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Indirect boundary stabilization with distributed delay of coupled multi-dimensional wave equations

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Abstract. In this article, our main concern is the study of the effect of a distributed time-delay in boundary stabilization of a strongly coupled multi-dimensional wave equations. We will establish that the system with time-delay inherits the same exponential decay rate from the corresponding one without delay.

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1. Introduction

Several practical processes might be modelled by distributed delay systems which present a wide range of applications in various fields such as micro-organism growth [24], hematopoiesis [1, 2], logistics [5] and traffic flow [20]. In the past four decades, many researchers have extensively investigated on the subject, and successfully applied them in more widespread other areas. They have developed mathematical tools in order to establish polynomial or exponential decays of these systems. We refer readers to [19] for a list of early works, and to [7, 8, 9, 12, 13, 14, 15, 21, 25, 26] and the references therein, for some other relevant results.

It is well known that the boundary delay term, which appears in certain practical problems, generates some instability effects that are largely studied in the literature. For more details, we refer the reader to the work of Nicaise and Pignotti [16] and the references therein. In the so called work, the authors considered the following wave equation with delay concentrated at τ for the system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \ \text{ in } \ \Omega \times (0,+\infty) \\ u(x,t) = 0 \ \text{ on } \ \Gamma_D \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \ \text{ on } \ \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \ \text{ in } \Omega \\ u(x,t-\tau) = f_0(x,t-\tau) \ \text{ on } \Gamma_N \times (0,\tau). \end{cases} \tag{1}$$

Under the condition $\mu_2 < \mu_1$, they investigated exponential stability results by combining inequalities due to Carleman estimates and compactness-uniqueness arguments. Later, they also obtain in [17] the exponential stability with distributed

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delay of the system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \ \text{ in } \ \Omega \times (0,+\infty) \\ u(x,t) = 0 \ \text{ on } \ \Gamma_D \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x,t-s)\, ds = 0 \ \text{ on } \ \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \ \text{ in } \Omega \\ u(x,t-\tau) = f_0(x,t-\tau) \ \text{ on } \Gamma_N \times (0,\tau), \end{cases} \tag{2}$$

under the assumption

$$\int_{\tau_1}^{\tau_2} \mu_2(s) ds < \mu_1 \tag{3}$$

by introducing suitable energies and by proving some observability inequalities.

In this paper, we investigate the study of the strongly coupled multi-dimensional wave equations with distributed delay in an open bounded domain Ω of $\mathbb{R}^n$ with smooth boundary Γ of class C^2 such that $\Gamma = \Gamma_D \cup \Gamma_N$ and $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$. More precisely, the main purpose of this work is to outline the stability results of the following abstract problem :

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - b y_t(x,t) = 0 \text{ in } \ \Omega \times (0,+\infty) \\ y_{tt}(x,t) - \Delta y(x,t) + b u_t(x,t) = 0 \text{ in } \ \Omega \times (0,+\infty) \\ u = 0 \ \text{ on } \ \Gamma_D \times (0,+\infty) \\ y = 0 \ \text{ on } \ \Gamma \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \beta_1 u_t(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) u_t(x,t-s) ds = 0 \ \text{ on } \ \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \ \text{ and } \ u_t(x,0) = u_1(x) \text{ in } \ \Omega \\ y(x,0) = y_0(x) \ \text{ and } \ y_t(x,0) = y_1(x) \text{ in } \ \Omega \\ u_t(x,-t) = f_0(x,-t) \ \text{ on } \Gamma_N \times (0,\tau_2). \end{cases} \tag{4}$$

Here and throughout the work, τ_1 and τ_2 are two real numbers with

$$0 \leq \tau_1 < \tau_2.$$

Moreover, β_1 and b are positive constants and the initial data $(u_0, u_1, y_0, y_1, f_0)^\top$ belong to a suitable space. Also, we assume that $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a positive L^∞ function verifying

$$\beta_1 > \int_{\tau_1}^{\tau_2} \beta_2(s) ds. \tag{5}$$

In the absence of delay (i.e., $\beta_2 = 0$), the system (4) becomes

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - b y_t(x,t) = 0 \text{ in } \ \Omega \times (0,+\infty) \\ y_{tt}(x,t) - \Delta y(x,t) + b u_t(x,t) = 0 \text{ in } \ \Omega \times (0,+\infty) \\ u = 0 \ \text{ on } \ \Gamma_D \times (0,+\infty) \\ y = 0 \ \text{ on } \ \Gamma \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \beta u_t(x,t) = 0 \ \text{ on } \ \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \ \text{ and } \ u_t(x,0) = u_1(x) \text{ in } \ \Omega \\ y(x,0) = y_0(x) \ \text{ and } \ y_t(x,0) = y_1(x) \text{ in } \ \Omega, \end{cases}$$

and it has been proven in [23] that the above system when $\beta = 1$ decays exponentially. In [3], Ammar-Khodja and Bader have studied the simultaneous stability of the system (4) in the one-dimensional case. They established that the system is exponentially stable. But in our knowledge, none of the authors cited above has inserted any distributed delay in the control. In this paper, our attention is focused on strongly coupled multi-dimensional wave equations with distributed delay involving in the boundary control for b small enough. Our goal in this paper is to prove that the system (4) inherits the same exponential decay from the system without delay.

The paper is organized as follows: the section 2 is devoted to the well-posedness of the problem (4) while the section 3 deals with the strong stability of problem (4). Finally, the section 4 gives an exponential stability result.

2. Well-posedness

In this section, we will give the well-posedness for the problem (4) using the semigroup theory, and establish strong stability result. Let us set

$$w(x,\rho,t,s) = u_t(x,t-\rho s), \quad \rho \in (0,1),\ s \in (\tau_1,\tau_2),\ t>0. \tag{6}$$

Then, the problem (4) is now equivalent to the following abstract problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - by_t(x,t) = 0 \text{ on } \ \Omega \times (0,+\infty) \\ y_{tt}(x,t) - \Delta y(x,t) + bu_t(x,t) = 0 \text{ in } \ \Omega \times (0,+\infty) \\ sw_t(x,\rho,t,s) + w_\rho(x,\rho,t,s) = 0 \ \text{ on } \ \Gamma_N \times (0,1) \times (0,+\infty) \times (\tau_1,\tau_2) \\ u = 0 \ \text{ on } \ \Gamma_D \times (0,+\infty) \\ y = 0 \ \text{ on } \ \Gamma \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \beta_1 u_t(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) w(x,1,t,s)ds = 0 \ \text{ on } \ \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \ \text{ and } \ u_t(x,0) = u_1(x) \text{ in } \ \Omega \\ y(x,0) = y_0(x) \ \text{ and } \ y_t(x,0) = y_1(x) \text{ in } \ \Omega \\ w(x,0,t,s) = u_t(x,t) \ \text{ on } \ \Gamma_N \times (0,+\infty) \times (\tau_1,\tau_2) \\ w(x,\rho,0,s) = f_0(x,-\rho s) \ \text{ on } \Gamma_N \times (0,\tau_2). \end{cases} \tag{7}$$

