Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 47(2), 2020, Pages 397–413
ISSN: 1223-6934

Finite dimensional null-controllability of a fractional parabolic equation

Constantin Niţă and Laurenţiu Emanuel Temereancă

Abstract. In this article we analyze some controllability properties of a fractional equation which serves as a model for anomalous diffusive phenomena. It is known that this equation is not spectrally controllable. Our aim is to study the behavior of the control when only the projection of the solution over a finite dimensional space is driven to zero in finite time.

2010 Mathematics Subject Classification. 30E05,93B05.
Key words and phrases. finite dimensional controllability, moment problem, biorthogonals.

1. Introduction

The aim of this paper is to study the controllability properties of the following parabolic type equation:

$$\begin{cases} u_t(t,x) + (-\partial_{xx})^{\alpha/2} u(t,x) = g(t)f(x), & t \in (0,T),\, x \in (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u^0(x), & x \in (0,1). \end{cases} \tag{1.1}$$

In (1.1), $(-\partial_{xx})^{\alpha/2}$ denotes the operatorial fractional power of order $\alpha/2 > 0$ of the Dirichlet Laplacian in the interval $(0,1)$. More precisely, $(-\partial_{xx})^{\alpha/2}$ is the linear unbounded operator in $L^2(0,1)$ defined as follows

$$(-\partial_{xx})^{\alpha/2} : D((-\partial_{xx})^{\alpha/2}) \subset L^2(0,1) \to L^2(0,1),$$

$$D((-\partial_{xx})^{\alpha/2}) = \left\{ u \in L^2(0,1) : u = \sum_{j\geq 1} \sqrt{2} a_j \sin(j\pi x) \text{ and } \sum_{j\geq 1} |a_j|^2 j^{2\alpha} < +\infty \right\},$$

$$u(x) = \sum_{j\geq 1} \sqrt{2} a_j \sin(j\pi x) \to (-\partial_{xx})^{\alpha/2} u(x) = \sum_{j\geq 1} \sqrt{2} a_j (j\pi)^\alpha \sin(j\pi x).$$

The eigenvalues of the operator $(-\partial_{xx})^{\alpha/2}$ are given by

$$\lambda_j = (j\pi)^\alpha \qquad (j \geq 1), \tag{1.2}$$

and the corresponding eigenfunctions are

$$v_j = \sin(j\pi x) \qquad (j \geq 1). \tag{1.3}$$

For any $\alpha > 0$ equation (1.1) is of parabolic type. When $\alpha = 2$ we recover the classical heat equation.

Received November 21, 2020.

A "generalized" diffusion equation which reads, in Fourier space, as follows

$$\frac{\partial \mathcal{F}P}{\partial t}(t,q) = -q^{\mu}\mathcal{F}P(t,q), \tag{1.4}$$

is introduced by [17] in the context of the anomalously enhanced diffusion in systems of elongated polymerlike breakable micelles. In (1.4), $\mathcal{F}P$ is the Fourier transform of the probability distribution P and $\mu < 2$. Due to reptation, short micelles diffuse much more rapidly than long ones. As time goes on, shorter and shorter micelles are encountered and the effective diffusion increases with time. This corresponds to a random walk for which the second moment of the jump-size distribution fails to exist ("Lévy flight").

In [12, Section 3.5. Long jumps: Lévy flights] the following equation is proposed as a simple model for the description of transport processes in complex systems quicker than the Brownian diffusion,

$$\frac{\partial P}{\partial t}(t,x) = K^{\mu}\ {}_{-\infty}D_x^{\mu}P(t,x) \qquad (x \in \mathbb{R},\ t > 0), \tag{1.5}$$

where $\mu \in (1,2)$, ${}_{-\infty}D_x^{\mu}$ is the Weyl operator which, in one dimension, is equivalent to the Riesz operator $-D^{\mu}$ and K is a positive constant. We recall that the Riesz fractional differentiation, D^{μ}, is defined by:

$$D^{\mu}f := (-\Delta)^{\frac{\mu}{2}} f = \mathcal{F}^{-1}\left[|z|^{\mu}\mathcal{F}f(z)\right], \tag{1.6}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Consequently, by taking the Fourier transform in (1.5), we obtain that $\mathcal{F}P$ verifies an equation similar to (1.4).

It is easy to see that the fundamental solution of (1.5) is given by

$$\zeta(t,x) = \mathcal{F}^{-1}\left[e^{-K^{\mu}|z|^{\mu}t}\right](x) = \frac{1}{2\pi K t^{\frac{1}{\mu}}} L_{\mu}\left[\frac{x}{K t^{\frac{1}{\mu}}}\right], \tag{1.7}$$

where

$$L_{\mu}(x) = \int_{\mathbb{R}} e^{-iyx-|y|^{\mu}}\,\mathrm{d}y, \tag{1.8}$$

is the Fourier transform of the function $e^{-|y|^{\mu}}$ which is known as the Lévy symmetric μ-stable distribution. We recall that L_{μ} is a "bell-shaped" function [6] but, unlike the Gaussian, it is a heavy-tailed distribution [16]:

$$L_{\mu}(x) \sim \frac{\Gamma(1+\mu)\sin(\pi\mu/2)}{\pi}\frac{1}{x^{1+\mu}} \text{ as } x \to \infty. \tag{1.9}$$

In the case $\mu = 2$ the function L_2 can be computed explicitly and represents the well-known normal distribution. In this particular case, the fundamental solution depends on the grouping $x/t^{1/2}$ of the independent variables, which allowed A. Einstein to show in [3] that the mean square displacement is proportional with time. Although in the case $\mu < 2$ the mean square displacement cannot be defined (the second moment of the distribution is not finite) the facts that the fundamental solution (1.7) depends on the grouping $x/t^{1/\mu}$ and $t^{1/\mu} \gg t^{1/2}$ as t tends to infinity indicate that we are in the presence of a much quicker diffusion phenomenon.

Notice that, for $\alpha \in (0,2)$, our equation (1.1) represents precisely a controlled version of (1.5) stated in a finite one-dimensional interval and it may be considered as a simple model for controlled parabolic dynamical system with enhanced diffusivity (super-diffusion or quicker propagation of the concentration front). The list of systems displaying such anomalous dynamic behavior is quite extensive: special domains of rotating flows, collective slip diffusion on solid surfaces, Richardson turbulent diffusion, bulk-surface exchange controlled dynamics in porous glasses, transport in micelle systems and in heterogeneous rocks, etc. (see, for instance, [12, 13] and the references therein).

Other mathematical models have been proposed for the anomalous diffusion phenomena. For instance, in [12] (see, also, [18]), the following fractional diffusion is considered

$$u_t(t,x) = {}_0D_t^{1-\alpha}\partial_x^2 u(t,x), \tag{1.10}$$

where the Riemann-Liouville operator ${}_0D_t^{1-\alpha}$ is defined, for $\alpha \in (0,1)$, by

$${}_0D_t^{1-\alpha}u(t,x) = \frac{1}{\Gamma(\alpha)}\partial_t\left(\int_0^t \frac{u(\tau,x)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\right).$$

To our knowledge, the controllability properties of (1.10) have not yet been studied but the following fractional (in time) equation is considered in [11]

$$\partial_{0+}^{\alpha}u(t,x) = \partial_x^2 u(t,x), \tag{1.11}$$

where, for $\alpha \in (0,1)$, ∂_{0+}^{α} is the left-sided Caputo derivative in zero,

$$\partial_{0+}^{\alpha}u(t,x) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial_s u(s,x)}{(t-s)^{\alpha}}\,\mathrm{ds}.$$

The fractional time derivative introduces some memory effects on the system that need to be taken into account when defining the notion of null controllability. As proved in [11], when the full control problem is considered for both the value of the state at the final time and the memory accumulated by the long-tail effects introduced by the fractional derivative, controllability cannot be achieved in finite time.

