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Lukasiewicz-Moisil algebra of fractions relative to an $\wedge$-closed system

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Abstract. The aim of this paper is to introduce the notion of n-valued Lukasiewicz-Moisil algebra of fractions relative to an $\wedge$-closed system.

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1. Definitions and preliminaries

Let n a natural number, $n \geq 2$.

Definition 1.1. *([1]) An n-***valued Lukasiewicz-Moisil algebra** *(or LM_n-***algebra***) is a structure $(L, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ of type $(2, 2, 1, 0, 0, \{1\}_{1\leq i\leq n-1})$, satisfying the following conditions:*
a) $(L, \wedge, \vee, N, 0, 1)$ is a De Morgan algebra,
b) $\varphi_1, ..., \varphi_{n-1} : L \to L$ are bounded distributive lattices morphisms such that for every $x \in L$:
b_1) $\varphi_i(x) \vee N\varphi_i(x) = 1$ for every $i = 1, ..., n-1$,
b_2) $\varphi_i(x) \wedge N\varphi_i(x) = 0$ for every $i = 1, ..., n-1$,
b_3) $\varphi_i\varphi_j(x) = \varphi_j(x)$ for every $i, j = 1, ..., n-1$,
b_4) $\varphi_i(Nx) = N\varphi_j(x)$ for every $i, j = 1, ..., n-1 cu$ $i + j = n$,
b_5) $\varphi_1(x) \leq \varphi_2(x) \leq ... \leq \varphi_{n-1}(x)$,
b_6) If $\varphi_i(x) = \varphi_i(y)$ for every $i = 1, ..., n-1$, then $x = y$.
The relation b_6) is called **the determination principle**.

As consequences of the determination principle we obtain:
b_7) If $x, y \in L$ then $x \leq y$ iff $\varphi_i(x) \leq \varphi_i(y)$ for every $i = 1, ..., n-1$,
b_8) $\varphi_1(x) \leq x \leq \varphi_{n-1}(x)$ for every $x \in L$.

We denote an LM_n-algebra $(L, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ by its universe L.

Remark 1.1. *The endomorphisms $\{\varphi_i\}_{1\leq i\leq n-1}$ are called* **chrysippian endo-morphisms**.

Examples:
E_1) Let $L_n = \{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\}$.We define $x \vee y = \max\{x, y\}, x \wedge y = \min\{x, y\}$, $Nx = 1 - x$ ($N(\frac{j}{n-1}) = \frac{n-1-j}{n-1}$) and $\varphi_i : L_n \to L_n, \varphi_i(\frac{j}{n-1}) = 0$ if $i + j < n$ and 1 if $i + j \geq n$, for $i, j = 1, ..., n-1$.
Then $(L_n, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ is an LM_n-algebra.

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E_2) Let $(B, \vee, \wedge, ', 0, 1)$ a Boolean algebra and $D(B) = \{(x_1, ..., x_{n-1}) \in B^{n-1}, x_1 \geq ... \geq x_{n-1}\}$. We define pointwise the infimum and the supremum, $N(x_1, ..., x_{n-1}) = (x'_{n-1}, ..., x'_1)$ and $\varphi_i(x_1, ..., x_{n-1}) = (x_i, ..., x_i)$ for all $i = 1, ..., n-1$.Then

$$(D(B), \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1 \leq i \leq n-1})$$

is an LM_n-algebra.

In the rest of this paper, by L we denote an LM_n-algebra.

We denote by $C(L)$ the set of all complemented elements of L and we call it *the center of* L. It is easy to see that $(B, \vee, \wedge, N, 0, 1)$ is a Boolean algebra.

Lemma 1.1. *(*[1]*) Let L an LM_n-algebra. The following are equivalent:*
(i) $x \in C(L)$,
(ii) *there are* $i \in \{1, ..., n-1\}$ *and* $y \in L$ *such that* $x = \varphi_i(y)$,
(iii) *there is* $i \in \{1, ..., n-1\}$ *such that* $x = \varphi_i(x)$,
(iv) $x = \varphi_i(x)$ *for every* $i = 1, ..., n-1$,
(v) $\varphi_i(x) = \varphi_j(x)$ *for every* $i, j = 1, ..., n-1$.