Setting

$$\mathcal{U} = \left(u, u_t, y, y_t, w\right)^{\mathsf{T}}.$$

Then we have

$$\mathcal{U}_t = (u_t, u_{tt}, y_t, y_{tt}, w_t)^{\mathsf{T}} = \left(u_t, \Delta u + by_t, y_t, \Delta y - bu_t, -s^{-1}w_\rho\right)^{\mathsf{T}}.$$

Therefore, the problem (7) can be rewritten in an abstract framework:

$$\begin{cases} \mathcal{U}_t = \mathcal{A}\mathcal{U} \\ \mathcal{U}(0) = (u_0, u_1, y_0, y_1, f_0(-\cdot s))^{\mathsf{T}}, \end{cases} \tag{8}$$

where the linear unbounded operator $\mathcal{A}$ is defined by

$$\mathcal{A}\left(u,v,y,z,w\right)^{\mathsf{T}} = \left(v, \Delta u + bz, z, \Delta y - bv, -s^{-1}w_\rho\right)^{\mathsf{T}},$$

with domain

$$\mathcal{D}(\mathcal{A}) =$$

$$= \left\{ \begin{array}{l} (u,v,y,z,w)^{\mathsf{T}} \in \left(H^2(\Omega) \cap V\right) \times V \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega) \times L^2\left((\tau_1,\tau_2); H^1(\Omega)\right) \\ w(x,0,s) = v(x) \text{ on } \Gamma_N \text{ and } \dfrac{\partial u}{\partial \nu}(x) = -\beta_1 v(x) - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) w(x,1,s)ds \text{ on } \Gamma_N \end{array} \right\},$$

such that

$$V = \left\{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_D\right\}.$$

Let us now introduce the Hilbert space

$$\mathcal{H} = V \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2\left((\tau_1, \tau_2); L^2(\Omega)\right)$$

endowed with the following norm

$$\begin{aligned}\left\|(u, v, y, z, w)^{\mathsf{T}}\right\|_{\mathcal{H}}^2 =& \|\nabla u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 + \|\nabla y\|_{L^2(\Omega)}^2 + \|z\|_{L^2(\Omega)}^2 \\ &+ \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |w(\cdot, \rho, s)|^2 d\rho\right) ds d\Gamma.\end{aligned}$$

So, the the natural associated inner product is the following

$$\begin{aligned}\left\langle \begin{pmatrix} u \\ v \\ y \\ z \\ w \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ y^* \\ z^* \\ w^* \end{pmatrix} \right\rangle_{\mathcal{H}} &= \int_\Omega \left(\nabla u \nabla \overline{u^*} + v\overline{v^*} + \nabla y \nabla \overline{y^*} + z\overline{z^*}\right) dx \\ &+ \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 w\overline{w^*} d\rho\right) ds d\Gamma.\end{aligned}$$

Proposition 2.1. *The operator $\mathcal{A}$ defined above is m-dissipative.*

Proof. Take $U = (u, v, y, z, w)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$. Then we have

$$\begin{aligned}\left\langle \mathcal{A} \begin{pmatrix} u \\ v \\ y \\ z \\ w \end{pmatrix}, \begin{pmatrix} u \\ v \\ y \\ z \\ w \end{pmatrix} \right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} v \\ \Delta u + bz \\ z \\ \Delta y - bv \\ -s^{-1} w_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ y \\ z \\ w \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \int_\Omega \left(\nabla v \nabla \overline{u} + (\Delta u + bz)\overline{v} + \nabla z \nabla \overline{y} + (\Delta y - bv)\overline{z}\right) dx \\ &- \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 w_\rho \overline{w} d\rho\right) ds d\Gamma.\end{aligned}$$

Using Green formula, Cauchy Schwarz's inequality and the definition of $\mathcal{D}(\mathcal{A})$ we obtain

$$\begin{aligned}
&\Re\left\langle \mathcal{A}\begin{pmatrix} u\\ v\\ y\\ z\\ w\end{pmatrix}, \begin{pmatrix} u\\ v\\ y\\ z\\ w\end{pmatrix}\right\rangle_{\mathcal{H}} \\
= \;& \Re\left(\int_\Omega \nabla v\nabla\overline{u}dx + (\Delta u + bz)\overline{v} + \nabla z\nabla\overline{y} + (\Delta y - bv)\overline{z}dx\right. \\
& \left. - \int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 w_\rho\overline{w}d\rho\right)dsd\Gamma\right) \\
= \;& \Re\left(\int_\Omega \nabla v\nabla\overline{u}dx - \int_\Omega \nabla u\nabla\overline{v}dx + \int_{\Gamma_N}\frac{\partial u}{\partial\nu}\overline{v}d\Gamma + b\int_\Omega z\overline{v}dx + \int_\Omega \nabla z\nabla\overline{y}dx\right. \\
& \left. - \int_\Omega \nabla y\nabla\overline{z}dx + \int_\Gamma \frac{\partial y}{\partial\nu}\overline{z}d\Gamma - b\int_\Omega v\overline{z}dx\right) - \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left[|w(x,\rho,s)|^2\right]_0^1 dsd\Gamma \\
= \;& \Re\left(\int_{\Gamma_N}\frac{\partial u}{\partial\nu}\overline{v}d\Gamma\right) - \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(x,1,s)|^2 dsd\Gamma \\
& + \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(x,0,s)|^2 dsd\Gamma \\
= \;& \Re\left(\int_{\Gamma_N}\left(-\beta_1 v - \int_{\tau_1}^{\tau_2}\beta_2(s)w(x,1,s)dsd\Gamma\right)\overline{v}d\Gamma\right) \\
& - \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(x,1,s)|^2 dsd\Gamma + \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(x,0,s)|^2 dsd\Gamma \\
= \;& -\beta_1\int_{\Gamma_N}|v(x)|^2 d\Gamma - \Re\left(\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)w(x,1,s)\overline{v}dsd\Gamma\right) \\
& - \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(x,1,s)|^2 dsd\Gamma + \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(x,0,s)|^2 dsd\Gamma.
\end{aligned}$$

Hence reminding that $w(x,0,s) = v(x)$ and using Young's inequality we find that

$$\begin{aligned}
\Re\left\langle \mathcal{A}\begin{pmatrix} u\\ v\\ y\\ z\\ w\end{pmatrix}, \begin{pmatrix} u\\ v\\ y\\ z\\ w\end{pmatrix}\right\rangle_{\mathcal{H}} \leq & -\beta_1\int_{\Gamma_N}|v(x)|^2 d\Gamma + \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|v(x)|^2 dsd\Gamma \\
& + \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(x,1,s)|^2 dsd\Gamma \\
& - \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(x,1,s)|^2 dsd\Gamma + \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|v(x)|^2 dsd\Gamma \\
\leq & \left(-\beta_1 + \int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)\int_{\Gamma_N}|v(x)|^2 d\Gamma.
\end{aligned}$$

We deduce that $\mathcal{A}$ is dissipative thanks to the relation (5).