On the other hand, an equation similar to (1.1) has been analyzed in [2] with the fractional Laplace operator defined by the following singular integral

$$(-\partial_{xx})^{\alpha/2}u(x) = c_{1,\alpha}P.V.\int_{\mathbb{R}} \frac{\tilde{u}(x)-\tilde{u}(y)}{|x-y|^{1+\alpha}}\,\mathrm{dy}, \tag{1.12}$$

where $\tilde{u}$ is the extension by zero of u outside the interval $(0,1)$ and $c_{1,\alpha} = \frac{\alpha 2^{\alpha}\Gamma\left(\frac{1+\alpha}{2}\right)}{2\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)}$. For this fractional Laplace operator, in [2], positive controllability results are obtained only if $\alpha > 1$. In fact, as shown in [8], the eigenvalues of the above operator are of the form $\left(j\pi + \frac{(2-\alpha)\pi}{4}\right)^{\alpha} + \mathcal{O}\left(\frac{1}{j}\right)$ which, asymptotically, are similar to λ_j from (1.2). Therefore, it is not surprising to see that we have the same qualitative behavior of (1.1) when the fractional Laplace operator is given as a singular integral (1.12) or as in the initial fractional operatorial form.

In (1.1) u is the state of the system and the control $g(t)f(x)$ is the product of separated variables functions in time and space. The space shape $f(x)$ of the control is fixed. Then one only acts on the system by means of tuning the time-intensity $g(t)$ of the control. Such types of controls are referred to in the literature as "lumped" or "bilinear" (see, for instance, [1]) and are of great interest being closer to the engineering applications.

Given $T > 0$ and $f \in L^2(0,1)$, we say that equation (1.1) is *null-controllable in time* $T > 0$ if, for any $u^0 \in L^2(0,1)$, there exists a control $g \in L^2(0,T)$ such that the solution u verifies

$$u(T,x)=0 \qquad (x\in(0,1))\,. \tag{1.13}$$

The null-controllability property of (1.1) in the case $\alpha>1$ is discussed and solved in [5]. On the other hand, the case $\alpha\le 1$ is studied in [14] (see, also, [2, 10, 15]). It is proved that the problem is not null-controllable (not even spectrally controllable) in any time $T>0$. Let us explain the main reason for this negative result. Since the eigenvalues λ_j have the form given by (1.2) then, according to Müntz Theorem, it follows that the family of exponential functions $\left(e^{-\lambda_j t}\right)_{j\ge 1}$ is complete in $L^2(0,T)$ for any $T\in(0,\infty]$. Moreover, it continues to be so even if a finite number of its elements are eliminated. Consequently, the family $\left(e^{-\lambda_j t}\right)_{j\ge 1}$ is not minimal in $L^2(0,T)$, for any $T\in(0,\infty]$ and, as it follows from Theorem 2.2, not even initial data consisting of one mode only can be controlled.

In this paper, we consider the case $\alpha\in(0,1)$ and we concentrate our attention on a different controllability notion. We introduce the following definition:

Definition 1.1. Given $N\in\mathbb{N}^*$, we say that equation (1.1) is *N-finite dimensional null-controllable in time $T>0$* if, for any $u^0\in L^2(0,1)$, there exists a control $g:=g_N\in L^2(0,T)$ such that the solution u verifies

$$\Pi_N u(T,x)=0 \qquad (x\in(0,1))\,, \tag{1.14}$$

where Π_N represents the projection operator over the space generated by the first N eigenfunctions:

$$\Pi_N\left(\sum_{j\ge 1}\sqrt{2}a_j v_j\right)=\sum_{j=1}^{N}\sqrt{2}a_j v_j.$$

For a study of the finite dimensional null controllability problem for semilinear heat equations in bounded domains with Dirichlet boundary conditions the interested reader is referred to [20]. Our aim is to prove that problem (1.1) is N-finite dimensional null-controllable in any time $T>0$. Moreover, we study the behavior of the controllability cost when N tends to infinity. Since we are interested to control as many frequencies as possible, we shall consider in the sequel that $N>N_0$ for a convenient choice of N_0. The main result of this article reads as follows:

Theorem 1.1. *Let $T>0$ and $\alpha\in(0,1)$. Suppose that the function $f=\sum_{j=1}^{\infty}\sqrt{2}f_j v_j\in L^2(0,1)$ verifies*

$$\begin{cases} f_j\ne 0 \qquad (j\in\mathbb{N}^*),\\ \liminf_{j\to\infty}|f_j|e^{\nu j^{\alpha}}>0 \qquad (\nu>0). \end{cases} \tag{1.15}$$

Then there exists $N_0\in\mathbb{N}$ with the property that, for any $N\ge N_0$ and $u^0\in L^2(0,1)$, there exists a control $g_N\in L^2(0,T)$ such that the solution u of (1.1) *verifies* (1.14) *and*

$$\|g_N\|_{L^2(0,T)}\le C\exp(\omega N\ln N)\|u_0\|_{L^2(0,1)}, \tag{1.16}$$

where C and ω are two positive constants independent of N and u_0.

Notice that, according to Theorem 1.1, equation (1.1) is N-finite dimensional null-controllable in any time $T>0$, but the norm of the corresponding control may explode exponentially as N goes to infinity. Moreover, we'll show that estimate (1.16) is, in some sense, optimal and the norm of the control is bounded from below in a similar way.

This paper is organized as follows. In Section 2 we prove that the N-finite dimensional null-controllability problem (1.1)-(1.14) is equivalent with a moment problem and the concept of a biorthogonal sequence is introduced to solve this moment problem. In Section 3 we construct a biorthogonal sequence, we estimate its norm in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$, and we discuss the optimality of our results. Finally, Section 4 is devoted to prove several controllability properties, among them our main result Theorem 1.1.

2. The moment problem

Let E_N be the space generated by the first N eigenfunctions:

$$E_N := \text{Span}\left\{(v_j)_{1\leq j\leq N}\right\}, \tag{2.1}$$

and let $\Lambda_N := \left(e^{-\lambda_j t}\right)_{1\leq j\leq N}$.

First, we have the following variational result.