Remark 1.2. *If* $x \in L$ *then* $\varphi_i(x) \in C(L)$ *for every* $i = 1, ..., n-1$.

Lemma 1.2. *(*[1]*) Let L be an LM_n-algebra. The following are equivalent:*
(i) $x \in C(L)$,
(ii) $Nx \in C(L)$,
(iii) $x \wedge Nx = 0$,
(iv) $x \vee Nx = 1$.

Proposition 1.1. *(*[1]*) Every LM_n-algebra L is completely chrysippian, provided for every* $(a_s)_{s \in S} \subseteq L$ *we have:*
(c_1) *if* $\bigvee_{s \in S} a_s$ *exists then* $\varphi_i\left(\bigvee_{s \in S} a_s\right) = \bigvee_{s \in S} \varphi_i(a_s)$ *for every* $i = 1, ..., n-1$,
(c_2) *if* $\bigwedge_{s \in S} a_s$ *exists then* $\varphi_i\left(\bigwedge_{s \in S} a_s\right) = \bigwedge_{s \in S} \varphi_i(a_s)$ *for every* $i = 1, ..., n-1$.

Based on this proposition and using the determination principle it is easy to prove that the following properties hold for any LM_n-algebra L and for every $(a_s)_{s \in S} \subseteq L$:
(c_3) if $\bigvee_{s \in S} a_s$ exists then $a \wedge \left(\bigvee_{s \in S} a_s\right) = \bigvee_{s \in S} (a \wedge a_s)$ for every $a \in L$,
(c_4) if $\bigwedge_{s \in S} a_s$ exists then $a \vee \left(\bigwedge_{s \in S} a_s\right) = \bigwedge_{s \in S} (a \vee a_s)$ for every $a \in L$.

Definition 1.2. *(*[1]*) Let L and L' be LM_n-algebras. A function $f : L \to L'$ is a* **morphism of LM_n-algebras** *iff it satisfies the following conditions, for every* $x, y \in L$:
(i) $f(x \vee y) = f(x) \vee f(y)$,
(ii) $f(x \wedge y) = f(x) \wedge f(y)$,
(iii) $f(0) = 0, f(1) = 1$,
(iv) $f(\varphi_i(x)) = \varphi_i(f(x))$ *for every* $i = 1, ..., n-1$.

Remark 1.3. *It follows that:*

$$f(Nx) = Nf(x)$$

for every $x \in L$.

Definition 1.3. *([?]) Let L be an LM_n-algebra. We say that $I \subseteq L$ is an $n-$***ideal** *if I is an ideal of the lattice L and if $x \in I$ then $\varphi_{n-1}(x) \in I$.*

We denote by $Idn(L)$ the set of all $n-ideals$ of the algebra L.

Remark 1.4. *([?]) If I is an $n-$ ideal, then $I \cap C(L)$ is an ideal of the Boolean algebra $C(L)$.*

Remark 1.5. *([?]) If J is an ideal of the Boolean algebra $C(L)$, then $\varphi_{n-1}^{-1}(J)$ is an $n-$ideal.*

Definition 1.4. *([1])* **A congruence on LM_n-algebra** *L is an equivalence relation on L, compatible with the operations $\wedge, \vee, N, \varphi_i$, for every $i = 1, ..., n-1$.*

Proposition 1.2. *([1]) For a congruence relation ρ on LM_n-algebra L the follwing conditions are equivalent:*
(1) *ρ is a congruence on L,*
(2) *ρ is compatible with $\wedge, \vee, \varphi_i$, for every $i = 1, ..., n-1$.*

2. n-valued Lukasiewicz-Moisil algebra of fractions relative to an $\wedge$-closed system

Definition 2.1. *([?]) A nonempty subset $S \subseteq L$ is called $\wedge-$***closed system in** *L if:*
b_9) $1 \in S$,
b_{10}) *$x, y \in S$ implies $x \wedge y \in S$,*

We denote by $S(L)$ the set of all $\wedge-$closed systems of L (clearly $\{1\}$, $L \in S(L)$).
For $S \in S(L)$, on the LM_n-algebra L we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there is } e \in S \cap C(L) \text{ such that } x \wedge e = y \wedge e.$$

