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The Reticulation of a Heyting Algebra

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Abstract. Through this paper H denotes a Heyting algebra. Following the construction method presented by L.P. Belluce for non-commutative rings and for semisimple algebras of infinite valued logic and bold fuzzy set theory, we can point out the reticulation of a Heyting algebra which means a distributive lattice so that its prime spectrum is homeomorphic to the prime spectrum of H.

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1. Basic notions

Definition 1.1. *Let L be a lattice and let $x, y \in L$. If $\sup\{z \in L \,|\, x \wedge z \leq y\}$ exists, we say that it is the relative pseudocomplement of x with respect to y and we denote it by $x \to y$.*

Hence, the definition of the relative pseudocomplement is equivalent to the existence of an element $x \to y$ so that

$$x \wedge z \leq y \;\Leftrightarrow\; z \leq x \to y.$$

Definition 1.2. *A lattice with 0 in which there exists $x \to y$ for all $x, y \in L$ is called a Heyting algebra.*

Heyting algebras, considered as lattices are distributive pseudocomplemented lattices but, considered as algebras, they are algebras of similarity type (2,2,2,0) instead of type (2,2,1,0) as for pseudocomplemented lattices.

Theorem 1.1. *Let **H** be the equational class of algebras $(L, (\wedge, \vee, \to, 0))$ that verifies the following identities:*
(1) *A set of identities which defines a lattice with 0.*
(2) $x \wedge (x \to y) = x \wedge y$.
(3) $x \wedge (y \to z) = x \wedge (x \wedge y \to x \wedge z)$.
(4) $z \wedge (x \wedge y \to x) = z$.
*Then, **H** is exactly the class of Heyting algebras.*

Theorem 1.2. *Let $x, y, z \in H$ with H a Heyting algebra. Then,*

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(i) $x \wedge (x \to y) \leq y$.
(ii) $x \wedge y \leq z \Leftrightarrow y \leq x \to z$.
(iii) $x \leq y \Leftrightarrow x \to y = 1$.
(iv) $y \leq x \to y$.
(v) $x \leq y$ *then* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$.
(vi) $x \to (y \to z) = x \wedge y \to z$.
(vii) $x \wedge (y \to z) = x \wedge (x \wedge y \to x \wedge z)$.
(viii) $x \wedge (x \to y) = x \wedge y$.
(ix) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.
(x) $x \to y \wedge z = (x \to y) \wedge (x \to z)$.
(xi) $(x \to y)^* = x^{**} \wedge y^*$ *where* $x^* = x \to 0$.

Example 1.1. *If* $(A, (\vee, \wedge, ^-, 0, 1))$ *is a Boole algebra, A becomes a Heyting algebra by considering* $x \to y = \overline{x} \vee y$.

Example 1.2. *The chains with 0 and 1 are Heyting algebras if we define*

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x. \end{cases}$$

Example 1.3. *Let* (X, τ) *be a topological space. Then* τ *is a Heyting algebra with* $0 = \emptyset$ *and for* $D_1, D_2 \in \tau$ *we define* $D_1 \to D_2 = int\,((X - D_1) \cup D_2)$.

Example 1.4. *Let L be a distributive lattice with 0. Then,* $Id\,(L)$ *the lattice ideals of L is a Heyting algebra if we define for* $I, J \in Id(L)$ $I \to J = \{x \in L \,|x \wedge i \in J, \text{ for all } i \in I\}$.

2. The reticulation of a Heyting algebra

Let H a Heyting algebra. We denote

$$Spec\,(H) = \{P\,|P \text{ prime ideal of } H\}.$$

We recall that P is a prime ideal if it is proper and if $x \wedge y \in P$ it results that $x \in P$ or $y \in P$. For each ideal I of H we define the set $r\,(I) = \{P \in Spec\,(H)\,|I \subsetneqq P\}$. $Spec\,(H)$ is a Stone space considering the topology generated by this sets. Each open compact set of this space has the form $r\,(x) = \{P \in Spec\,(H)\,|x \notin P\}$ for all $x \in H$. It is obtained in this case a bounded Stone space and for each open compact set $U \in St\,(H)$, $St\,(H) \setminus Cl\,(U)$ is compact. Moreover, $r\,(a^*) = St\,(H) \setminus Cl\,(r\,(a))$ for each $a \in H$.