Now we will show that $\lambda I - \mathcal{A}$ is surjective for at least one $\lambda > 0$. For that purpose, putting $(f_1, f_2, f_3, f_4, f_5)^{\mathsf{T}} \in \mathcal{H}$, we look for $(u, v, y, z, w)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ solution of the equation

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix}. \tag{9}$$

Therefore,

$$\begin{cases} \lambda u - v = f_1 \\ \lambda v - \Delta u - bz = f_2 \\ \lambda y - z = f_3 \\ \lambda z - \Delta y + bv = f_4 \\ \lambda w + s^{-1} w_\rho = f_5. \end{cases} \tag{10}$$

Suppose that we have found u with the right regularity. Then, we set

$$v = \lambda u - f_1 \tag{11}$$

and we can determine z.

From the definition of $\mathcal{D}(\mathcal{A})$ we recall that

$$w(x, 0, s) = v(x), \text{ for } x \in \Gamma_N, \; s \in (\tau_1, \tau_2); \tag{12}$$

and from (10),

$$\lambda w(x, \rho, s) + s^{-1} w_\rho(x, \rho, s) = f_5(x, \rho, s), \text{ for } x \in \Gamma_N, \; \rho \in (0, 1), \; s \in (\tau_1, \tau_2). \tag{13}$$

Consequently, by (12) and (13) we obtain

$$w(x, \rho, s) = e^{-\lambda \rho s} v(x) + s e^{-\lambda \rho s} \int_0^\rho f_5(x, \sigma, s) e^{-\lambda \sigma s} d\sigma. \tag{14}$$

Using (11) we get

$$w(x, \rho, s) = \lambda u(x) e^{-\lambda \rho s} - f_1(x) e^{-\lambda \rho s} + s e^{-\lambda \rho s} \int_0^\rho f_5(x, \sigma, s) e^{-\lambda \sigma s} d\sigma \tag{15}$$

and in particular

$$w(x, 1, s) = \lambda u(x) e^{-\lambda s} + g(x, s) \tag{16}$$

where g is a $L^2(\Gamma_N \times (\tau_1, \tau_2))$ function defined by

$$g(x, s) = -f_1(x) e^{-\lambda s} + s e^{-\lambda s} \int_0^1 f_5(x, \sigma, s) e^{-\lambda \sigma s} d\sigma. \tag{17}$$

Eliminating v and z in (10) we get

$$\lambda^2 u - \Delta u - \lambda b y = \lambda f_1 + f_2 - b f_3 \in L^2(\Omega) \tag{18}$$

$$\lambda^2 y - \Delta y + \lambda b u = b f_1 + \lambda f_3 + f_4 \in L^2(\Omega) \tag{19}$$

Let $\phi = (\varphi_1, \varphi_2) \in V \times H_0^1(\Omega)$ be a test function. Then (18) can be reformulated in the following variational form

$$\int_\Omega \left(\lambda^2 u - \Delta u - \lambda b y\right) \varphi_1 dx = \int_\Omega (\lambda f_1 + f_2 - b f_3) \varphi_1 dx. \tag{20}$$

Integrating the left hand side of (20) by parts, and using (16) it follows that

$$\begin{aligned}
&\int_\Omega \left(\lambda^2 u - \Delta u - \lambda b y\right) \varphi_1 dx = \\
&= \lambda^2 \int_\Omega u\varphi_1 dx + \int_\Omega \nabla u \nabla \varphi_1 dx - \int_{\Gamma_N} \frac{\partial u}{\partial \nu} \varphi_1 d\Gamma - \lambda b \int_\Omega y \varphi_1 dx \\
&= \lambda^2 \int_\Omega u\varphi_1 dx + \int_\Omega \nabla u \nabla \varphi_1 dx \\
&\quad - \int_{\Gamma_N} \left(-\beta_1 v - \int_{\tau_1}^{\tau_2} \beta_2(s) w(x,1,s) ds\right) \varphi_1 d\Gamma - \lambda b \int_\Omega y\varphi_1 dx \\
&= \lambda^2 \int_\Omega u\varphi_1 dx + \int_\Omega \nabla u \nabla \varphi_1 dx + \beta_1 \int_{\Gamma_N} v\varphi_1 d\Gamma \\
&\quad + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) \left(\lambda u(x) e^{-\lambda s} + g(x,s)\right) \varphi_1 ds d\Gamma - \lambda b \int_\Omega y \varphi_1 dx \\
&= \lambda^2 \int_\Omega u\varphi_1 dx + \int_\Omega \nabla u \nabla \varphi_1 dx + \beta_1 \int_{\Gamma_N} \left(\lambda u - f_1\right) \varphi_1 d\Gamma \\
&\quad + \lambda \int_{\Gamma_N} u\varphi_1 \int_{\tau_1}^{\tau_2} \beta_2(s) e^{-\lambda s} ds d\Gamma + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) g(x,s) \varphi_1 ds d\Gamma - \lambda b \int_\Omega y\varphi_1 dx \\
&= \lambda^2 \int_\Omega u\varphi_1 dx + \int_\Omega \nabla u \nabla \varphi_1 dx + \lambda\beta_1 \int_{\Gamma_N} u\varphi_1 d\Gamma - \beta_1 \int_{\Gamma_N} f_1 \varphi_1 d\Gamma \\
&\quad + \lambda \int_{\Gamma_N} u\varphi_1 \int_{\tau_1}^{\tau_2} \beta_2(s) e^{-\lambda s} ds d\Gamma + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) g(x,s) \varphi_1 ds d\Gamma - \lambda b \int_\Omega y\varphi_1 dx.
\end{aligned}$$