Lemma 2.1. *θ_S is a congruence on L.*

Proof. The reflexivity (since $1 \in S \cap C(L)$) and the symmetry of θ_S are immediately. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$. Thus there are $e, f \in S \cap C(L)$ such that $x \wedge e = y \wedge e$ and $y \wedge f = z \wedge f$. If denote $g = e \wedge f \in S \cap C(L)$, then $x \wedge g = x \wedge (e \wedge f) = (x \wedge e) \wedge f = (y \wedge e) \wedge f = (y \wedge f) \wedge e = (z \wedge f) \wedge e = z \wedge (f \wedge e) = z \wedge g$, hence $(x, z) \in \theta_S$.

To prove the compatibility of θ_S with the operations $\wedge, \vee$, and φ_i for every $i = 1, ..., n-1$, let $x, y, z, t \in L$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$. Thus there are $e, f \in S \cap C(L)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap C(L)$. Then $(x \wedge z) \wedge g = (y \wedge t) \wedge g$, hence $(x \wedge z, y \wedge t) \in \theta_S$.

From $x \wedge g = y \wedge g$ and $z \wedge g = t \wedge g$ we deduce $(x \vee z) \wedge g = (y \vee t) \wedge g$, that is $(x \vee z, y \vee t) \in \theta_S$.

From $x \wedge e = y \wedge e$ (with $e \in S \cap C(L)$) we deduce that for every $i = 1, ..., n-1$, $\varphi_i(x) \wedge \varphi_i(e) = \varphi_i(y) \wedge \varphi_i(e)$, that is $(\varphi_i(x), \varphi_i(y)) \in \theta_S$ (since $\varphi_i(e) = e \in S \cap C(L)$). □

For $x \in L$ we denote by x/S the equivalence class of x relative to θ_S and by

$$L[S] = L/\theta_S.$$

By $p_S : L \to L[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in L$. Clearly, in $L[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in L$,

$$x/S \wedge y/S = (x \wedge y)/S$$

$$x/S \vee y/S = (x \vee y)/S$$
$$N(x/S) = (Nx)/S$$
$$\bar{\varphi}_i : L[S] \to L[S], \bar{\varphi}_i(x/S) = (\varphi_i(x))/S \text{ for every } i = 1, ..., n-1.$$

So, p_S is an onto morphism of LM_n-algebras.

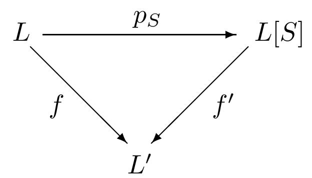
Remark 2.1. *Since for every $s \in S \cap C(L)$, $s \wedge s = s \wedge 1$, we deduce that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap C(L)) = \{\mathbf{1}\}$.*

Proposition 2.1. *If $a \in L$, then $a/S \in C(L[S])$ iff there is $e \in S \cap C(L)$ such that $e \wedge a \in C(L)$. So, if $e \in C(L)$, then $e/S \in C(L[S])$.*

Proof. For $a \in L$, we have $a/S \in C(L[S]) \Leftrightarrow \bar{\varphi}_i(a/S) = a/S$ for all $i = 1,..., n-1 \Leftrightarrow \varphi_i(a)/S = a/S$ for all $i = 1, ..., n-1 \Leftrightarrow (\varphi_i(a), a) \in \theta_S \Leftrightarrow$ there is $e \in S \cap C(L)$ such that $\varphi_i(a) \wedge e = a \wedge e \Leftrightarrow \varphi_i(a \wedge e) = a \wedge e \Leftrightarrow a \wedge e \in C(L)$.

If $e \in C(L)$, since $1 \in S \cap C(L)$ and $1 \wedge e = e \in C(L)$ we deduce that $e/S \in C(L[S])$. ☐

Theorem 2.1. *If L is an LM_n-algebra and $f : L \to L'$ is a morphism of LM_n-algebras such that $f(S \cap C(L)) = \{1\}$, then there is an unique morphism of LM_n-algebras $f' : L[S] \to L'$, such that the diagram*



is commutative (i.e. $f' \circ p_S = f$).