Proposition 2.1. *If H is a Heyting algebra,* $\left(\{r\,(x)\}_{x \in H}, \wedge, \vee, \to, \emptyset, Spec\,(H)\right)$ *is a Heyting algebra isomorphic to H.*

Proof. We know that all open sets of a topological space define a Heyting algebra from Example 1.3. Let I, J be ideals of H.

We prove that $r\,(I) \to r\,(J) = r\,(I \to J)$. To realize that, we have to show that $Int\,[(St\,(H) \setminus r\,(I)) \cup r\,(J)] = r\,(I \to J)$. Let $P \in r\,(I \to J)$ which means that $I \to J \subsetneqq P$. Then, there exists $a \in I \to J$ and $a \notin P$. If $I \subseteq P$ then $P \notin r\,(I)$ and so, $r\,(I \to J) \subseteq Int\,[(St\,(H) \setminus r\,(I)) \cup r\,(J)]$. If $I \subsetneqq P$ there exists $b \in I$ and $b \notin P$. But, $a \wedge b \in J$ and $a \wedge b \notin P$ since P is a prime ideal. It results $J \subsetneqq P$ hence, $P \in r\,(J)$ and then, $r\,(I \to J) \subseteq Int\,[(St\,(H) \setminus r\,(I)) \cup r\,(J)]$.

Let us consider now that $r\,(K) \subseteq (St\,(H) \setminus r\,(I)) \cup r\,(J)$ for K an ideal of H and we prove that $r\,(K) \subseteq r\,(I \to J)$. That is equivalent to show that $K \subseteq I \to J$ which means that $I \cap K \subseteq J$. It remains to prove that $r\,(I \cap K) \subseteq r\,(J)$. To realize this, let $P \in r\,(I \cap K)$ so, $I \cap K \subsetneqq P$. Since P is a prime ideal, it result $I \subsetneqq P$ and $K \subsetneqq P$ and so, $P \in r\,(K)$, $P \notin St\,(H) \setminus r\,(I)$ hence, $P \in r\,(J)$. ∎

We intend to show that the topological space $Spec\,(H)$ is homeomorphic to the prime space of a distributive lattice.

On a Heyting algebra H we define the relation

$$x \equiv y \Leftrightarrow \text{for each } P \in \mathcal{P}(H)\ ,\ x \in P \Leftrightarrow y \in P$$

where $\mathcal{P}(H)$ is the set of all prime ideals of H.

Lemma 2.1. *The relation defined as above is a congruence relation on H with respect to the distributive lattice structure of H.*

Proof. Let $x \equiv y$ and $x_1 \equiv y_1$. Then, for a prime ideal P , if $x \wedge x_1 \in P$ it implies that $x \in P$ or $x_1 \in P$ which is equivalent with $y \in P$ or $y_1 \in P$ and then $y \wedge y_1 \in P$. If $x \vee x_1 \in P$ it results $x, x_1 \in P \Leftrightarrow y, y_1 \in P$ and then, $y \vee y_1 \in P$. □

In this way, $H/\equiv$, which we denote by $H_{\mathcal{P}}$, becomes a bounded distributive lattice. With $\widehat{x}$ we denote the congruence class of x in $H_{\mathcal{P}}$.

Definition 2.1. *Let I be an ideal of H and J an ideal in $H_{\mathcal{P}}$. We define the sets:*

$$I^* = \{\widehat{x}\,|\text{there exists } y \in I \text{ so that } y \in \widehat{x}\}$$
$$J_* = \bigcup_{\widehat{x} \in J} \widehat{x}.$$

Proposition 2.2. *With the previous notations, for each ideal I of H and J an ideal in $H_{\mathcal{P}}$, the following statements hold:*