So (20) can be rewritten as

$$\begin{aligned}
&\lambda^2 \int_\Omega u\varphi_1 dx + \int_\Omega \nabla u \nabla \varphi_1 dx + \lambda\beta_1 \int_{\Gamma_N} u\varphi_1 d\Gamma - \beta_1 \int_{\Gamma_N} f_1 \varphi_1 d\Gamma \\
&\quad + \lambda \int_{\Gamma_N} u\varphi_1 \int_{\tau_1}^{\tau_2} \beta_2(s) e^{-\lambda s} ds d\Gamma + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) g(x,s) \varphi_1 ds d\Gamma - \lambda b \int_\Omega y\varphi_1 dx \\
&= \int_\Omega \left(\lambda f_1 + f_2 - b f_3\right) \varphi_1 dx
\end{aligned}$$

that is

$$\begin{aligned}
&\lambda^2 \int_\Omega u\varphi_1 dx + \int_\Omega \nabla u \nabla \varphi_1 dx + \lambda\beta_1 \int_{\Gamma_N} u\varphi_1 d\Gamma + \lambda \int_{\Gamma_N} u\varphi_1 \int_{\tau_1}^{\tau_2} \beta_2(s) e^{-\lambda s} ds d\Gamma - \lambda b \int_\Omega y\varphi_1 dx \\
&= \int_\Omega \left(\lambda f_1 + f_2 - b f_3\right) \varphi_1 dx + \beta_1 \int_{\Gamma_N} f_1 \varphi_1 d\Gamma - \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) g(x,s) \varphi_1 ds d\Gamma. \qquad (21)
\end{aligned}$$

Analogously (19) can be reformulated in the following variational form

$$\int_\Omega \left(\lambda^2 y - \Delta y + \lambda b u\right) \varphi_2 dx = \int_\Omega \left(b f_1 + \lambda f_3 + f_4\right) \varphi_2 dx. \qquad (22)$$

Integrating the left hand side of (22) by parts it follows that

$$\begin{aligned}\int_\Omega \left(\lambda^2 y-\Delta y+\lambda bu\right)\varphi_2 dx = &\ \lambda^2\int_\Omega y\varphi_2 dx+\int_\Omega \nabla y\nabla\varphi_2 dx-\int_\Gamma \frac{\partial y}{\partial\nu}\varphi_2 d\Gamma+\lambda b\int_\Omega u\varphi_2 dx\\ = &\ \lambda^2\int_\Omega y\varphi_2 dx+\int_\Omega \nabla y\nabla\varphi_2 dx+\lambda b\int_\Omega u\varphi_2 dx.\end{aligned}$$

Then (22) can be rewritten as

$$\lambda^2\int_\Omega y\varphi_2 dx+\int_\Omega \nabla y\nabla\varphi_2 dx+\lambda b\int_\Omega u\varphi_2 dx=\int_\Omega \left(bf_1+\lambda f_3+f_4\right)\varphi_2 dx. \tag{23}$$

Combining (21) and (23) we get

$$a_\lambda\left((u,y),(\varphi_1,\varphi_2)\right)=L_\lambda(\varphi_1,\varphi_2) \tag{24}$$

where a_λ is a bilinear form defined by

$$\begin{aligned}a_\lambda\left((u,y),(\varphi_1,\varphi_2)\right)=&\lambda^2\int_\Omega u\varphi_1 dx+\int_\Omega \nabla u\nabla\varphi_1 dx+\lambda\beta_1\int_{\Gamma_N} u\varphi_1 d\Gamma\\ &+\lambda\int_{\Gamma_N} u\varphi_1\int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}dsd\Gamma-\lambda b\int_\Omega y\varphi_1 dx+\lambda^2\int_\Omega y\varphi_2 dx\\ &+\int_\Omega \nabla y\nabla\varphi_2 dx+\lambda b\int_\Omega u\varphi_2 dx\end{aligned} \tag{25}$$

and L_λ a linear form defined by

$$\begin{aligned}L_\lambda(\varphi_1,\varphi_2)=&\int_\Omega \left(\lambda f_1+f_2-bf_3\right)\varphi_1 dx+\beta_1\int_{\Gamma_N} f_1\varphi_1 d\Gamma\\ &-\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)g(x,s)\varphi_1 dsd\Gamma+\int_\Omega \left(bf_1+\lambda f_3+f_4\right)\varphi_2 dx.\end{aligned} \tag{26}$$

It is clear that a_λ is continuous and coercive in $\left(V\times H_0^1(\Omega)\right)^2$ and L_λ is continuous in $V\times H_0^1(\Omega)$. Thanks to the Lax-Milgram theorem the variational equation (24) admits a unique $(u,y)\in V\times H_0^1(\Omega)$. Furthermore (u,y) is a weak solution of (18)-(19) associated to the following boundary conditions

$$\begin{cases} u=0 \text{ on } \Gamma_D\\ \dfrac{\partial u}{\partial\nu}=-\beta_1 v-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)w(x,1,s)ds \ \text{ on } \Gamma_N\\ y=0 \text{ on } \Gamma.\end{cases} \tag{27}$$

Then the classical elliptic theory (see [11], chapter 2), implies that the weak solution (u,y) of (18)-(19) associated to the boundary conditions (27) belongs to the space $H^2(\Omega)\times H^2(\Omega)$. Finally we have found $(u,v,y,z,w)^\top\in\mathcal{D}(\mathcal{A})$ which verifies (9). This shows that the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and then generates a $\mathcal{C}_0$-semigroup of contractions in $\mathcal{H}$. Thanks to Lumer-Phillips' theorem, problem (4) is well posed. □

3. Strong stability

In this section, we will prove that the system (4) is strongly stable using the spectral decomposition theory of Sz-Nagy-Foias and Foguel [4, 6, 22]. Following this theory, it suffices to establish that $\mathcal{A}$ has no eigenvalue on the imaginary axis, since the resolvent of $\mathcal{A}$ is compact.

First of all we assume that the following geometric control condition is satisfied, that is: there exist $\delta > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \geq \delta, \ \forall \ x \in \Gamma_N \quad \text{and} \quad m \cdot \nu \leq 0, \ \forall \ x \in \Gamma_D \tag{28}$$

where $m = x - x_0$.

Lemma 3.1. *Assume that b is small enough. There is no eigenvalue of $\mathcal{A}$ on the imaginary axis, that is*

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

Proof. Let $i\lambda$ be an eigenvalue of $\mathcal{A}$ and $U = (u, v, y, z, w)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ the associated eigenvector. Then we have

$$\mathcal{A}U = i\lambda U. \tag{29}$$

Using (29) and the dissipativeness of $\mathcal{A}$, we get

$$0 = \Re\left(i\lambda \|U\|_{\mathcal{H}}^2\right) = \Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} \leq \left(-\beta_1 + \int_{\tau_1}^{\tau_2} \beta_2(s) ds\right) \int_{\Gamma_N} |v(x)|^2 d\Gamma \leq 0.$$

We deduce that

$$\int_{\Gamma_N} |v(x)|^2 d\Gamma = 0.$$

Consequently,

$$v = 0 \quad \text{on} \quad \Gamma_N. \tag{30}$$

The equation (29) can be formulated as

$$\begin{cases} v = i\lambda u \\ \Delta u + bz = i\lambda v \\ z = i\lambda y \\ \Delta y - bv = i\lambda z \\ s^{-1} w_\rho = i\lambda w \end{cases} \tag{31}$$

with boundary conditions

$$\begin{cases} u = 0 \ \text{ on } \ \Gamma_D \\ \dfrac{\partial u}{\partial \nu} = -\beta_1 v - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) w(x, 1, s) ds \ \text{ on } \ \Gamma_N \\ y = 0 \ \text{ on } \ \Gamma. \end{cases} \tag{32}$$