Lemma 2.1. *Let $T > 0$, $\alpha \in (0,1)$ and the initial data $u^0 \in L^2(0,1)$. The function $g_N \in L^2(0,T)$ is a control which drives to zero the projection of the solution of* (1.1) *at time T over the space E_N if and only if, the following relation holds*

$$\int_0^T \int_0^1 g_N(t) f(x)\overline{\varphi}(t,x)\,\mathrm{dx}\,\mathrm{dt} + \int_0^1 u^0(x)\overline{\varphi}(0,x)\,\mathrm{dx} = 0, \tag{2.2}$$

for every $\varphi^T \in E_N$, where $\varphi \in L^2(0,1)$ is the solution of the following adjoint backward problem

$$\begin{cases} -\varphi_t(t,x) + (-\partial_{xx})^{\alpha/2}\varphi(t,x) = 0 & t\in(0,T),\ x\in(0,1), \\ \varphi(t,0) = \varphi(t,1) = 0 & t\in(0,T), \\ \varphi(T,x) = \varphi^T(x) & x\in(0,1). \end{cases} \tag{2.3}$$

Proof. If we multiply in (1.1) by $\overline{\varphi}$ and we integrate by parts over $(0,T)\times(0,1)$, we obtain that $g_N \in L^2(0,T)$ is a control for (1.1) if and only if it verifies (2.2). □

The following result gives us the moment problem associated with the N-finite dimensional null-controllability property of (1.1).

Theorem 2.2. *Problem* (1.1) *is N-finite dimensional null-controllable in time $T > 0$ if and only if, for initial data $u^0 \in L^2(0,1)$, there exists $g_N \in L^2(0,T)$ such that*

$$\int_0^T g_N(T-t)e^{-\lambda_j t}\,\mathrm{dt} = -\frac{u_j^0}{f_j}e^{-\lambda_j T} \qquad (1\leq j\leq N), \tag{2.4}$$

where

$$f_j = \sqrt{2}\int_0^1 f(x)\sin(j\pi x)\,\mathrm{dx},\ \ u_j^0 = \sqrt{2}\int_0^1 u^0(x)\sin(j\pi x)\,\mathrm{dx}, \tag{2.5}$$

are the Fourier coefficients of the functions f and u^0, respectively.

Proof. From the above Lemma 2.1, the system (1.1) is N-finite dimensional null-controllable if and only if, there exists $g_N \in L^2(0,T)$ such that (2.2) holds for any initial data $u^0 \in L^2(0,1)$. If $\varphi_T = \sqrt{2}\sin(j\pi x)$, the corresponding solution of (2.3) is given by

$$\varphi(t,x) = \sqrt{2}\, e^{(t-T)\lambda_j} \sin(j\pi x) \qquad (1 \le j \le N).$$

Hence, by using this particular choice in (2.2) we deduce that (2.4) holds. □

We remark that

$$\sum_{j=1}^{N} \frac{1}{\lambda_j} < \infty,$$

and, consequently, it follows that the family Λ_N is incomplete in $L^2(0,T)$. Hence, it is minimal and we deduce that there exists a biorthogonal sequence to it in $L^2(0,T)$.

If there exists a biorthogonal sequence $(\theta_k(T,\cdot))_{1\le k\le N}$ to the family Λ_N in $L^2(0,T)$ then the problem of moments (2.4) may be solved immediately by setting

$$g_N(T-t) = \sum_{k=1}^{N} -\frac{u_k^0}{f_k} e^{-\lambda_k T} \theta_k(T,t).$$

In order to evaluate the norm of the control g_N we have to study the behavior of the norms of the elements $(\theta_k(T,\cdot))_{1\le k\le N}$ from the biorthogonal sequence.

3. A biorthogonal sequence in a finite interval

In this section we construct and we evaluate a biorthogonal sequence to the family of exponential functions Λ_N in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$.

3.1. The product. In this section we introduce a finite product P_k , with the property that $P_k(-i\lambda_j) = \delta_{kj}$ and we obtain an estimate for this product on the real axis. For every $1 \le k \le N$, we define the function

$$P_k(z) = \prod_{\substack{1\le p\le N\\ p\ne k}} \frac{\lambda_p - iz}{\lambda_p - \lambda_k} = \underbrace{\prod_{\substack{1\le p\le N\\ p\ne k}} \frac{\lambda_p}{\lambda_p - \lambda_k}}_{Q_k} \underbrace{\prod_{\substack{1\le p\le N\\ p\ne k}} \left(1 - \frac{iz}{\lambda_p}\right)}_{\widetilde{P}_k(z)}. \tag{3.1}$$

We estimate the first part of the product P_k.

Lemma 3.1. *Given $\alpha \in (0,1)$, for any $1 \le k \le N$, we have that*

$$|Q_k| \le \exp\left[\left(\ln 2 + \frac{2-\alpha}{\alpha(1-\alpha)}\right) k^\alpha N^{1-\alpha}\right]. \tag{3.2}$$

Proof. We remark that

$$
\begin{aligned}
|Q_k| = \prod_{\substack{1\le p\le N\\ p\neq k}} \frac{p^\alpha}{|p^\alpha-k^\alpha|} &= \exp\left[\sum_{1\le p\le k-1}\ln\left(\frac{p^\alpha}{k^\alpha-p^\alpha}\right)+\sum_{k+1\le p\le N}\ln\left(\frac{p^\alpha}{p^\alpha-k^\alpha}\right)\right]\\
&= \exp\left[\sum_{1\le p\le k-1}\ln\left(1+\frac{2p^\alpha-k^\alpha}{k^\alpha-p^\alpha}\right)+\sum_{k+1\le p\le N}\ln\left(1+\frac{k^\alpha}{p^\alpha-k^\alpha}\right)\right]\\
&\le \exp\left[\int_1^k \ln\left(1+\frac{2x^\alpha-k^\alpha}{k^\alpha-x^\alpha}\right)\mathrm{dx}+\int_k^N \ln\left(1+\frac{k^\alpha}{x^\alpha-k^\alpha}\right)\mathrm{dx}\right]\\
&\le \exp\left[k\underbrace{\int_0^1 \ln\left(1+\frac{1}{1-t^\alpha}\right)\mathrm{dt}}_{I_1}+k\underbrace{\int_1^{\frac{N}{k}} \ln\left(1+\frac{1}{t^\alpha-1}\right)\mathrm{dt}}_{I_2}\right]. \qquad (3.3)
\end{aligned}
$$

We evaluate now each one of the two integrals I_1 and I_2. We have that

$$I_1=\ln 2+\alpha\int_0^1 \frac{t^{\alpha-1}(1-t)}{(2-t^\alpha)(1-t^\alpha)}\,\mathrm{dt}\le \ln 2+\int_0^1 t^{\alpha-1}\,\mathrm{dt},$$

where we have used the fact that

$$1\le \frac{1-t}{1-t^\alpha}\le\frac{1}{\alpha}\qquad (t\in[0,1]). \qquad (3.4)$$

It follows that

$$I_1\le \ln 2+\frac{1}{\alpha}. \qquad (3.5)$$

Let us pass to analyze I_2. We have that

$$
\begin{aligned}
I_2 =&(t-1)\ln\left(1+\frac{1}{t^\alpha-1}\right)\Big|_1^{\frac{N}{k}}+\alpha\int_1^{\frac{N}{k}}\frac{t^{\alpha-1}(t-1)}{t^\alpha(t^\alpha-1)}\,\mathrm{dt}\\
\le&\frac{1}{\alpha}\left(\frac{N}{k}\right)^{1-\alpha}+\int_1^{\frac{N}{k}}\frac{1}{t^\alpha}\,\mathrm{dt}=\frac{1}{\alpha(1-\alpha)}\left(\frac{N}{k}\right)^{1-\alpha}-\frac{1}{1-\alpha}, \qquad (3.6)
\end{aligned}
$$

where we have used the estimates

$$(t-1)\ln\left(1+\frac{1}{t^\alpha-1}\right)\le\frac{1}{\alpha}t^{1-\alpha}\qquad (t\ge 1),$$

and

$$1<\frac{t^{\alpha-1}(t-1)}{t^\alpha-1}\le\frac{1}{\alpha}\qquad (t\ge 1). \qquad (3.7)$$

From (3.3)-(3.6) it follows that (3.2) holds.

