Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 49(1), 2022, Pages 35–51, DOI:
ISSN: 1223-6934

MV-pseudo metrics on MV-algebras

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Abstract. In this paper, we define the notions of MV-pseudo norm and MV-pseudo metric on MV-algebras and study some of their algebraic properties. The notion of uniform MV-algebra is also introduced and its relationship to MV-pseudo metrics is studied.

2010 Mathematics Subject Classification. 06D35, 54E35.
Key words and phrases. MV-algebra, MV-pseudo norm, MV-pseudo metric, Uniform MV-algebra.

1. Introduction

Probably the first thing that comes to mind about a function is its *continuity*. The behavior of a continuous function at a point is similar to the behavior of the function in a small neighborhood of that point. In calculus the definition of continuity depends only on the notion of distance between two points. So if we can measure the distance between the points of sets, we can study the continuity of the functions between those sets. In 1905, Maurice René Fréchet introduced *metric* as a real valued function on a set to measure distance between points of the set. The concept of a *pseudo metric* is a minor abstraction of a metric. Indeed, a pseudo metric is a generalization of a metric in which the distance between two distinct points can be zero. In fact pseudo metric spaces behavior exactly like metric spaces except for the fact that they need not be Hausdorff.

BCK-algebras, BCC-algebras, BL-algebras and MV-algebras are of the most important algebraic structures related to logic which have been introduced to the mathematics community around the second half of the last century, and their algebraic properties have been studied. One research area of recent decades is the study of the aforementioned structures equipped with topology. (See [10], [13] and [12].) Algebraic structures related to logic which are endowed with uniformity have also been discussed in the recent years. For example, Khanegir et al., in [11], introduced the notion of uniform BL-algebra and studied some of its properties. See [14], [6] and [5] for some other examples.

MV-algebras, which were introduced by Chang in [7] in 1958, prove the completeness theorem for $\aleph_0$-valued Lukasiewicz logic. Our aim in this article is to introduce and study *MV-pseudo metrics* on MV-algebras. To this end, we first define MV-pseudo norms on MV-algebras, and study their algebraic properties. Then, in Section 4, we will introduce MV-pseudo metrics and study the relation between them and uniform continuity of the operations of MV-algebras.

Received January 14, 2021. Accepted April 7, 2022.

The article is organized as follows: in Section 2 we present some definitions and results of the MV-algebra theory and uniform spaces which will be used later in the paper.

In Section 3 we define the concept of MV-pseudo norm, and discuss its algebraic properties and its relation to filters and ideals. Also, the relationship between MV-pseudo norm on MV-algebras and quotient MV-algebras will be examined in this section. Finally, we show that if $f : A_1 \to A_2$ is an isomorphism between MV-algebras, and N_{A_1} is an MV-pseudo norm on A_1, then $N_{A_2} = N_{A_1} \circ f^{-1}$ is an MV-pseudo norm on A_2.

In Section 4, we define MV-pseudo metrics and examine their relations to MV-pseudo norms. There are also a few theorems about the relationship between MV-pseudo metrics and uniform MV-algebras. Theorem 4.12 in particular provides an efficient way to construct an MV-pseudo metric on MV-algebras. Proposition 4.16, shows a connection between ideals, MV-pseudo norms and uniform MV-algebras.

2. Preliminaries

In this section, we present some definitions and results of the MV-algebra theory and uniform spaces which will be used later in the paper.

MV-algebras

An *MV-algebra* is an algebra $(A, \oplus, *, 0)$ of type $(2, 1, 0)$ such that for every $x, y \in A$,
$(M1)$ $(A, \oplus, 0)$ is a commutative monoid,
$(M2)$ $x \oplus 0^* = 0^*$,
$(M3)$ $(x^*)^* = x$, and
$(M4)$ $(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$.[9]

In an MV-algebra A, for every $x, y \in A$, define
$(M5)$ $1 := 0^*$;
$(M6)$ $x \odot y := (x^* \oplus y^*)^*$;
$(M7)$ $x \ominus y := x \odot y^*$;
$(M8)$ $x \to y := (x \odot y^*)^*$;
$(M9)$ $x \rightsquigarrow y := (x \oplus y^*)^*$.