Recalling the definition of $\mathcal{D}(\mathcal{A})$ and using (30) it follows that $w(x, 0, s) = 0$ on Γ_N. Then from the last equation of (31) we have the system

$$\begin{cases} w_\rho - i\lambda s w = 0 \\ w(x, 0, s) = 0 \end{cases} \tag{33}$$

which admits a unique solution $w = 0$. Consequently the boundary conditions become

$$\begin{cases} u = 0 \ \text{ on } \ \Gamma_D \\ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \Gamma_N \\ y = 0 \ \text{ on } \ \Gamma. \end{cases} \tag{34}$$

If $\lambda = 0$ then $v = z = 0$. Consequently, using (34) we obtain from (31)

$$\begin{cases} \Delta u = 0 \\ u = 0 \ \text{ on } \ \Gamma_D \\ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \Gamma_N \end{cases} \tag{35}$$

and

$$\begin{cases} \Delta y = 0 \\ y = 0 \ \text{ on } \ \Gamma. \end{cases} \tag{36}$$

It is obvious that (35) and (36) admit respectively a unique solution $u = 0$ and $y = 0$. In short we get $U = 0$ which contradicts the fact that U is an eigenvector.
In the sequel, we assume that $\lambda \neq 0$. Using (30) and the first equation of (31) it follows that

$$u = 0 \ \text{ on } \ \Gamma_N. \tag{37}$$

Eliminating v and z in (31) we get the following system

$$\begin{cases} \lambda^2 u + \Delta u + ib\lambda y = 0 \\ \lambda^2 y + \Delta y - ib\lambda u = 0 \\ u = 0 \ \text{ on } \ \Gamma \\ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \Gamma_N \\ y = 0 \ \text{ on } \ \Gamma. \end{cases} \tag{38}$$

If we use the fact that b is small enough and proceeding as in Toufayli (see [23]), we get $u = y = v = z = 0$. Finally $U = 0$ which contradicts the fact that U is an eigenvector. The proof is thus completed. □

4. Exponential stability

In this section, we will show that the system (4) is exponentially stable. Our future computations are based on frequency domain approach for exponential stability (see Huang [10] and Pruss [18]), more precisely on the below result.

Lemma 4.1. *A $\mathcal{C}_0$-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ of contractions on a Hilbert space $\mathcal{H}$ is exponentially stable, namely satisfies*

$$\left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} \leq Ce^{-\omega t}\|U_0\|_{\mathcal{H}} \quad \forall\ U_0 \in \mathcal{H},\ \forall\ t \geq 0, \tag{39}$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{A}) \supset \{i\beta,\ \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{40}$$

and

$$\sup_{\beta\in\mathbb{R}} \left\|(i\beta - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{41}$$

where $\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

The main result of current section is the following.

Theorem 4.2. *Assume that $(u_0, u_1, y_0, y_1, f_0)^\top \in \mathcal{D}(\mathcal{A})$. Then, the system (4) is exponentially stable in the energy space $\mathcal{H}$.*

Proof. As the condition (40) is guaranteed by Lemma 3.1, it suffices now to check the condition (41) in other words, the boundedness of the resolvent on the imaginary axis. For that, we will establish that for any $\lambda \in \mathbb{R}$ and $F = (f, g, h, k, l)^\top \in \mathcal{H}$, the solution $U = (u, v, y, z, w)^\top \in \mathcal{D}(\mathcal{A})$ of

$$(i\lambda I - \mathcal{A})\, U = F \tag{42}$$

satisfies

$$\|U\|_{\mathcal{H}} \leq C \|F\|_{\mathcal{H}} \tag{43}$$

where C is a positive constant (not dependent on λ and F).
Problem (4) without delay (corresponding to $\beta_2 = 0$) is the following one

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - b y_t(x,t) = 0 \text{ in } \Omega \times (0, +\infty) \\ y_{tt}(x,t) - \Delta y(x,t) + b u_t(x,t) = 0 \text{ in } \Omega \times (0, +\infty) \\ y = 0 \text{ on } \Gamma \times (0, +\infty) \\ u = 0 \text{ on } \Gamma_D \times (0, +\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \beta_1 u_t(x,t) = 0 \text{ on } \Gamma_N \times (0, +\infty) \\ u(x,0) = u_0(x) \text{ and } u_t(x,0) = u_1(x) \text{ in } \Omega \\ y(x,0) = y_0(x) \text{ and } y_t(x,0) = y_1(x) \text{ in } \Omega \end{cases} \tag{44}$$

This problem is well-posed in

$$\mathcal{H}_0 = V \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \tag{45}$$

endowed with the norm

$$\left\| (u, v, y, z)^\top \right\|_{\mathcal{H}_0}^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 + \|\nabla y\|_{L^2(\Omega)}^2 + \|z\|_{L^2(\Omega)}^2. \tag{46}$$

The generator of its semigroup is $\mathcal{A}_0$ defined by

$$\mathcal{A}_0 \,(u, v, y, z)^\top = (v, \Delta u + bz, z, \Delta y - bv)^\top \tag{47}$$

with domain

$$\mathcal{D}(\mathcal{A}_0) = \\ = \left\{ (u, v, y, z)^\top \in \left(H^2(\Omega) \cap V\right) \times V \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega) : \frac{\partial u}{\partial \nu} = -\beta_1 v \text{ on } \Gamma_N \right\}. \tag{48}$$

The system (42) has been studied in [23] by Toufayli where it has been proved that $\mathcal{A}_0$ generates an exponentially stable semigroup. So, according to this study we have $i\mathbb{R} \subset \rho(\mathcal{A}_0)$ and there exist a constant $C_0 > 0$ such that

$$\left\| (i\xi - \mathcal{A}_0)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_0)} \leq C_0, \quad \forall\, \xi \in \mathbb{R}. \tag{49}$$

The relation (49) implies that the solution $U^* = (u^*, v^*, y^*, z^*)^\mathsf{T} \in \mathcal{D}(\mathcal{A}_0)$ of

$$(i\lambda I - \mathcal{A}_0)\begin{pmatrix} u^* \\ v^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} u \\ v \\ y \\ z \end{pmatrix} \tag{50}$$

verifies

$$\left\| (u^*, v^*, y^*, z^*)^\mathsf{T} \right\|_{\mathcal{H}_0} \le C_0 \left\| (u, v, y, z)^\mathsf{T} \right\|_{\mathcal{H}_0}. \tag{51}$$