□

Theorem 3.2. *For each* $1\le k\le N$*, the product* P_k *defined by* (3.1) *is an entire function of exponential type zero which verifies*

$$P_k(-i\lambda_j)=\delta_{kj}\qquad (1\le j\le N). \qquad (3.8)$$

Proof. From the definition of the product P_k in (3.1), the relation (3.8) follows immediately. To prove that the product is an entire function of exponential type zero we remark that

$$\left|\widetilde{P}_k(z)\right| \leq \exp\left[\sum_{\substack{1\leq p\leq N\\ p\neq k}} \ln\left(1+\frac{|z|}{\lambda_p}\right)\right] \leq \exp\left[\underbrace{\int_0^N \ln\left(1+\frac{|z|}{s^\alpha}\right) \mathrm{ds}}_{I}\right],$$

and

$$I = N\ln\left(1+\frac{|z|}{N^\alpha}\right) + \alpha\,|z| \int_0^N \frac{1}{s^\alpha+|z|}\,\mathrm{ds} \leq N\ln\left(1+\frac{|z|}{N^\alpha}\right) + \alpha N.$$

Let $\varepsilon > 0$ be an arbitrary positive number. Taking into account that $\ln(1+t) \leq \sqrt{t}$, for $t \geq 0$, we have

$$N\ln\left(1+\frac{|z|}{N^\alpha}\right) \leq N^{1-\frac{\alpha}{2}}\,|z|^{1/2} \leq \frac{1}{2\varepsilon}N^{2-\alpha} + \frac{\varepsilon}{2}|z|.$$

Hence,

$$I \leq \alpha N + \frac{1}{2\varepsilon}N^{2-\alpha} + \frac{\varepsilon}{2}|z|,$$

and it follows that

$$\left|\widetilde{P}_k(z)\right| \leq \exp\left(\frac{1}{2\varepsilon}N^{2-\alpha} + \alpha N\right)\exp\left(\frac{\varepsilon}{2}\,|z|\right). \tag{3.9}$$

From relations (3.1), (3.2) and (3.9) we obtain

$$|P_k(z)| \leq \exp\left[\frac{1}{2\varepsilon}N^{2-\alpha} + \alpha N + \left(\ln 2 + \frac{2-\alpha}{\alpha(1-\alpha)}\right)N\right]\exp\left(\frac{\varepsilon}{2}\,|z|\right). \tag{3.10}$$

From (3.10) we deduce that

$$|P_k(z)| \leq \exp\left(\varepsilon\,|z|\right) \qquad \left[|z| \geq \frac{2}{\varepsilon}\left(\frac{1}{2\varepsilon}N^{2-\alpha} + \alpha N + \left(\ln 2 + \frac{2-\alpha}{\alpha(1-\alpha)}\right)N\right)\right].$$

From the above inequality it follows that P_k is an entire function of exponential type zero. The proof is complete. □

We pass to estimate the second part of the function P_k on the real axis. We have the following result.

Lemma 3.3. *Given $\alpha \in (0,1)$ and $1 \leq k \leq N$, for every $x \in \mathbb{R}$, we have that*

$$\left|\widetilde{P}_k(x)\right| \leq \begin{cases} \exp\left(\frac{5}{2}N\ln\left(1+\frac{x^2}{N^{2\alpha}\pi^{2\alpha}}\right)\right) & \text{if } |x| \geq N^\alpha\pi^\alpha, \\ \exp\left(\left(\frac{1+2\alpha}{2-4\alpha}\right)N\ln\left(1+\frac{x^2}{N^{2\alpha}\pi^{2\alpha}}\right)\right) & \text{if } |x| < N^\alpha\pi^\alpha \text{ and } \alpha < \frac{1}{2}, \\ \exp\left(\frac{N^{1/2}}{\pi^{1/2}}\,|x|\right) & \text{if } |x| < N^{1/2}\pi^{1/2} \text{ and } \alpha = \frac{1}{2}, \\ \exp\left(\left(\frac{4\alpha^2+2\alpha-1}{4\alpha-2}\right)\frac{1}{\pi}\,|x|^{1/\alpha}\right) & \text{if } |x| < N^\alpha\pi^\alpha \text{ and } \alpha > \frac{1}{2}. \end{cases} \tag{3.11}$$

Proof. Since $\left|\widetilde{P}_k(x)\right|$ is an even function we study only the case $x \geq 0$. We have

$$\left|\widetilde{P}_k(x)\right|^2 \leq \exp\left[\sum_{1\leq p\leq N} \ln\left(1+\frac{y^2}{p^{2\alpha}}\right)\right] \leq \exp\left[\underbrace{\int_0^N \ln\left(1+\frac{y^2}{s^{2\alpha}}\right) \mathrm{ds}}_{I_N(y)}\right],$$

where $y = \frac{x}{\pi^\alpha}$. Since we have that

$$I_N(y) = N\ln\left(1+\frac{y^2}{N^{2\alpha}}\right) + 2\alpha y^{1/\alpha}\int_0^{\frac{N}{y^{1/\alpha}}} \frac{1}{t^{2\alpha}+1}\,\mathrm{dt},$$

we are lead to study the properties of the function $f : [0,\infty) \to \mathbb{R}$,

$$f(r) = \int_0^r \frac{1}{t^{2\alpha}+1}\,\mathrm{dt}.$$

Since the function f has the following properties

$$f(r) \leq \begin{cases} r & \text{if } r \leq 1 \\ \dfrac{1}{1-2\alpha} r^{1-2\alpha} & \text{if } r > 1 \text{ and } \alpha < \frac{1}{2} \\ \dfrac{2\alpha}{2\alpha-1} - \dfrac{1}{2\alpha-1} r^{1-2\alpha} & \text{if } r > 1 \text{ and } \alpha > \frac{1}{2} \\ \ln(1+r) & \text{if } r > 1 \text{ and } \alpha = \frac{1}{2}, \end{cases} \tag{3.12}$$

by taking $r = \frac{N}{y^{1/\alpha}}$, it follows that:

a) If $y \geq N^\alpha$, by using the first inequality from (3.12), we have that

$$I_N(y) \leq N\ln\left(1+\frac{y^2}{N^{2\alpha}}\right) + 2\alpha N \leq 5N\ln\left(1+\frac{y^2}{N^{2\alpha}}\right),$$

which gives the first inequality in (3.11).

b) If $y < N^\alpha$ and $\alpha < \frac{1}{2}$, by taking into account the second inequality from (3.12) and the fact that $\ln(1+t) \geq \frac{t}{2}$, for every $t < 1$, it follows that

$$I_N(y) \leq N\ln\left(1+\frac{y^2}{N^{2\alpha}}\right) + \frac{2\alpha}{1-2\alpha} N^{1-2\alpha} y^2 \leq \left(\frac{1+2\alpha}{1-2\alpha}\right) N\ln\left(1+\frac{y^2}{N^{2\alpha}}\right),$$

and thus, the second inequality in (3.11) holds.

