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Superstability of higher-order fractional differential equations

Abdellatif Ben Makhlouf

Abstract. Using generalized Taylor's formula, this work investigate the superstability for a class of fractional differential equations with Caputo derivative. In this way, some interesting results are generalized.

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1. Introduction and preliminaries

One of the important main research area in the theory of Functional Equations (FE) is the Hyers-Ulam stability (HUS). In the past, in 1940, the researcher Ulam proposed a problem regarding the stability of FE to give conditions for a linear mapping near an approximately linear mapping be in the talk at the University of Wisconsin. In 1941, author in [7] solved it. Recently, by replacing FE with Differential Equations (DE), a generalization of Ulam's Problem (UP) has been made and many studies obtained the HUS of DE [17, 18].

Fractional differential equations (FDE) is an important research field, recent investigation has been recorded in this area, this includes stability [3, 12, 13, 20], finite-time stability (FTS) [15], stabilization [14], observer design [9, 14] and fault estimation [10]. Nevertheless, the concept of Fractional Derivative (FD) is not new and is much as old as DE. First of all, in 1695, L'Hospital proposed the question regarding FD in a letter written to Leibniz and connected his generalization of DE. In the past few years, many researchers have investigated on the study of HUS of FDE and published an important number of works [1, 4, 5, 19].

Authors in [3] have proposed a novel concept named superstability (SS) which is a special case of HUS, they have studied the stability of the following FE: $\xi(\chi_1+\chi_2)=\xi(\chi_1)\xi(\chi_2)$. It is important to know that the earliest works related to SS of DE appeared in [6, 8]. To the best of our knowledge, there is no works in the literature which treats the same concept for the fractional order systems.

In this work, we will study the SS of the following initial value problem

$$^{C}D_r^{p\lambda}E(x)+A(x)E(x)=0, \tag{1}$$

with initial conditions (IC):

$$E(r)={}^{C}D_r^{\lambda}E(r)={}^{C}D_r^{2\lambda}E(r)=...={}^{C}D_r^{(p-1)\lambda}E(r)=0, \tag{2}$$

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where $p \in \mathbb{N}^*$, ${}^C D_r^{s\lambda} E \in C\big([r, r+u]\big)$, for each $s \in \{0, 1, ..., p\}$, $A \in C\big([r, r+u]\big)$, $u > 0$ and ${}^C D_r^{s\lambda} = {}^C D_r^{\lambda}.{}^C D_r^{\lambda}...{}^C D_r^{\lambda}$ ($s-$times).

Motivated by [6, 8], we introduce the following definition.

Definition 1.1. Suppose that E satisfies:

$$|\psi\big(A, E, {}^C D_r^{\lambda} E, {}^C D_r^{2\lambda} E, ..., {}^C D_r^{p\lambda} E\big)| \leq \nu, \ \forall \omega \in [r, r+u], \tag{3}$$

for some $\nu \geq 0$ with IC therefore either

$$|E(\omega)| \leq \vartheta \nu, \ \forall \omega \in [r, r+u],$$

where $\vartheta > 0$, or

$$\psi\big(A, E, {}^C D_r^{\lambda} E, {}^C D_r^{2\lambda} E, ..., {}^C D_r^{p\lambda} E\big) = 0.$$

Then, we say that (1) has SS with IC.

Definition 1.2. [11] Given $0 < l < 1$. The Caputo fractional derivative of an absolutely continuous function f is defined as,

$${}^C D_c^l f(s) = \frac{1}{\Gamma(1-l)} \int_c^s (s-\tau)^{-l} f'(\tau) d\tau. \tag{4}$$

Theorem 1.1. [16]*(Generalized Taylor's formulat) Let $0 < \eta < 1$. Assume that ${}^C D_{r_1}^{t\eta} h \in C\big([r_1, r_2]\big)$, for each $t \in \{0, 1, ..., s\}$, with $s \in \mathbb{N}^*$, then we have*

$$h(x) = \sum_{t=0}^{s-1} {}^C D_{r_1}^{t\eta} h(r_1) \frac{(x-r_1)^{t\eta}}{\Gamma(t\eta+1)} + {}^C D_{r_1}^{s\eta} h(c) \frac{(x-r_1)^{s\eta}}{\Gamma(s\eta+1)},$$

with $c \in [r_1, x]$, for each $x \in (r_1, r_2]$.