Also, the system (50) can be rewritten as

$$\begin{cases} i\lambda u^* - v^* = u \\ i\lambda v^* - \Delta u^* - bz^* = v \\ i\lambda y^* - z^* = y \\ i\lambda z^* - \Delta y^* + bv^* = z. \end{cases} \tag{52}$$

We have

$$\begin{aligned}
&\left\langle (i\lambda I - \mathcal{A})\begin{pmatrix} u \\ v \\ y \\ z \\ w \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ y^* \\ z^* \\ \alpha w \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} i\lambda u - v \\ i\lambda v - \Delta u - bz \\ i\lambda y - z \\ i\lambda z - \Delta y + bv \\ i\lambda w + s^{-1} w_\rho \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ y^* \\ z^* \\ \alpha w \end{pmatrix} \right\rangle_{\mathcal{H}} \\
= \; & \int_\Omega \nabla (i\lambda u - v) \nabla \overline{u^*} dx + \int_\Omega (i\lambda v - \Delta u - bz) \overline{v^*} dx + \int_\Omega \nabla (i\lambda y - z) \nabla \overline{y^*} dx \\
& + \int_\Omega (i\lambda z - \Delta y + bv) \overline{z^*} dx + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 (i\lambda w + s^{-1} w_\rho) \overline{w} d\rho \right) ds d\Gamma \\
= \; & i\lambda \int_\Omega \nabla u \nabla \overline{u^*} dx - \int_\Omega \nabla v \nabla \overline{u^*} dx + i\lambda \int_\Omega v \overline{v^*} dx - \int_\Omega \Delta u \, \overline{v^*} dx - b \int_\Omega z \overline{v^*} dx \\
& + i\lambda \int_\Omega \nabla y \nabla \overline{y^*} dx - \int_\Omega \nabla z \nabla \overline{y^*} dx + i\lambda \int_\Omega z \overline{z^*} dx - \int_\Omega \Delta y \, \overline{z^*} dx + b \int_\Omega v \overline{z^*} dx \\
& + i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |w|^2 d\rho \right) ds d\Gamma + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 w_\rho \overline{w} d\rho \right) ds d\Gamma \\
= \; & i\lambda \int_\Omega \nabla u \nabla \overline{u^*} dx + \int_\Omega v \Delta \overline{u^*} dx - \int_{\Gamma_N} v \frac{\partial \overline{u^*}}{\partial \nu} d\Gamma + i\lambda \int_\Omega v \overline{v^*} dx + \int_\Omega \nabla u \nabla \overline{v^*} dx \\
& - \int_{\Gamma_N} \frac{\partial u}{\partial \nu} \overline{v^*} d\Gamma - b \int_\Omega z \overline{v^*} dx + i\lambda \int_\Omega \nabla y \nabla \overline{y^*} dx - \int_\Omega \nabla z \nabla \overline{y^*} dx + i\lambda \int_\Omega z \overline{z^*} dx \\
& + \int_\Omega \nabla y \nabla \overline{z^*} dx + b \int_\Omega v \overline{z^*} dx + i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |w|^2 d\rho \right) ds d\Gamma \\
& + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 w_\rho \overline{w} d\rho \right) ds d\Gamma
\end{aligned}$$

$$
\begin{aligned}
= \quad & i\lambda \int_\Omega \nabla u \nabla \overline{u^*} dx + \int_\Omega v \Delta \overline{u^*} dx - \int_{\Gamma_N} v\left(-\beta_1 \overline{v^*}\right) d\Gamma + i\lambda \int_\Omega v\overline{v^*} dx + \int_\Omega \nabla u \nabla \overline{v^*} dx \\
& - \int_{\Gamma_N} \left(-\beta_1 v - \int_{\tau_1}^{\tau_2} \beta_2(s) w(\cdot,1,s) ds\right) \overline{v^*} d\Gamma - b\int_\Omega z\overline{v^*} dx + i\lambda \int_\Omega \nabla y \nabla \overline{y^*} dx \\
& + \int_\Omega z \Delta \overline{y^*} dx + i\lambda \int_\Omega z\overline{z^*} dx + \int_\Omega \nabla y \nabla \overline{z^*} dx + b\int_\Omega v\overline{z^*} dx \\
& + i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |w|^2 d\rho\right) ds d\Gamma + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 w_\rho \overline{w} d\rho\right) ds d\Gamma.
\end{aligned}
$$

Then, using (52) we get

$$
\begin{aligned}
& \left\langle (i\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ y \\ z \\ w \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ y^* \\ z^* \\ \alpha w \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega \nabla u \nabla \overline{(-i\lambda u^* + v^*)} dx + \int_\Omega v \overline{(-i\lambda v^* + \Delta u^* + bz^*)} dx \\
& \quad + \int_\Omega \nabla y \nabla \overline{(-i\lambda y^* + z^*)} dx + \int_\Omega z \overline{(-i\lambda z^* + \Delta y^* - bv^*)} dx + 2\beta_1 \int_{\Gamma_N} v\overline{v^*} d\Gamma \\
& \quad + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) \overline{v^*} w(\cdot,1,s) ds d\Gamma + i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |w|^2 d\rho\right) ds d\Gamma \\
& \quad + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 w_\rho \overline{w} d\rho\right) ds d\Gamma \\
= \quad & -\int_\Omega |\nabla u|^2 dx - \int_\Omega |v|^2 dx - \int_\Omega |\nabla y|^2 dx - \int_\Omega |z|^2 dx + 2\beta_1 \int_{\Gamma_N} v\overline{v^*} d\Gamma \\
& + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) \overline{v^*} w(\cdot,1,s) ds d\Gamma + i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |w|^2 d\rho\right) ds d\Gamma \\
& + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 w_\rho \overline{w} d\rho\right) ds d\Gamma \\
= \quad & -\|\nabla u\|^2_{L^2(\Omega)} - \|v\|^2_{L^2(\Omega)} - \|\nabla y\|^2_{L^2(\Omega)} - \|z\|^2_{L^2(\Omega)} + 2\beta_1 \int_{\Gamma_N} v\overline{v^*} d\Gamma \\
& + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) \overline{v^*} w(\cdot,1,s) ds d\Gamma + i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |w|^2 d\rho\right) ds d\Gamma \\
& + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 w_\rho \overline{w} d\rho\right) ds d\Gamma.
\end{aligned}
$$

Now using (46), we can rewrite the above relation as

$$\left\langle (i\lambda I-\mathcal{A})\begin{pmatrix}u\\v\\y\\z\\w\end{pmatrix},\begin{pmatrix}u^*\\v^*\\y^*\\z^*\\\alpha w\end{pmatrix}\right\rangle_{\mathcal{H}} = -\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2+2\beta_1\int_{\Gamma_N} v\overline{v^*}d\Gamma$$
$$+\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\overline{v^*}w(\cdot,1,s)dsd\Gamma+i\lambda\alpha\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|w|^2d\rho\right)dsd\Gamma$$
$$+\alpha\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 w_\rho\overline{w}d\rho\right)dsd\Gamma. \tag{53}$$