c) If $y < N^\alpha$ and $\alpha > \frac{1}{2}$, by considering the third inequality from (3.12) and taking into account that $\ln(1+t) \leq t^{\frac{1}{2\alpha}}$, for every $t < 1$ and $\frac{1}{2\alpha} < 1$, we obtain that

$$I_N(y) \leq N\ln\left(1+\frac{y^2}{N^{2\alpha}}\right) + \frac{4\alpha^2}{2\alpha-1} y^{1/\alpha} \leq \left(\frac{4\alpha^2+2\alpha-1}{2\alpha-1}\right) y^{1/\alpha},$$

which gives the last inequality in (3.11).

d) If $y < N^{1/2}$ and $\alpha = \frac{1}{2}$, from the last equality in (3.12) and taking into account that $\ln(1+t) \leq \sqrt{t}$, for every $t \geq 0$, we deduce that

$$I_N(y) = N\ln\left(1+\frac{y^2}{N}\right) + y^2\ln\left(1+\frac{N}{y^2}\right) \leq 2y^2\ln\left(1+\frac{N}{y^2}\right) \leq 2yN^{1/2},$$

and the proof of lemma is complete.

□

3.2. Construction of the multiplier. In this section we consider a multiplier function M_k with rapid decay on the real axis, such that the product P_kM_k is bounded on the real axis and $M_k(-i\lambda_k) = 1$. This multiplier is obtained by following step by step the construction gives in [7, Pages 19-20]. For every $a > 0$ we set

$$H_a := \frac{1_{[-a,a]}}{2a},$$

and we recall that

$$\widehat{H}_a(z) = \int_{\mathbb{R}} H_a(t)e^{-itz}\,\mathrm{d}t = \frac{\sin(az)}{az}.$$

Let $\delta \in (0, 1)$ be a sufficiently small number to be chosen latter on and let $a_0 = a_1 = ... = a_{N+1} = \frac{\delta}{N+2}$. We consider the convolution $u = H_{a_0} * ... * H_{a_{N+1}}$ and we have the following result.

Lemma 3.4. *For $N \geq 0$, the function $u = H_{a_0} * ... * H_{a_{N+1}}$ belongs to $\mathcal{C}^N(\mathbb{R})$ and it is supported in $[-\delta, \delta]$. Moreover, the following estimates hold*

$$\int_{-\delta}^{\delta} u = \|u\|_1 = 1, \tag{3.13}$$

$$\int_{\mathbb{R}} \left|u^{(j)}(x)\right| \,\mathrm{dx} \leq \frac{1}{a_0 ... a_{j-1}} \qquad (1 \leq j \leq N). \tag{3.14}$$

Proof. It is similar to [7, Theorem 1.3.5] and we omit it. □

Remark 3.1. If u is the function defined in Lemma 3.4, then its support is $[-\delta, \delta]$. By taking into account that $u \in \mathcal{C}^N(\mathbb{R})$, it follows that

$$u(-\delta) = u(\delta) = u^{(j)}(-\delta) = u^{(j)}(\delta) = 0 \qquad (1 \leq j \leq N). \tag{3.15}$$

We introduce the function M_k given by

$$M_k(z) = \int_{-\delta}^{\delta} u(t)e^{-it(z+i\lambda_k)}\,\mathrm{d}t = \left(\frac{\sin\left(\frac{\delta}{N+2}(z + i\lambda_k)\right)}{\frac{\delta}{N+2}(z + i\lambda_k)} \right)^{N+2}. \tag{3.16}$$

The properties of the multiplier M_k are given in the following lemma.

Lemma 3.5. *M_k is an entire function of exponential type δ. Moreover, we have that*

$$M_k(-i\lambda_k) = 1, \tag{3.17}$$

and, for every $x \in \mathbb{R}$, we have the following estimate

$$|M_k(x)| \leq \min\left\{ \exp\left[\left(\frac{6}{\delta} + \delta\pi\right) N\right], \exp\left[-N\ln|x| + N\ln\left(\frac{3N}{\delta}\right) + \delta\lambda_k\right] \right\}. \tag{3.18}$$

Proof. Firstly, by using (3.13), we deduce the following estimate

$$|M_k(z)| \leq e^{\delta|z+i\lambda_k|}, \tag{3.19}$$

which implies that M_k is an entire function of exponential type δ.

Relation (3.17) follows from definition (3.16) of M_k. On the other hand, estimate (3.18) is obtained by using Lemma 3.4. Indeed, by performing several integrations by parts (justified by the regularity of u) and taking into account (3.15), we deduce that

$$M_k(x) = \frac{1}{-i(x+i\lambda_k)} \int_{-\delta}^{\delta} u(t) \left(e^{-it(x+i\lambda_k)}\right)' \mathrm{dt} = \frac{1}{i(x+i\lambda_k)} \int_{-\delta}^{\delta} u'(t) e^{-it(x+i\lambda_k)} \mathrm{dt}$$

$$= \frac{1}{i^2(x+i\lambda_k)^2} \int_{-\delta}^{\delta} u''(t) e^{-it(x+i\lambda_k)} \mathrm{dt} = \dots = \frac{1}{i^N(x+i\lambda_k)^N} \int_{-\delta}^{\delta} u^{(N)}(t) e^{-it(x+i\lambda_k)} \mathrm{dt}.$$

By using the above relations and (3.14) with $j=2$, it follows that

$$|M_k(x)| \leq \frac{1}{x^2+\lambda_k^2} \int_{-\delta}^{\delta} |u''(t)|\, e^{t\lambda_k} \mathrm{dt} \leq \frac{1}{x^2+\lambda_k^2} \frac{(N+2)^2}{\delta^2} e^{\delta\lambda_k} \leq \exp\left[\left(\frac{6}{\delta}+\delta\pi\right) N\right].$$

Moreover, (3.14) with $j=N$ implies that

$$|M_k(x)| \leq \frac{1}{(x^2+\lambda_k^2)^{\frac{N}{2}}} \int_{-\delta}^{\delta} \left|u^{(N)}(t)\right| e^{t\lambda_k} \,\mathrm{dt} \leq \frac{1}{(x^2+\lambda_k^2)^{\frac{N}{2}}} \frac{(N+2)^N}{\delta^N} e^{\delta\lambda_k}$$
$$\leq \exp\left[-N\ln|x| + N\ln\left(\frac{3N}{\delta}\right) + \delta\lambda_k\right],$$

which gives (3.18) and completes the proof of the lemma. □

3.3. The biorthogonal sequence. Now, we have all the ingredients needed to construct a biorthogonal sequence to the family of exponential functions Λ_N in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ and to estimate its norm.

Theorem 3.6. *Let $T>0$, $\alpha \in (0,1)$ and $(\lambda_j)_{1\leq j\leq N}$ be given by (1.2). There exists a biorthogonal sequence $(\theta_k(T,\cdot))_{1\leq k\leq N}$ to the family of exponential functions Λ_N in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ with the following property*

$$\|\theta_k(T,\cdot)\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leq C\exp\left(\omega N \ln N\right) \qquad (1\leq k\leq N), \tag{3.20}$$

where C and ω are positive constants independent of k and N.

