X)$, we have

$$st - \lim_n \left\| \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k} q_k L_k(f) - f \right\| = 0.$$

Proof. Let $f \in C(X)$ and $x \in X$ be fixed. By the continuity of f at the point x, we may write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for all $t \in X$ satisfying $|t - x| < \delta$. Since

$$|f(t) - f(x)| = |f(t) - f(x)| \chi_{X_\delta}(t) + |f(t) - f(x)| \chi_{X \setminus X_\delta}(t),$$

where $X_\delta = [x - \delta, x + \delta] \cap X$ and χ_{X_δ} denotes the characteristic function of the set X_δ. Then we have

$$|f(t) - f(x)| \leq \varepsilon + 2M \frac{(t - x)^2}{\delta^2},$$

for all $t \in X$, where $M := \|f\|$. This means

$$-\varepsilon - \frac{2M}{\delta^2} (t - x)^2 \leq f(t) - f(x) \leq \varepsilon + \frac{2M}{\delta^2} (t - x)^2.$$

Using the linearity and the positivity of the operators (L_n), we get,

$$\begin{aligned}
&\left|\frac{1}{u_n}\sum_{k=0}^{n} p_{n-k}q_k L_k(f;x) - f(x)\right| \\
&= \left|\frac{1}{u_n}\sum_{k=0}^{n} p_{n-k}q_k \left(L_k(f(t);x) - L_k(f(x);x) + L_k(f(x);x)\right) - f(x)\right| \\
&\le \frac{1}{|u_n|}\sum_{k=0}^{n} |p_{n-k}q_k| L_k(|f(t)-f(x)|;x) + |f(x)| \left|\frac{1}{|u_n|}\sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0;x) - e_0(x)\right| \\
&\le \frac{1}{|u_n|}\sum_{k=0}^{n} |p_{n-k}q_k| L_k\left(\varepsilon + \frac{2M}{\delta^2}(t-x)^2;x\right) + M\left|\frac{1}{|u_n|}\sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0;x) - e_0(x)\right| \\
&\le \varepsilon + \left(\varepsilon + M + \frac{2Mc^2}{\delta^2}\right)\left|\frac{1}{|u_n|}\sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0;x) - e_0(x)\right| \\
&\quad + \frac{4Mc}{\delta^2}\left|\frac{1}{|u_n|}\sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_1;x) - e_1(x)\right| \\
&\quad + \frac{2M}{\delta^2}\left|\frac{1}{|u_n|}\sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_2;x) - e_2(x)\right|
\end{aligned}$$

where $c := \max_{x\in X}|x|$. Then taking supremum over $x \in X$, we have

$$\left\|\frac{1}{u_n}\sum_{k=0}^{n} p_{n-k}q_k L_k(f) - f\right\| \le \varepsilon + T\left\{\sum_{i=0}^{2}\left\|\frac{1}{|u_n|}\sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_i) - e_i\right\|\right\} \qquad (6)$$

where $T := \max\left\{\varepsilon + M + \frac{2Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$. Now, for a given $\epsilon > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \epsilon$. Then,

$$S_n(\epsilon) := \left\{m \le n : \left\|\frac{1}{u_m}\sum_{k=0}^{m} p_{m-k}q_k L_k(f) - f\right\| \ge \epsilon\right\}$$

and

$$S_{i,n}(\epsilon) := \left\{m \le n : \left\|\frac{1}{|u_m|}\sum_{k=0}^{m} |p_{m-k}q_k| L_k(e_i) - e_i\right\| \ge \frac{\epsilon - \varepsilon}{3T}\right\}, \quad i = 0,1,2.$$

It follows from (6) that $S_n(\epsilon) \subset \bigcup_{i=0}^{2} S_{i,n}(\epsilon)$ and hence, $\lim_n \frac{1}{n}|S_n(\epsilon)| \le \sum_{i=0}^{2} \lim_n \frac{1}{n}|S_{i,n}(\epsilon)|$. Then using the hypothesis (5), we get

$$st - \lim_n \left\|\frac{1}{u_n}\sum_{k=0}^{n} p_{n-k}q_k L_k(f) - f\right\| = 0.$$

The proof is complete. □

3. An application

In the following example, we prove that our new convergence method is stronger than the classical ones.