In the sequel we set $\alpha=-\dfrac{1}{\varepsilon}$. Then recalling (42) and taking the real part in (53), we obtain

$$\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2=-\Re\left\langle F,\begin{pmatrix}u^*\\v^*\\y^*\\z^*\\-\frac{1}{\varepsilon}w\end{pmatrix}\right\rangle_{\mathcal{H}}+\Re\left(2\beta_1\int_{\Gamma_N}v\overline{v^*}d\Gamma\right)$$
$$+\Re\left(\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\overline{v^*}w(\cdot,1,s)dsd\Gamma\right)-\Re\left(\frac{1}{\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 w_\rho\overline{w}d\rho\right)dsd\Gamma\right). \tag{54}$$

Using (51) and the Cauchy-Schwarz inequality we have

$$\begin{aligned}-\Re\left\langle F,\begin{pmatrix}u^*\\v^*\\y^*\\z^*\\-\frac{1}{\varepsilon}w\end{pmatrix}\right\rangle_{\mathcal{H}} &\le \|F\|_{\mathcal{H}}\left\|(u^*,v^*,y^*,z^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}\left\|(0,0,0,0,w)^{\mathsf{T}}\right\|_{\mathcal{H}}\\ &\le \|F\|_{\mathcal{H}}\left\|(u^*,v^*,y^*,z^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}\left\|(u,v,y,z,w)^{\mathsf{T}}\right\|_{\mathcal{H}}\\ &\le C_0\|F\|_{\mathcal{H}}\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \end{aligned}\tag{55}$$

Applying the Young's inequality one obtains

$$\Re\left(2\beta_1\int_{\Gamma_N}v\overline{v^*}d\Gamma\right)\le\frac{2\beta_1^2}{\varepsilon}\int_{\Gamma_N}|v|^2d\Gamma+\varepsilon\int_{\Gamma_N}|v^*|^2d\Gamma,\quad\text{with } \varepsilon>0. \tag{56}$$

From the dissipativeness of $\mathcal{A}$, we deduce using (42) and the Cauchy-Schwarz inequality that

$$\left(\beta_1-\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)\int_{\Gamma_N}|v|^2d\Gamma\le\langle(i\lambda I-\mathcal{A})U,U\rangle_{\mathcal{H}}\le\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{57}$$

Note further that (51) and the dissipativeness of $\mathcal{A}_0$ directly yield

$$\begin{aligned}\beta_1 \int_{\Gamma_N} |v^*|^2 \, d\Gamma &\leq \Re \left\langle (i\lambda I - \mathcal{A}_0)\, U^*, U^* \right\rangle_{\mathcal{H}_0} \\ &\leq \left\|(u,v,y,z)^\top\right\|_{\mathcal{H}_0} \|U^*\|_{\mathcal{H}_0} \leq C_0 \left\|(u,v,y,z)^\top\right\|^2_{\mathcal{H}_0}. \end{aligned} \tag{58}$$

Consequently using (57) and (58) in (56), we get

$$\Re\left(2\beta_1 \int_{\Gamma_N} v\overline{v^*} d\Gamma\right) \leq \frac{2\beta_1^2}{\varepsilon\left(\beta_1 - \int_{\tau_1}^{\tau_2} \beta_2(s)ds\right)} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \frac{\varepsilon C_0}{\beta_1} \left\|(u,v,y,z)^\top\right\|^2_{\mathcal{H}_0}. \tag{59}$$

Thanks to the Young's inequality, we get

$$\begin{aligned}\Re\left(\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)\overline{v^*} w(\cdot,1,s) ds d\Gamma\right) &\leq \frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s) |w(\cdot,1,s)|^2 ds d\Gamma \\ &\quad + \varepsilon \int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s) |v^*|^2 ds d\Gamma.\end{aligned}$$

That is using (58)

$$\begin{aligned}\Re\left(\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)\overline{v^*} w(\cdot,1,s) ds d\Gamma\right) &\leq \frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s) |w(\cdot,1,s)|^2 ds d\Gamma \\ &\quad + \frac{\varepsilon C_0 \int_{\tau_1}^{\tau_2} \beta_2(s)ds}{\beta_1} \left\|(u,v,y,z)^\top\right\|^2_{\mathcal{H}_0}.\end{aligned} \tag{60}$$

Furthermore, we have

$$\begin{aligned} -\frac{1}{\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 w_\rho \overline{w} d\rho\right) ds d\Gamma &= -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left[|w|^2\right]_0^1 ds d\Gamma \\ = \; -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(\cdot,1,s)|^2 ds d\Gamma &+ \frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(\cdot,0,s)|^2 ds d\Gamma \\ = \; -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(\cdot,1,s)|^2 ds d\Gamma &+ \frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|v|^2 ds d\Gamma.\end{aligned}$$

Thus,

$$\begin{aligned} -\Re\left(\frac{1}{\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 w_\rho \overline{w} d\rho\right) ds d\Gamma\right) &= -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(\cdot,1,s)|^2 ds d\Gamma \\ &+ \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)ds\int_{\Gamma_N}|v|^2 d\Gamma.\end{aligned}$$

Using (57), one can write

$$\begin{aligned} -\Re\left(\frac{1}{\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 w_\rho \overline{w} d\rho\right) ds d\Gamma\right) &\leq -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)|w(\cdot,1,s)|^2 ds d\Gamma \\ &+ \frac{\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\end{aligned} \tag{61}$$

Now adding (55), (59), (61) and (60) one gets

$$\begin{aligned}
\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 &\leq C_0\left\|F\right\|_{\mathcal{H}}\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}\\
&+\frac{2\beta_1^2}{\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}+\frac{\varepsilon C_0}{\beta_1}\left\|(u,v,y,z)^{\top}\right\|_{\mathcal{H}_0}^2\\
&+\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|w(\cdot,1,s)\right|^2 dsd\Gamma+\frac{\varepsilon C_0\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1}\left\|(u,v,y,z)^{\top}\right\|_{\mathcal{H}_0}^2\\
&-\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|w(\cdot,1,s)\right|^2 dsd\Gamma+\frac{\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}.
\end{aligned}$$

That is

$$\begin{aligned}
\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 &\leq C_0\left\|F\right\|_{\mathcal{H}}\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}\\
&+\frac{2\beta_1^2}{\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}+\frac{\varepsilon C_0}{\beta_1}\left(1+\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)\left\|(u,v,y,z)^{\top}\right\|_{\mathcal{H}_0}^2\\
&+\frac{\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}.
\end{aligned}$$