Proof. Let $\delta \in (0,1)$ such that $T \geq 12\delta$. Let P_k and M_k be the functions defined by (3.1) and (3.16), respectively. We consider the function

$$\psi_k(z) := P_k(z)\left(M_k(z)\right)^5 \frac{\sin\left(\delta\left(z+i\lambda_k\right)\right)}{\delta(z+i\lambda_k)} \qquad (z\in\mathbb{C}). \tag{3.21}$$

Since P_k is a function of exponential type zero and M_k, $\dfrac{\sin\left(\delta\left(z+i\lambda_k\right)\right)}{\delta(z+i\lambda_k)}$ are functions of exponential type δ, we deduce that ψ_k is an entire function of exponential type $\frac{T}{2}$. Moreover, we have that

$$\psi_k(-i\lambda_j) = P_k(-i\lambda_j)\left(M_k(-i\lambda_j)\right)^5 \frac{\sin\left(\delta\left(-i\lambda_j+i\lambda_k\right)\right)}{\delta(-i\lambda_j+i\lambda_k)} = \delta_{kj}.$$

Next, we prove that $\psi_k(x) \in L^2(\mathbb{R})$. The estimate of the function $\widetilde{P}_k$ on the real axis from Lemma 3.3 combined with (3.18), allow us to evaluate the quantity $\left|\widetilde{P}_k(x)\right| |M_k(x)|^5$, by considering the following two cases:

a) The case $|x| \geq N^\alpha \pi^\alpha$. By using (3.11) and the second estimate of M_k in (3.18), we deduce that there exists $\omega_1 > 0$ such that

$$\begin{aligned} \left|\widetilde{P}_k(x)\right| |M_k(x)|^5 &\leq \exp\left[\frac{5N}{2}\ln\left(1+\frac{x^2}{N^{2\alpha}\pi^{2\alpha}}\right) - 5N\ln|x| + 5N\ln\left(\frac{3N}{\delta}\right) + 5\delta\lambda_k\right] \\ &= \exp\left[\frac{5N}{2}\ln\left(\frac{9N^2}{N^{2\alpha}\pi^{2\alpha}\delta^2}\right) + \frac{5N}{2}\ln\left(\frac{N^{2\alpha}\pi^{2\alpha}+x^2}{x^2}\right) + 5\delta\lambda_k\right] \\ &\leq \exp\left[\frac{5N}{2}\ln\left(\frac{9N^2}{N^{2\alpha}\pi^{2\alpha}\delta^2}\right) + \frac{5N}{2}\ln 2 + 5\delta\lambda_k\right] \leq \exp\left(\omega_1 N\ln N\right). \end{aligned} \tag{3.22}$$

b) The case $|x| < N^\alpha\pi^\alpha$. Estimate (3.11) of $\left|\widetilde{P}_k(x)\right|$ and the first estimate in (3.18) give that there exists $\omega_2 > 0$ such that

$$\left|\widetilde{P}_k(x)\right| |M_k(x)|^5 \leq \exp(\omega_2 N). \tag{3.23}$$

From (3.2), (3.22) and (3.23) it follows that

$$\begin{aligned} \int_{\mathbb{R}} |\psi_k(x)|^2 \,\mathrm{dx} &\leq \exp\left(2\omega_3 N\ln N\right)\int_{\mathbb{R}}\left|\frac{\sin\left(\delta\left(x+i\lambda_k\right)\right)}{\delta(x+i\lambda_k)}\right|^2 \mathrm{dx} \\ &\leq \frac{1}{\delta}\exp\left(2\omega_3 N\ln N + 2\lambda_k\right)\int_{\mathbb{R}}\left|\frac{\sin t}{t}\right|^2 \mathrm{dt} \leq 2\pi C^2\exp\left(2\omega N\ln N\right), \end{aligned}$$

where $\omega > \omega_3 := \ln 2 + \frac{2-\alpha}{\alpha(1-\alpha)} + \max\{\omega_1,\, \omega_2\}$ and C is a positive constant. From the last inequality we deduce that $\psi_k \in L^2(\mathbb{R})$ and

$$\|\psi_k\|_{L^2(\mathbb{R})} \leq \sqrt{2\pi}C\exp\left(\omega N\ln N\right). \tag{3.24}$$

Let us introduce the inverse Fourier transform of the function ψ_k:

$$\theta_k(T,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_k(x)e^{ixt}\,\mathrm{dx} \qquad (1 \leq j \leq N). \tag{3.25}$$

From Paley-Wiener Theorem we obtain that $(\theta_k(T,\cdot))_{1\leq k\leq N}$ given by (3.25) is a biorthogonal sequence to the family of exponential functions Λ_N in $L^2(-\frac{T}{2},\frac{T}{2})$. To obtain the norm estimate (3.20), we use (3.24) and Plancherel's Theorem. The proof of the theorem is complete. □

Remark 3.2. The norm of the biorthogonal sequence $(\theta_k(T,\cdot))_{1\leq k\leq N}$ given by (3.25) in Theorem 3.6 increases as $\exp(\omega N\ln N)$ as $N\to\infty$. What can be said about the norms of many other biorthogonals which can be found? In the following theorem we prove that we can find a positive constant ω' such that the norm of any biorthogonal sequence to the family Λ_N is bounded from below by $\exp(\omega' N\ln N)$ and in this sense, (3.20) is optimal.

Theorem 3.7. *Let $\alpha\in(0,1)$, $T>0$ and $(\xi_k)_{1\leq k\leq N}$ be a biorthogonal sequence to the family of exponential functions Λ_N in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$. There exists $N_0\in\mathbb{N}^*$, with the property that we can find two positive constants C and ω' such that*

$$\|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \geq C\exp\left[-\lambda_k\frac{T}{2} + \omega'(1-\alpha)N\ln N\right] \qquad (1\leq k\leq N,\, N\geq N_0). \tag{3.26}$$

Proof. We will give the proof in several steps and similar arguments from [4] are used.
Step 1. We define the following function:

$$\tau_k(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \xi_k(t) e^{-itz}\,\mathrm{dt} \qquad (1 \le k \le N). \tag{3.27}$$

From Paley-Wiener Theorem we deduce that τ_k is an entire function of exponential type $\frac{T}{2}$. Furthermore, we have that

$$|\tau_k(x)| \le \sqrt{T}\,\|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \qquad (x \in \mathbb{R}). \tag{3.28}$$

Since τ_k is a function of exponential type we deduce, from Hadamard's factorization theorem (see [19, Chapter 2, p.74]), that

$$\tau_k(z) = a z^p e^{bz} \prod_{z_m \in E} \left(1 - \frac{z}{z_m}\right) e^{z/z_m}, \tag{3.29}$$

where E is the set of the zeros z_m of τ_k with $z_m \ne 0$, $E = \{z_m \in \mathbb{C} |\, \tau_k(z_m) = 0,\ z_m \ne 0\}$.
From (3.27) and since $(\xi_k)_{1 \le k \le N}$ is a biorthogonal sequence to the family of exponential functions Λ_N in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ it follows that $\tau_k(-i\lambda_j) = \delta_{kj}$. Hence, $\{-i\lambda_j |\, 1 \le j \le N, j \ne k\} \subseteq E$.
Next, we introduce the function $\phi_k(z)$ defined by

$$\phi_k(z) := \frac{\tau_k(z)}{P_k(z)}, \tag{3.30}$$

where P_k is the product given by (3.1). Moreover, the function ϕ_k has the following properties:

- it is an entire function of exponential type $\frac{T}{2}$,
- $\phi_k(-i\lambda_k) = 1$,
- $\tau_k(z) = P_k(z)\phi_k(z)$.