Example 3.1. Let us consider the real sequences $\{p_n, q_n, u_n\}$ with $u_n = n^2 + 1$, $p_n = 1$, $q_n = -\frac{2}{n+1}$. *Observe now that* $\sum_{k=0}^{n} |p_{n-k}q_k| < 2|u_n|$. Then, consider a sequence

$$\{s_n\} = \begin{cases} 1, & n \text{ is square}, \\ n(n+1), & \text{otherwise}, \end{cases}$$

and the following classical Bernstein operators:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

where $x \in [0,1]$, $f \in C[0,1]$ and $n \in \mathbb{N}$. *Using these polynomials, we introduce the following positive linear operators on* $C[0,1]$:

$$D_n(f;x) = s_n B_n(f;x), \qquad x \in [0,1], \quad f \in C[0,1]. \tag{7}$$

We claim that

$$\lim_n \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| D_k(e_i) - e_i \right\| = 0 \text{ on } [0,1] \text{ for each } i = 0,1,2. \tag{8}$$

Indeed, first observe that

$$\begin{aligned} D_n(e_0;x) &= s_n e_0(x), \\ D_n(e_1;x) &= s_n e_1(x), \\ D_n(e_2;x) &= s_n \left[e_2(x) + \frac{e_1(x) - e_2(x)}{n} \right]. \end{aligned}$$

So,

$$\begin{aligned} \left| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| D_k(e_0;x) - e_0(x) \right| &= \left| \frac{1}{n^2+1} \sum_{k=0}^{n} \frac{2}{k+1} s_k - 1 \right| \\ &= \begin{cases} \left| \frac{1}{n^2+1} \sum_{k=0}^{n} \frac{2}{k+1} - 1 \right|, & n \text{ is square}, \\ \left| \frac{1}{n^2+1} \sum_{k=0}^{n} \frac{2}{k+1} k(k+1) - 1 \right|, & \text{otherwise}, \end{cases} \\ &= \begin{cases} \left| \frac{1}{n^2+1} \sum_{k=0}^{n} \frac{2}{k+1} - 1 \right|, & n \text{ is square}, \\ \left| \frac{n(n+1)}{(n^2+1)} - 1 \right|, & \text{otherwise}, \end{cases} \end{aligned}$$

then, we get

$$st - \lim_n \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| D_k(e_0) - e_0 \right\| = 0$$

that guarantees (8) holds true for $i=0$. Also, since

$$\left|\frac{1}{|u_n|}\sum_{k=0}^{n}|p_{n-k}q_k|\,D_k\left(e_1;x\right)-e_1\left(x\right)\right| = \left|e_1\left(x\right)\right|\left|\frac{1}{n^2+1}\sum_{k=0}^{n}\frac{2}{k+1}s_k-1\right| \leq \left|\frac{1}{n^2+1}\sum_{k=0}^{n}\frac{2}{k+1}s_k-1\right| \overset{st}{\to} 0,$$

whence we find

$$st-\lim_n\left\|\frac{1}{|u_n|}\sum_{k=0}^{n}|p_{n-k}q_k|\,D_k\left(e_1\right)-e_1\right\|=0,$$

that guarantees (8) holds true for $i=1$. Finally, we have

$$\begin{aligned}&\left|\frac{1}{|u_n|}\sum_{k=0}^{n}|p_{n-k}q_k|\,D_k\left(e_2;x\right)-e_2\left(x\right)\right|\\ &\quad=\left|\frac{1}{n^2+1}\sum_{k=0}^{n}\frac{2}{k+1}s_k\left[e_2\left(x\right)+\frac{e_1\left(x\right)-e_2\left(x\right)}{n}\right]-e_2\left(x\right)\right|\\ &\quad\leq\left|\frac{1}{n^2+1}\sum_{k=0}^{n}\frac{2}{k+1}s_k-1\right|+\left|\frac{1}{n^2+1}\sum_{k=0}^{n}\frac{2}{k+1}\frac{s_k}{k}\right|\end{aligned}$$

and since

$$\left|\frac{1}{n^2+1}\sum_{k=0}^{n}\frac{2}{k+1}\frac{s_k}{k}\right| \overset{st}{\to} 0,$$

then it is easy to check that

$$st-\lim_n\left\|\frac{1}{|u_n|}\sum_{k=0}^{n}|p_{n-k}q_k|\,D_k\left(e_2\right)-e_2\right\|=0.$$

So, our claim (8) holds true for each $i=0,1,2$. Now, we can say that our sequence $\{D_n\}$ defined by (7) satisfy all assumptions of Theorem 2.1. Using these facts, we conclude that

$$st-\lim_n\left\|\frac{1}{u_n}\sum_{k=0}^{n}p_{n-k}q_kD_k\left(f\right)-f\right\|=0$$

holds for any $f\in C\left[0,1\right]$. However, since $\left\|D_n\left(e_0\right)-e_0\right\|=\left|s_n-1\right|$ and a sequence $\{s_n\}$ does not ordinary or statistically convergent to 1, $\{\left\|D_n\left(e_0\right)-e_0\right\|\}$ does not ordinary or statistically convergent to 0 and hence, classical Korovkin theorem ([9]) or the statistical Korovkin theorem ([7]) does not work for the sequence $\{D_n\}$.