In an MV-algebra A, for every $x, y \in A$, we write $x \leq y$ if and only if $x^* \oplus y = 1$. It is well-know that $\leq$ is a partial order on A, which gives A the structure of a distributive lattice, where the join and meet are defined by $x \wedge y = y \odot (y^* \oplus x)$ and $x \vee y = x \oplus (y \ominus x)$, respectively, 0 is the least element and 1 is the greatest element. By $(M6)$ and $(M7)$, for every $x, y \in A, x \leq y \Longleftrightarrow x \ominus y = 0$.

Proposition 2.1. [9] *The following hold in an MV-algebra A.*
$(M10)$ $x \oplus x^* = 1, x \odot x^* = 0$.
$(M11)$ $(A, \odot, 1)$ *is a commutative monoid.*
$(M12)$ $x \odot 0 = x \odot x^* = 0$.
$(M13)$ $x \oplus y = 0 \Longrightarrow x = y = 0$.
$(M14)$ $x \odot y = 1 \Longrightarrow x = y = 1$.
$(M15)$ $(x \wedge y)^* = x^* \vee y^*, (x \vee y)^\star = x^\star \wedge y^*$.
$(M16)$ $x \leq y \Longleftrightarrow y^* \leq x^*$.
$(M17)$ $x \leq y \Longrightarrow x \oplus z \leq y \oplus z, x \odot z \leq y \odot z$.
$(M18)$ $x \odot y \leq x \wedge y \leq x \leq x \vee y \leq x \oplus y$.

$(M19)$ $x \ominus y \leq x \leq x \oplus y$.
$(M20)$ $y \odot (x \oplus z) \leq x \oplus (y \odot z)$.
$(M21)$ $z \odot x^* \leq (x^* \odot y) \oplus (y^* \odot z)$.
$(M22)$ $(z \oplus y) \odot y^* \leq z$.
$(M23)$ $(y \odot (z \oplus y))^* \leq y^*$.
$(M24)$ $(x^* \odot y)^* \odot y = (y^* \odot x)^* \odot x$.
$(M25)$ $x \odot (y \odot z) = (x \odot y) \odot z$.
$(M26)$ $(x \oplus y) \ominus y \leq y$.
$(M27)$ $x \odot z \leq y \Longleftrightarrow x \leq z^* \oplus y$.
$(M28)$ $x \odot (y \to z) \leq (x \odot y) \to (x \odot z)$.
$(M29)$ $(x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \odot x_2) \to (y_1 \odot y_2)$.
$(M30)$ $(x \ominus y) \leq y^*$.
$(M31)$ $(x \oplus y) \ominus (a \oplus b) \leq (x \ominus a) \oplus (y \ominus b)$.
$(M32)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.
$(M33)$ $(x \ominus y) \oplus y = (y \ominus x) \oplus x$.
$(M34)$ $x \odot (x^* \oplus y) = y \odot (y^* \oplus x)$.
$(M35)$ $x \ominus 0 = x, x \ominus x = 0 \ominus x = x \ominus 1 = 0, 1 \ominus x = x^*$.
$(M36)$ $x \oplus x = x \Longleftrightarrow x \odot x = x$.
$(M37)$ $x \oplus (\wedge_{i\in I} x_i) = \wedge(x \oplus x_i), x \odot (\vee_{i\in I} x_i) = \vee(x \odot x_i)$.
$(M38)$ $x \leq y \Longleftrightarrow x \wedge y = x \Longleftrightarrow x \vee y = y$.
$(M39)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z), x \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z)$.
$(M40)$ $(x \to a) \odot (b \to y) \leq (a \to b) \to (x \to y)$.