Step 2. We will give estimates from below for $|P_k(z)|$. We have that

$$|P_k(z)| = \left| \prod_{\substack{1 \le p \le N \\ p \ne k}} \frac{\lambda_p - iz}{\lambda_p - \lambda_k} \right| = \left(\prod_{\substack{1 \le p \le N \\ p \ne k}} |\lambda_p - iz| \right) \left(\prod_{\substack{1 \le p \le N \\ p \ne k}} |\lambda_p - \lambda_k| \right)^{-1}.$$

For $z \in \mathbb{C}$ such that $|z| \ge 3N^\alpha \pi^\alpha$ it follows that

$$\prod_{\substack{1 \le p \le N \\ p \ne k}} |\lambda_p - iz| \ge \prod_{\substack{1 \le p \le N \\ p \ne k}} (|z| - \lambda_p) \ge \prod_{\substack{1 \le p \le N \\ p \ne k}} (|z| - N^\alpha \pi^\alpha) = (|z| - N^\alpha \pi^\alpha)^{N-1}. \tag{3.31}$$

Next,

$$\prod_{\substack{1 \le p \le N \\ p \ne k}} |\lambda_p - \lambda_k| \le \prod_{\substack{1 \le p \le N \\ p \ne k}} (|\lambda_p| + |\lambda_k|) \le \prod_{\substack{1 \le p \le N \\ p \ne k}} 2N^\alpha \pi^\alpha = (2N^\alpha \pi^\alpha)^{N-1}. \tag{3.32}$$

From (3.31) and (3.32) it follows that

$$|P_k(z)| \ge \left(\frac{|z| - N^\alpha \pi^\alpha}{2N^\alpha \pi^\alpha} \right)^{N-1} \qquad (z \in \mathbb{C},\ |z| \ge 3N^\alpha \pi^\alpha). \tag{3.33}$$

Step 3. The following estimate holds

$$|\tau_k(z)| \le \int_{-\frac{T}{2}}^{\frac{T}{2}} |\xi_k(t)|\, e^{t|\Im(z)|}\mathrm{dt} \le e^{\frac{T}{2}|\Im(z)|} \int_{-\frac{T}{2}}^{\frac{T}{2}} |\xi_k(t)|\,\mathrm{dt} \le e^{\frac{T}{2}|\Im(z)|}\sqrt{T}\, \|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}, \tag{3.34}$$

and from (3.33) and (3.34) we obtain that

$$|\phi_k(z)| \le \frac{\sqrt{T}e^{\frac{T}{2}|\Im(z)|}\, \|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}}{\left(\frac{|z|-N^\alpha\pi^\alpha}{2N^\alpha\pi^\alpha}\right)^{N-1}} \qquad (z \in \mathbb{C},\ |z| \ge 3N^\alpha\pi^\alpha). \tag{3.35}$$

We consider the function

$$\zeta_k : \mathbb{C} \to \mathbb{C}, \quad \zeta_k(z) = \phi_k(-i\lambda_k - z),$$

and we remark that ζ_k is an entire function with the property that $\zeta_k(0) = 1$. We apply Theorem 11 from [9, p. 21] and we deduce that, for any $R > 0$ and $\eta \in (0, \frac{3e}{2})$,

$$\ln(|\zeta_k(z)|) > -\left(2 + \ln\left(\frac{3e}{2\eta}\right)\right) \ln\left(M_{\zeta_k}(2eR)\right) \qquad (z \in \mathbb{C},\ |z| \le R), \tag{3.36}$$

outside of a set of circles the sum of whose radii is not greater than $4\eta R$, where $M_{\zeta_k}(2eR) = \max\limits_{|z|=2eR} |\zeta_k(z)|$. From (3.35) we have that

$$\begin{aligned} M_{\zeta_k}(2eR) &= \max_{|z|=2eR} |\phi_k(-i\lambda_k - z)| \le \sqrt{T}\, \|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \max_{|z|=2eR} e^{\frac{T}{2}|-\lambda_k - \Im(z)|} \\ &\le \sqrt{T}\, \|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} e^{\frac{T}{2}\lambda_k} e^{eRT}, \end{aligned}$$

if $|z| = 2eR \ge 3N^\alpha\pi^\alpha$. We denote $\beta := 2 + \ln\left(\frac{3e}{2\eta}\right) > 1$ and using the above estimate, for any $R > 0$ and $\eta \in (0, \frac{3e}{2})$ such that $2eR \ge 3N^\alpha\pi^\alpha$, it follows that

$$\ln(|\zeta_k(z)|) > -\beta \ln\left(\sqrt{T}\, \|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} e^{\frac{T}{2}\lambda_k + eRT}\right) \qquad (z \in \mathbb{C},\ |z| \le R), \tag{3.37}$$

outside of a set of circles the sum of whose radii is not greater than $4\eta R$.
We consider $R > 8N^\alpha\pi^\alpha$ and $\eta \in (0, \frac{1}{16})$. We deduce that there exists $x_0 \in [\frac{R}{2}, R]$ such that

$$\ln(|\zeta_k(x_0)|) > -\beta \ln\left(\sqrt{T}\, \|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} e^{\frac{T}{2}\lambda_k + eRT}\right). \tag{3.38}$$

On the other hand, from (3.35), we have that

$$|\zeta_k(x_0)| = |\phi_k(-i\lambda_k - x_0)| \le \frac{\sqrt{T}e^{\frac{T}{2}\lambda_k}\, \|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}}{\left(\frac{|-i\lambda_k - x_0| - N^\alpha\pi^\alpha}{2N^\alpha\pi^\alpha}\right)^{N-1}}. \tag{3.39}$$

Choosing, $R = 8N\pi > 8N^\alpha\pi^\alpha$ we obtain the estimate

$$|-i\lambda_k - x_0| - N^\alpha\pi^\alpha \ge ||x_0| - |-i\lambda_k|| - N^\alpha\pi^\alpha = x_0 - \lambda_k - N^\alpha\pi^\alpha \ge 4N\pi - 2N\pi = 2N\pi, \tag{3.40}$$

and for this, it follows that in (3.39),

$$|\zeta_k(x_0)| \le \frac{\sqrt{T}e^{\frac{T}{2}\lambda_k}\, \|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \left(N^\alpha\pi^\alpha\right)^{N-1}}{(N\pi)^{N-1}}. \tag{3.41}$$

From (3.38) and (3.41) the following estimate is obtained:

$$\ln\left(\frac{\sqrt{T}e^{\frac{T}{2}\lambda_k}\,\|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}\,(N\pi)^{\alpha(N-1)}}{(N\pi)^{N-1}}\right) > -\beta\ln\left(\sqrt{T}\,\|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}\,e^{\frac{T}{2}\lambda_k+8\pi eTN}\right),$$

which is equivalent to

$$(1+\beta)\ln\left(\|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}\right)$$
$$> -(1+\beta)\ln(\sqrt{T}) - (1+\beta)\frac{T}{2}\lambda_k + (1-\alpha)(N-1)\ln(N\pi) - \beta 8\pi eTN.$$