Proof. If $x, y \in L$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there is $e \in S \cap C(L)$ such that $x \wedge e = y \wedge e$. Since f is morphism of LM_n-algebras, we obtain that $f(x \wedge e) = f(y \wedge e) \Leftrightarrow f(x) \wedge f(e) = f(y) \wedge f(e) \Leftrightarrow f(x) \wedge 1 = f(y) \wedge 1 \Leftrightarrow f(x) = f(y)$.

From this observation we deduce that the map $f' : L[S] \to L'$ defined for $x \in L$ by $f'(x/S) = f(x)$ is correctly defined. Clearly, f' is a morphism of LM_n-algebras. The unicity of f' follows from the fact that p_S is an onto map. ☐

Remark 2.2. *The previous theorem allows us to call $L[S]$ the* **Lukasiewicz-Moisil $n-$valued algebra of fractions relative to the $\wedge-$closed system** *S.*

Examples:

(1) If $S = \{1\}$ or is such that $1 \in S$ and $S \cap (C(L)\backslash\{1\}) = \emptyset$, then for $x, y \in L, (x, y) \in \theta_S \Leftrightarrow 1 \wedge x = 1 \wedge y \Leftrightarrow x = y$, hence in this case $L[S] = L$.

(2) If S is an *$\wedge-$closed system* such that $0 \in S$ (for example $S = L$ or $S = C(L)$), then for every $x, y \in L, (x, y) \in \theta_S$ (since $x \wedge 0 = y \wedge 0$ and $0 \in S \cap C(L)$), hence in this case $L[S] = 0$.

(3) If $\mathcal{P}$ is a prime ideal of L (that is $\mathcal{P} \neq L$ and if $x \wedge y \in \mathcal{P}$ implies $x \in \mathcal{P}$ or $y \in \mathcal{P}$), such that $L\backslash P \subseteq C(L)$, then $S = L\backslash\mathcal{P}$ is an $\wedge-$closed system. We denote $L[S]$ by $L_{\mathcal{P}}$. The set $M = \{x/S : x \in \mathcal{P}\}$ is a maximal n-ideal of $L_{\mathcal{P}}$. Indeed, if $x, y \in \mathcal{P}$, then $x/S \vee y/S = (x \vee y)/S \in M$ (since $x \vee y \in \mathcal{P}$). If $x, y \in L$ such that $x \in \mathcal{P}$ and $y/S \leq x/S \Rightarrow y/S \wedge x/S = y/S \Rightarrow (y \wedge x, y) \in \theta_S$, so there exists $e \in S \cap C(L) = S$ such that $y \wedge x \wedge e = y \wedge e$, hence $y \wedge e \leq x$. Since $x \in \mathcal{P}$, then $y \wedge e \in \mathcal{P}$, hence $y \in \mathcal{P}$ (since $e \notin \mathcal{P}$), so $y/S \in M$. If $x \in \mathcal{P} \Rightarrow \varphi_{n-1}(x) \in \mathcal{P} \Rightarrow \bar{\varphi}_{n-1}(x/S) = (\varphi_{n-1}(x))/S \in M$. To prove the

maximality of M let I an n-ideal of $L_{\mathcal{P}}$ such that $M \subseteq M^{'}$ and $M \neq M^{'}$. Then there exists $x/S \in M^{'}$ such that $x/S \notin M$, (that is $x \notin \mathcal{P} \Longleftrightarrow x \in S$), hence $x/S = \mathbf{1}$ (see Remark 2.1) so $M^{'} = L_{\mathcal{P}}$. Moreover, M is the only maximal ideal of $L_{\mathcal{P}}$ (since if we have another maximal ideal $M^{''}$ of $L_{\mathcal{P}},$ then $M^{''} \nsubseteq M$ hence there exists $x/S \in M^{''}$ such that $x/S \notin M$, so $x/S = \mathbf{1}$ and $M^{''} = L_{\mathcal{P}}$, a contradiction!). In other words $L_{\mathcal{P}}$ is a *local* LM_n -algebra. The process of passing from L to $L_{\mathcal{P}}$ is called *localization* at $\mathcal{P}$.

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