At this level we chose ε sufficiently small such that $\varepsilon \ll \dfrac{\beta_1}{C_0\left(1+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}$ to obtain

$$\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 \leq \left(C_0+\frac{1}{\varepsilon}+\frac{4\beta_1^2+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}. \quad (62)$$

Since $\left\|(u,v,y,z,w)^{\mathsf{T}}\right\|_{\mathcal{H}}^2=\left\|(u,v,y,z)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2+\displaystyle\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|w(\cdot,\rho,s)|^2d\rho\right)dsd\Gamma$, we deduce that

$$\begin{aligned}
\left\|U\right\|_{\mathcal{H}}^2 &\leq \left(C_0+\frac{1}{\varepsilon}+\frac{4\beta_1^2+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}\\
&+\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|w(\cdot,\rho,s)|^2d\rho\right)dsd\Gamma. \quad (63)
\end{aligned}$$

Now we need a best estimation for $\displaystyle\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|w(\cdot,\rho,s)|^2d\rho\right)dsd\Gamma$.

Following (42) and solving the next Cauchy problem (64)

$$\begin{cases} s^{-1}w_\rho + i\lambda w = l \\ w(\cdot,0,s) = v \end{cases} \tag{64}$$

we obtain

$$w(\cdot,\rho,s) = ve^{-i\lambda s\rho} + s\int_0^\rho e^{-i\lambda s(\rho-\sigma)}l(\cdot,\sigma,s)\,d\sigma, \quad \forall\ \rho\in(0,1). \tag{65}$$

Using the triangular inequality, it follows from (65) that

$$|w(\cdot,\rho,s)| \le |v| + s\int_0^\rho |l(\cdot,\sigma,s)|\,d\sigma, \quad \forall\ \rho\in(0,1),$$

which leads to

$$|w(\cdot,\rho,s)|^2 \le |v|^2 + s^2\left(\int_0^\rho |l(\cdot,\sigma,s)|\,d\sigma\right)^2 + 2|v|\,s\left(\int_0^\rho |l(\cdot,\sigma,s)|\,d\sigma\right), \quad \forall\ \rho\in(0,1). \tag{66}$$

On the one hand, by Cauchy-Schwarz's inequality we obtain

$$\begin{aligned}\left(\int_0^\rho |l(\cdot,\sigma,s)|\,d\sigma\right)^2 &\le \left(\int_0^\rho |l(\cdot,\sigma,s)|^2\,d\sigma\right)\left(\int_0^\rho d\sigma\right)\\ &\le \rho\int_0^\rho |l(\cdot,\sigma,s)|^2\,d\sigma\\ &\le \int_0^\rho |l(\cdot,\sigma,s)|^2\,d\sigma;\end{aligned}$$

that is

$$\left(\int_0^\rho |l(\cdot,\sigma,s)|\,d\sigma\right)^2 \le \int_0^\rho |l(\cdot,\sigma,s)|^2\,d\sigma. \tag{67}$$

On the other hand, Young's inequality guarantees that

$$2|v|\,s\left(\int_0^\rho |l(\cdot,\sigma,s)|\,d\sigma\right) \le |v|^2 + s^2\left(\int_0^\rho |l(\cdot,\sigma,s)|\,d\sigma\right)^2. \tag{68}$$

Combining (66), (67) and (68) it follows that

$$|w(\cdot,\rho,s)|^2 \le 2|v|^2 + 2s^2\int_0^\rho |l(\cdot,\sigma,s)|^2\,d\sigma. \tag{69}$$

Integrating (69) on $\Gamma_N\times(\tau_1,\tau_2)\times(0,1)$ yields

$$\begin{aligned}&\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|w(\cdot,\rho,s)|^2d\rho\right)dsd\Gamma\\ &\quad\le 2\int_{\tau_1}^{\tau_2}s\beta_2(s)ds\int_{\Gamma_N}|v|^2\,d\Gamma + 2\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}s^3\beta_2(s)\int_0^1|l(\cdot,\rho,s)|^2\,dsd\rho d\Gamma\\ &\quad\le 2\tau_2\int_{\tau_1}^{\tau_2}\beta_2(s)ds\int_{\Gamma_N}|v|^2\,d\Gamma + 2\tau_2^2\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}s\beta_2(s)\int_0^1|l(\cdot,\rho,s)|^2\,dsd\rho d\Gamma.\end{aligned}$$

Then using (57) and the $\mathcal{H}$-norm definition, the above relation can be rewritten as

$$\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|w(\cdot,\rho,s)|^2d\rho\right)dsd\Gamma \leq \left(\frac{2\tau_2\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}\right)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+2\tau_2^2\|F\|_{\mathcal{H}}^2. \tag{70}$$

Putting (70) in (63), it follows that

$$\|U\|_{\mathcal{H}}^2 \leq \left(C_0+\frac{1}{\varepsilon}+\frac{4\beta_1^2+(1+4\varepsilon\tau_2)\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+2\tau_2^2\|F\|_{\mathcal{H}}^2$$

that is

$$\|U\|_{\mathcal{H}}^2 \leq C_\varepsilon\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+2\tau_2^2\|F\|_{\mathcal{H}}^2 \tag{71}$$

where C_ε is a positive constant which doesn't depend on λ. More precisely,

$$C_\varepsilon = C_0+\frac{1}{\varepsilon}+\frac{4\beta_1^2+(1+4\varepsilon\tau_2)\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}. \tag{72}$$

Applying Young's inequality to (71) it follows that

$$\|U\|_{\mathcal{H}}^2 \leq \frac{C_\varepsilon}{2\varepsilon'}\|F\|_{\mathcal{H}}^2+\frac{\varepsilon' C_\varepsilon}{2}\|U\|_{\mathcal{H}}^2+2\tau_2^2\|F\|_{\mathcal{H}}^2, \quad \text{with } \varepsilon'>0. \tag{73}$$

One can choose ε' small enough such that $\dfrac{\varepsilon' C_\varepsilon}{2}<1$. Consequently, (73) becomes

$$\|U\|_{\mathcal{H}}^2 \leq C_{\varepsilon\varepsilon'}\|F\|_{\mathcal{H}}^2, \tag{74}$$

where one sets

$$C_{\varepsilon\varepsilon'} = \frac{\dfrac{C_\varepsilon}{2\varepsilon'}+2\tau_2^2}{1-\dfrac{\varepsilon' C_\varepsilon}{2}}. \tag{75}$$

Finally (74) directly leads to (43) with

$$C=\sqrt{C_{\varepsilon\varepsilon'}}. \tag{76}$$

That means the resolvent of $\mathcal{A}$ is uniformly bounded on the imaginary axis. The proof of theorem 4.2 is thus completed. □

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