For sufficiently large N (depending on T and α) we deduce that

$$(1-\alpha)(N-1)\ln(N\pi) - \beta 8\pi eTN = (1-\alpha)(N-1)\ln N +$$
$$+(1-\alpha)(N-1)\ln\pi - \beta 8\pi eTN \geq \frac{(1-\alpha)}{2}N\ln N - \beta 8\pi eTN \geq \frac{1}{4}(1-\alpha)N\ln N.$$

Hence, we obtain

$$(1+\beta)\ln\left(\|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}\right) > -(1+\beta)\ln(\sqrt{T}) - (1+\beta)\frac{T}{2}\lambda_k + \frac{1}{4}(1-\alpha)N\ln N,$$

equivalent to

$$\|\xi_k\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} > \frac{1}{\sqrt{T}}e^{-\frac{T}{2}\lambda_k}e^{\frac{1-\alpha}{4(1+\beta)}N\ln N}. \tag{3.42}$$

Therefore (3.26) holds with $\omega' = \frac{1}{4(1+\beta)}$ and $C = \frac{1}{\sqrt{T}}$. The proof is complete. □

Now, we have all the ingredients needed to prove our main controllability result, Theorem 1.1.

4. Controllability results

In this section we firstly give the proof of the main result Theorem 1.1.

Proof of Theorem 1.1: We show the existence of a function g_N which verifies the moment problem (2.4). Indeed, let $(\theta_k(T,\cdot))_{1\leq k\leq N}$ be the biorthogonal sequence to the family of exponential functions Λ_N in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ given by Theorems 3.6. In this case the moment problem (2.4) is solved immediately by setting

$$g_N(T-t) = \sum_{k=1}^{N} -\frac{u_k^0}{f_k}e^{-\lambda_k\frac{T}{2}}\theta_k\left(T, t-\frac{T}{2}\right).$$

To estimate the norm of the control g_N we use proceed as follows:

$$\|g_N\|_{L^2(0,T)} \leq C\max\left\{\|\theta_k(T,\,\cdot\,)\|_{L^2(0,T)}\right\}\sum_{k=1}^{N}|u_k^0|e^{-(\nu+\frac{T\pi^\alpha}{2})k^\alpha},$$

and (1.16) follows immediately by taking into account (3.20). The proof of Theorem 1.1 is complete. □

The following result shows that the estimate given by Theorem 1.1 is optimal.

Theorem 4.1. *There exists initial datum $u^0 \in C^\infty(0,1)$ such that any N-finite dimensional control g_N of equation (1.1) verifies*

$$\|g_N\|_{L^2(0,T)} \geq C \exp\left[\omega'(1-\alpha)N \ln N\right], \tag{4.1}$$

where the constants C and ω' are given by Theorem 3.7.

Proof. Let $u^0(x) = -\sqrt{2} f_N e^{T\lambda_N} v_N$, where f_N is the N-th coefficient of the function $f = \sum_{j=1}^{\infty} \sqrt{2} f_j v_j \in L^2(0,1)$ and v_j are given by (1.3).

According to Theorem 2.2, any control g_N will verify

$$\int_0^T g_N(T-t) e^{-\lambda_j t}\, \mathrm{d}t = \delta_{jN} \qquad (1 \leq j \leq N),$$

since

$$u_j^0 = \sqrt{2} \int_0^1 u^0(x) \sin(j\pi x)\, \mathrm{d}x = \begin{cases} 0 & \text{if } j \neq N \\ -f_N e^{T\lambda_N} & \text{if } j = N. \end{cases}$$

Therefore,

$$g_N(T-t) = e^{\frac{T}{2}\lambda_N} \xi_N \left(t - \frac{T}{2}\right), \tag{4.2}$$

where ξ_N is the N-th element from a biorthogonal sequence $(\xi_k)_{1\leq k\leq N}$ to the family of exponential functions Λ_N in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$.

Using (3.26) from Theorem 3.7, and (4.2) we deduce that relation (4.1) holds, and the proof is complete. □

Remark 4.1. In this article we have considered that the exponent $\alpha \in (0,1)$. The case $\alpha = 1$ is also interesting and corresponds to the classical Cauchy distribution (see, for instance, [12]). Although the results in this case are qualitatively similar to the previous ones, the estimates are different and will be presented elsewhere.

References

[1] S.A. Avdonin, S.A. Ivanov, *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*, Cambridge Univ. Press, 1995.

[2] U. Biccari, V. Hernández-Santamaría, Controllability of a one-dimensional fractional heat equation: theoretical and numerical aspects, *IMA J. Math. Control Inf.* **36** (2019), 1199–1235.

[3] A. Einstein, Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, *Annalen der Physik* **322** (1905), 549–560.

[4] H.O. Fattorini, Estimates for sequences biorthogonal to certain complex exponentials and boundary control of the wave equation, *Lecture Notes in Control and Inform. Sci.* **2** (1979), 111–124.

[5] H.O. Fattorini, D.L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, *Arch. Ration. Mech. Anal.* **43** (1971), 272–292.

[6] W. Gawronski, On the Bell-Shape of Stable Densities, *Ann. Probab.* **12** (1984), 230–242.

[7] L. Hormander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis,* Classics in Mathematics, Springer-Verlag, Berlin, 2003.

[8] M. Kwaśnicki, Eigenvalues of the fractional Laplace operator in the interval, *J. Funct. Anal.* **262** (2012), 2379–2402.

[9] B.J. Levin, *Distribution of Zeros of Entire Functions*, AMS, Providence, RI, 1964.

[10] P. Lissy, *A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation*, 2020. hal-02420212v2.

[11] Q. Lü, E. Zuazua, On the lack of controllability of fractional in time ODE and PDE, *Math. Control Signals Syst.* **28** (2016), 1–21.
[12] R. Metzler, J. Klafter, The random walks guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* **339** (2000), 1–77.
[13] R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A: Math. Gen.* **37** (2004), no. 31, 161–208.
[14] S. Micu, E. Zuazua, On the controllability of a fractional order parabolic equation, *SIAM J. Control Optim.* **44** (2006), no. 6, 1950–1972.
[15] L. Miller, tOn the controllability of anomalous diffusions generated by the fractional Laplacian, *Math. Control Signals Systems* **18** (2006), 260–271.
[16] E.W. Montroll, J.T. Bendler, On Lévy (or stable) distributions and the Williams-Watts model of dielectric relaxation, *J. Stat. Phys.* ***34*** (1984), 129–162.
[17] A. Ott, J.-P. Bouchaud, D. Langevin, W. Urbach, Anomalous diffusion in "living polymers": A genuine Lévy flight?, *Phys. Rev. Lett.* **65** (1990), 2201–2204.
[18] W.R. Schneider, W. Wyss, Fractional diffusion and wave equations, *J. Math. Phys.* **30** (1989), no. 1, 134–144.
[19] R.M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New-York, 1980.
[20] E. Zuazua, Finite dimensional null controllability for the semilinear heat equation, *J. Math. Pures Appl.* **76** (1997), 237–264.

(Constantin Niţă) Department of Mathematics, University of Craiova, 200585, Romania.
E-mail address: nita_gyk@yahoo.com

(Laurenţiu Emanuel Temereancă) Department of Applied Mathematics, University of Craiova, 200585, Romania.
E-mail address: temereanca_laurentiu@yahoo.com