(i) *I^* is an ideal in $H_{\mathcal{P}}$.*

(ii) *J_* is an ideal in H.*

(iii) *$(J_*)^* = J$.*

(iv) *$1 \in J_*$ iff $\widehat{1} \in J$.*

Proof. (i) Let $\widehat{x}, \widehat{y} \in I^*$. There exist $x_1, y_1 \in I$ so that $x_1 \in \widehat{x}$, $y_1 \in \widehat{y}$. Then, $x \equiv x_1$, $y \equiv y_1$ and so, $x \vee y \equiv x_1 \vee y_1$. Since $x_1 \vee y_1 \in I$ it results that $\widehat{x} \vee \widehat{y} \in I^*$. Now we consider $\widehat{x} \leq \widehat{y}$ with $\widehat{y} \in I^*$. Then, there exists $y_1 \in I$ with $y_1 \in \widehat{y}$. But, $\widehat{x} \wedge \widehat{y} = \widehat{x}$ and then $x \equiv x \wedge y \equiv x \wedge y_1$. We get $x \wedge y_1 \in I$ and $x \wedge y_1 \in \widehat{x}$ which means that $\widehat{x} \in I^*$.

(ii) Let $x, y \in J_*$. This implies $\widehat{x}, \widehat{y} \in J$ and then $\widehat{x} \vee \widehat{y} = \widehat{x \vee y} \in J$. So, $x \vee y \in J_*$. Let $x \leq y$ with $y \in J_*$ hence, $\widehat{y} \in J$. But, $\widehat{x} \wedge \widehat{y} = \widehat{x \wedge y} = \widehat{x}$ implies that $\widehat{x} \in J$ and so $x \in J_*$.

(iii) Let $\widehat{x} \in (J_*)^*$ which means that there exists $y \in J_*$, $y \in \widehat{x}$. But then, $\widehat{x} = \widehat{y} \in J$ hence, $(J_*)^* \subseteq J$. Conversely, for $\widehat{x} \in J$ we obtain $x \in J_*$ and then, $\widehat{x} \in (J_*)^*$.

(iv) If $1 \in J_*$, $\widehat{1} \in (J_*)^* = J$ and for $\widehat{1} \in J$ we get $1 \in J_*$ from the previous definition. □

Proposition 2.3. *For a Heyting algebra H, the following statements are true:*

(i) *If P is a prime ideal in H, then P^* is a prime ideal in $H_{\mathcal{P}}$.*

(ii) *If P is a prime ideal in H, then $(P^*)_* = P$.*

(iii) *If J is a prime ideal in $H_{\mathcal{P}}$, then J_* is a prime ideal in H.*

Proof. (i) Let $\widehat{x}, \widehat{y}$ so that $\widehat{x} \wedge \widehat{y} \in P^*$. Then there exists $z \in P$ so that $z \in \widehat{x \wedge y}$. Since $z \in P$ and $z \equiv x \wedge y$ we obtain that $x \wedge y \in P$ which is a prime ideal. Hence, $x \in P$ or $y \in P$ which means that $\widehat{x} \in P^*$ or $\widehat{y} \in P^*$.

(ii) Let $x \in (P^*)_*$ so, $\widehat{x} \in P^*$ which implies that there exists $y \in P$ so that $y \in \widehat{x}$. $\widehat{x} = \widehat{y}$ leads us to $x \equiv y$ and $y \in P$. Hence, $x \in P$. Conversely, if $x \in P$, $\widehat{x} \in P^*$ and then $x \in (P^*)_*$.

(iii) Let x, y from H so that $x \wedge y \in J_*$. Then, $\widehat{x \wedge y} \in J$ which is a prime ideal in $H_{\mathcal{P}}$. Hence, $\widehat{x} \in J$ or $\widehat{y} \in J$ which implies that $x \in J_*$ or $y \in J_*$. □

Proposition 2.4. *The function $P \to P^*$ is a bijective map from $Spec(H)$ onto $Spec(H_{\mathcal{P}})$.*

Proof. The previous proposition assures us that this function is well defined. If $P^* = Q^*$, from Proposition 2.2, $P = (P^*)_* = (Q^*)_* = Q$ which means that the mapping is an injective one. If we consider J a prime ideal in $H_{\mathcal{P}}$, then, $J = (J_*)^*$ with J_* prime ideal in H as it is proved in Propositions 2.2 and 2.3. So, the mapping is onto. □

To have a unit of notation we shall denote now the set of all prime ideals of H with $\mathcal{P}$ and the set of all prime ideals of $H_{\mathcal{P}}$ with $\mathcal{P}^*$.