Definition 2.1. Let A be an MV-algebra.
(1) A non-empty subset I of A is called an *ideal* if it satisfies the following conditions.
(I1) For every $x, y \in I, x \oplus y \in I$.
(I2) If $x \in I$ and $y \leq x$, then $y \in I$. [7]
(2) A non-empty subset F of A is called a *filter* if it satisfies the following conditions.
(F1) For every $x, y \in F, x \odot y \in F$.
(F2) If $x \in F$ and $x \leq y$, then $y \in F$. [9]

Proposition 2.2. [9] *Let I and F be subsets of an MV-algebra A. Then I is an ideal if and only if*
(I3) $0 \in I$, *and*
(I4) $y \in I$ *and* $x \ominus y \in I$ *imply that* $x \in I$.
Also, F is a filter if and only if
(F3) $1 \in F$, *and*
(F4) $x \in F$ *and* $x \to y \in F$ *imply that* $y \in F$.

Proposition 2.3. [9] *Let F be a filter and I be an ideal of an MV-algebra A. Then the following are congruence relations on A.*

$$x \overset{F}{\equiv} y \iff x \to y \in F \text{ and } y \to x \in F.$$
$$x \overset{I}{\equiv} y \iff x \ominus y \in I \text{ and } y \ominus x \in I.$$

Moreover, if $x/F = \{y \in A : x \overset{F}{\equiv} y\}$, $A/F = \{x/F : x \in A\}$, $x/I = \{y \in A : x \overset{I}{\equiv} y\}$ and $A/I = \{x/I : x \in A\}$, then both A/F and A/I are quotient MV-algebras with the operations

$x/F \odot y/F = (x \odot y)/F,\ x/I \oplus y/I = (x \oplus y)/I,\ (x/F)^* = x^*/F$ and $(x/I)^* = x^*/I$.

Uniform Spaces

Let X be a nonempty set. A *uniformity* on X is a nonempty family $\mathcal{U}$ of subsets of $X \times X$ with the following properties.
(U_1) $\triangle = \{(x, x) : x \in X\} \subseteq U$, for each $U \in \mathcal{U}$.
(U_2) If $U \in \mathcal{U}$, then $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$ belongs to $\mathcal{U}$.
(U_3) If $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some $V \in \mathcal{U}$, where $V \circ V = \{(x, y) : \exists z \in X \text{ s.t. } (x, z), (z, y) \in V\}$.
(U_4) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
(U_5) If $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.
The pair $(X,\mathcal{U})$ is called a *uniform space.*

Let $(X,\mathcal{U})$ be a uniform space. We say that $U \in \mathcal{U}$ is *symmetric* if $U = U^{-1}$. A subfamily $\mathcal{B}$ of $\mathcal{U}$ is called a *base* for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. A subfamily $\mathcal{S}$ of $\mathcal{U}$ is called a *subbase* for $\mathcal{U}$ if the collection of all finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{U}$. [8]

Lemma 2.4. [8] *A nonempty family $\mathcal{B}$ of subsets of $X \times X$ is a base for the uniformity $\mathcal{U} = \{U \subseteq X \times X : \exists B \in \mathcal{B}, B \subseteq U\}$ if and only if the following hold.*
$(B1)$ $\triangle = \{(x, x) : x \in X\} \subseteq U$, *for each* $U \in \mathcal{B}$.
$(B2)$ *If U belongs to $\mathcal{B}$, then U^{-1} contains a member of $\mathcal{B}$.*
$(B3)$ *If U belongs to $\mathcal{B}$, then there exists V in $\mathcal{B}$ such that $V \circ V \subseteq U$.*
$(B4)$ *If U and V are in $\mathcal{B}$, then there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$.*

Suppose that $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces. *The product* of $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is a uniform space $(Z,\mathcal{W})$ with the underlying set $Z = X \times Y$ and the uniformity $\mathcal{W}$ on Z whose base consists of the sets

$$W_{U,V} = \{((x, y), (x', y')) \in Z \times Z : \ (x, x') \in U, (y, y') \in V\},$$

where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. The uniformity $\mathcal{W}$ is written as $\mathcal{W} = \mathcal{U} \times \mathcal{V}$.[8]

Definition 2.2. [8] Let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a map between uniform spaces. The map f is *uniformly continuous* if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ for all $(x, y) \in U$, that is, $(f \times f)(U) \subseteq V$. We denote $f \times f$ by $f^{(2)}$.