2. Main theorem

In this section, we present our main result.

Theorem 2.1. *Assume that $\sup_{\chi \in [r, r+u]} |A(\chi)| < \dfrac{\Gamma(p\lambda+1)}{u^{p\lambda}}$. Then, (1) has the SS with IC (2).*

Proof. Let $\nu > 0$, and $E \in C\big([r, r+u]\big)$ such that ${}^C D_r^{t\lambda} E \in C\big([r, r+u]\big)$ for each $t \in \{0, 1, ...p\}$, if

$$|{}^C D_r^{p\lambda} E(x) + A(x) E(x)| \leq \nu$$

and

$$E(r) = {}^C D_r^{\lambda} E(r) = {}^C D_r^{2\lambda} E(r) = ... = {}^C D_r^{(p-1)\lambda} E(r) = 0.$$

Using Theorem 1.1, we get

$$E(x) = \sum_{t=0}^{p-1} {}^C D_r^{t\lambda} E(r) \frac{(x-r)^{t\lambda}}{\Gamma(t\lambda+1)} + {}^C D_r^{p\lambda} E(c) \frac{(x-r)^{p\lambda}}{\Gamma(p\lambda+1)},$$

with $c \in [r, x]$, for every $x \in (r, r+u]$. Thus

$$\begin{aligned} |E(x)| &= |{}^C D_r^{p\lambda} E(c) \frac{(x-r)^{p\lambda}}{\Gamma(p\lambda+1)}| \\ &\leq \sup_{\chi \in [r, r+u]} |{}^C D_r^{p\lambda} E(\chi)| \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}. \end{aligned} \tag{5}$$

Then,

$$\begin{aligned}\sup_{\chi\in[r,r+u]}|E(\chi)| &\leq \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}\Big[\sup_{\chi\in[r,r+u]}|{}^{C}D_r^{p\lambda}E(\chi)-A(\chi)E(\chi)| \\ &\quad + \sup_{\chi\in[r,r+u]}|A(\chi)|\sup_{\chi\in[r,r+u]}|E(\chi)|\Big] \\ &\leq \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}\nu + \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}\sup_{\chi\in[r,r+u]}|A(\chi)|\sup_{\chi\in[r,r+u]}|E(\chi)|. \quad (6)\end{aligned}$$

Hence,

$$\sup_{\chi\in[r,r+u]}|E(\chi)|\Big(1-\frac{u^{p\lambda}}{\Gamma(p\lambda+1)}\sup_{\chi\in[r,r+u]}|A(\chi)|\Big) \leq \frac{u^{p\lambda}}{\Gamma(p\lambda+1)}\nu.$$

Therefore, there exists $K>0$ such that

$$|E(x)| \leq K\nu,$$

for all $x\in[r,r+u]$.
This complete the proof. □

Remark 2.1. It is important to note that in [8] authors have obtained the SS results for DE with integer-order derivatives while in our case, the main result is obtained for fractional-order derivatives. In this sense, our work present a full generalization of the interesting results in [8].

3. Conclusion

In this paper, the generalized Taylor formula is used to demonstrate the SS of FDE of higher-order under certain conditions.

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(Abdellatif Ben Makhlouf) Department of Mathematics, College of Science, Jouf University, P.O. Box: 2014, Sakaka, Saudi Arabia. ORCID ID 0000-0001-7142-7026
E-mail address: abmakhlouf@ju.edu.sa