Finally, we compare the topologies on $\mathcal{P}$ and on $\mathcal{P}^*$. We have already mentioned that $\{r(x)\}_{x \in H}$ generates a topology on $\mathcal{P}$. Let now consider the family of sets:

$$t(\widehat{x}) = \{J \in \mathcal{P}^* \,|\widehat{x} \notin J\} \ , \widehat{x} \in H_{\mathcal{P}}.$$

This defines a topology on $H_{\mathcal{P}}$. For each $\mathcal{U} \subseteq \mathcal{P}$ we define

$$\mathcal{U}^* = \{P^* \,|P \in \mathcal{U}\}.$$

Proposition 2.5. *With the previous notation, $(r(x))^* = t(\widehat{x})$ for each $x \in H$.*

Proof. $(r(x))^* = \{P^* \,|P \in r(x)\} = \{P^* \,|x \notin P\}$. Let P^* a prime ideal with $x \notin P$. If $\widehat{x} \in P^*$ there exists $y \in P$ with $y \in \widehat{x}$. Then, $x \equiv y$ which is in contradiction with $x \notin P$. Hence, $\widehat{x} \notin P^*$ and then $P^* \in t(\widehat{x})$. Conversely, if $J \in t(\widehat{x})$ then J is a prime ideal and $\widehat{x} \notin J$. Then, J_* is prime ideal in H from Proposition 2.3 and $x \notin J_*$. Hence, $J_* \in r(x)$. Since Proposition 2.2 proves that $(J_*)^* = J$ we obtain that $J \in (r(x))^*$. □

Let

$$\tau = \{r(x) \,|x \in H\} \ , \tau^* = \{t(\widehat{x}) \,|\widehat{x} \in H_{\mathcal{P}}\}$$

be the two topologies raised for discussion and we consider the mapping $f : \tau \to \tau^*$ defined by $f(r(x)) = (r(x))^*$.

Theorem 2.1. *$H_{\mathcal{P}}$ is the reticulation of the Heyting algebra H.*

Proof. We have to prove that the spectral spaces of H and $H_{\mathcal{P}}$ are homeomorphic. To realize that we show that the previous mapping is a bijective one, it preserves the arbitrary reunions and the finite intersections.

If $(r(x))^* = (r(y))^*$ we obtain that, for any J prime ideal in $H_{\mathcal{P}}$, $\widehat{x} \notin J \Leftrightarrow \widehat{y} \notin J$. From Proposition 2.4, for each J there exists P a prime ideal in $H_{\mathcal{P}}$ so that $J = P^*$. Hence, $x \notin (P^*)_* \Leftrightarrow y \notin (P^*)_*$ and since $(P^*)_* = P$ from Proposition 2.3, we get $r(x) = r(y)$. So, f is injective. Proposition 2.2 states that f is onto.

For $x, y \in H$ we remark that

$$\begin{aligned} f(r(x) \cap r(y)) &= (r(x) \cap r(y))^* = r(x \wedge y)^* = t\left(\widehat{x \wedge y}\right) \\ &= t(\widehat{x} \wedge \widehat{y}) = t(\widehat{x}) \cap t(\widehat{y}) = f(x) \cap f(y). \end{aligned}$$

If we consider now $x_i \in H$ for each $i \in I$ with I an arbitrary set, we prove that $f\left(\underset{i \in I}{\cup} r(x_i)\right) = \underset{i \in I}{\cup} f(r(x_i))$ which means that $\left(\underset{i \in I}{\cup} r(x_i)\right)^* = \underset{i \in I}{\cup} (r(x_i))^*$. To realize this, let $P^* \in \left(\underset{i \in I}{\cup} r(x_i)\right)^*$. Then $P \in \underset{i \in I}{\cup} r(x_i)$. There exists $i \in I$ so that $P \in r(x_i)$

and so, $P^* \in (r(x_i))^*$. Hence, $P^* \in \bigcup_{i \in I} (r(x_i))^*$. Conversely, if $P^* \in \bigcup_{i \in I} (r(x_i))^*$ there exists $i \in I$ so that $P^* \in (r(x_i))^*$ So, $P \in r(x_i)$ for an index $i \in I$. Hence, $P \in \bigcup_{i \in I} r(x_i)$ and then $P^* \in \left(\bigcup_{i \in I} r(x_i)\right)^*$. □

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