4. Rate of convergence

The main result in this section is a study the rate of statistical Voronoi mean with the aid of the modulus of continuity that is defined by

$$\omega(f,\delta)=\sup_{|t-x|\leq\delta,\ x,t\in X}|f(t)-f(x)| \quad (\delta>0),\ f\in C(X).$$

It is readily seen that, for any $\alpha>0$ and for all $f\in C(X)$

$$\omega(f,\alpha\delta)\leq\left(1+[\alpha]\right)\omega(f,\delta)$$

where $[\alpha]$ is defined to be the greatest integer less than or equal to α. Then the result is stated as follows.

Theorem 4.1. *Let the real sequences $\{p_n, q_n, u_n\}$ with $u_n \neq 0$ for $n \geq 0$, be given and $\sum_{k=0}^{n} |p_{n-k}q_k| < K|u_n|$, with K independent of n. Assume that $\{L_n\}$ is a sequence of positive linear operators acting from $C(X)$ into itself, satisfying the following conditions:*

(i) $st - \lim_n \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0) - e_0 \right\| = 0,$

(ii) $st - \lim_n \omega(f; \alpha_n) = 0$ *where* $\alpha_n := \sqrt{\left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k\left((t-.)^2\right) \right\|}.$

Then for all $f \in C(X)$,

$$st - \lim_n \left\| \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k}q_k L_k(f) - f \right\| = 0.$$

Proof. Let $f \in C(X)$ and $x \in X$ be fixed. Using the properties of ω, and the positivity and monotonicity of L_n, we get that

$$\begin{aligned}
&\left| \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k}q_k L_k(f;x) - f(x) \right| \\
&\leq \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(|f(t) - f(x)|; x) + M \left| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0; x) - e_0(x) \right| \\
&\leq \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k\left(\omega\left(f; \delta \frac{|t-x|}{\delta}\right); x\right) + M \left| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0; x) - e_0(x) \right| \\
&\leq \omega(f;\delta) \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k\left(1 + \frac{|t-x|}{\delta}; x\right) + M \left| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0; x) - e_0(x) \right| \\
&\leq \omega(f;\delta) \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k\left(1 + \frac{(t-x)^2}{\delta^2}; x\right) + M \left| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0; x) - e_0(x) \right| \\
&\leq \omega(f;\delta) \left[\frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(1;x) + \frac{1}{\delta^2} \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k\left((t-x)^2; x\right) \right] \\
&\quad + M \left| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0; x) - e_0(x) \right| \\
&\leq \omega(f;\delta) \left| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0; x) - e_0(x) \right| + \omega(f;\delta) \\
&\quad + M \left| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k(e_0; x) - e_0(x) \right| \\
&\quad + \frac{\omega(f;\delta)}{\delta^2} \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k}q_k| L_k\left((t-x)^2; x\right).
\end{aligned}$$

Then taking supremum over $x \in X$, we have

$$\begin{aligned}\left\| \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k} q_k L_k(f) - f \right\| &\leq \omega(f;\delta) \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k(e_0) - e_0 \right\| \\ &\quad + 2\omega(f;\delta) + M \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k(e_0) - e_0 \right\|\end{aligned}$$

where $\delta := \alpha_n = \sqrt{\left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k\left((t-.)^2\right) \right\|}$. Then, from the hypotheses we conclude that

$$st - \lim_n \left\| \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k} q_k L_k(f) - f \right\| = 0,$$

we obtain the assertion. □

Remark 4.1. If we replace the conditions (i), (ii) in Theorem 4.1 by the following condition:

$$st - \lim_n \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k(e_i) - e_i \right\| = 0 \quad (i = 0, 1, 2). \tag{9}$$

Then, since

$$L_k\left((t-x)^2; x\right) \leq N \left\{ |L_k(e_0;x) - e_0(x)| + |L_k(e_1;x) - e_1(x)| + |L_k(e_2;x) - e_2(x)| \right\} \tag{10}$$

where $N = \|e_2\| + 2\|e_1\| + e_0$. We get

$$\begin{aligned}\left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k\left((t-.)^2\right) \right\| &\leq N \left\{ \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k(e_0) - e_0 \right\| \right. \\ &\quad \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k(e_1) - e_1 \right\| \\ &\quad \left. \left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k(e_2) - e_2 \right\| \right\}.\end{aligned}$$

It follows that (9), (10) that

$$st - \lim_n \alpha_n = 0$$

where $\alpha_n := \sqrt{\left\| \frac{1}{|u_n|} \sum_{k=0}^{n} |p_{n-k} q_k| L_k\left((t-.)^2\right) \right\|}$. So, by Theorem 4.1 we get, for all $f \in C(X)$,

$$st - \lim_n \left\| \frac{1}{u_n} \sum_{k=0}^{n} p_{n-k} q_k L_k(f) - f \right\| = 0.$$

Hence, if we replace the conditions (i), (ii) with the condition (9) in Theorem 4.1, then we get the rates of statistical Voronoi mean in Theorem 2.1.

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