In Definition 2.2, if f is bijective and the maps f and f^{-1} are uniformly continuous, then the map f is called a *unimorphism*, and X and Y are said to be *uniformly equivalent*. [8].

3. MV-pseudo norms on MV-algebras

In this section, MV-pseudo norms on MV-algebras are defined and their algebraic properties are discussed. There are also propositions that show the connection between ideals, filters and MV-pseudo norms. The method of building an MV-pseudo norm on the quotient MV-algebras is also stated.

Definition 3.1. Let A be an MV-algebra. Then, we say that a map $N : A \longrightarrow \mathbb{R}$ is an *MV-pseudo norm* on A if the following hold.
$(N1)$ $N(x \oplus y) \leq N(x) + N(y)$.
$(N2)$ $N(x^*) \leq N(1) - N(x)$.
An MV-pseudo norm is an *MV-norm* if
$(N3)$ $N(x) = 0 \Leftrightarrow x = 0$.

Proposition 3.1. *Let N be an MV-pseudo norm on A. Then, the following hold.*
(i) $N(x^*) = N(1) - N(x)$.
(ii) $N(0) = 0$.
(iii) $x \leq y \Longrightarrow N(x) \leq N(y)$. *Moreover,* $N(x) \geq 0$ *for every* $x \in A$.
(iv) $|N(x) - N(y)| \leq N(1)$.

Proof. (i) Since $x \oplus x^* = 1$, by $(N1)$, $N(1) \leq N(x) + N(x^*)$. Hence $N(1) - N(x) \leq N(x^*)$. By $(N2)$, $N(x^*) = N(1) - N(x)$.
(ii) $N(0) = 0$ because $N(1) = N(0^*) = N(1) - N(0)$.
(iii) Let $x \leq y$. Then $x^* \oplus y = 1$. By $(N1)$ and $(N2)$, $N(1) \leq N(x^*) + N(y) = N(1) - N(x) + N(y)$. Therefore $N(x) \leq N(y)$.
(iv) Since $x \leq 1 = 1 \oplus y$, $N(x) \leq N(1) + N(y)$. So $N(x) - N(y) \leq N(1)$. Similarly, $N(y) - N(x) \leq N(1)$. Hence $|N(x) - N(y)| \leq N(1)$. □

Example 3.1. Let X be a finite set and $(P(X), \cup, *, \varnothing, X)$ be the MV-algebra in which for each $B \in P(X)$, B^* is the complement of B in X, i.e., $B^* = X \setminus B$. Define the map $N : P(X) \longrightarrow \mathbb{R}$ by $N(B) = cardB$. For any B and C of $P(X)$, $N(B \cup C) = card(B \cup C) = cardB + cardC - card(B \cap C) \leq cardB + cardC$. Since $B \cup B^* = X$ and $B \cap B^* = \varnothing$, $N(X) = card(B \cup B^*) = cardB + cardB^* = N(B) + N(B^*)$. Thus $N(B^*) = N(X) - N(B)$. Hence, N is an MV-pseudo norm. Finally, if $N(B) = 0$, then $cardB = 0 \Leftrightarrow B = \varnothing$. So, N is an MV-norm.

Example 3.2. (i) Define $\oplus : [0,1] \times [0,1] \longrightarrow [0,1]$ by $x \oplus y = \min\{x+y, 1\}$ and $* : [0,1] \longrightarrow [0,1]$ by $x^* = 1 - x$. Then, $([0,1], \oplus, *, 0)$ is an MV-algebra which is called the *standard MV-algebra* [7]. The map $N : [0,1] \longrightarrow \mathbb{R}$ given by $x \longmapsto |x|$ is an MV-norm because $|x \oplus y| = |\min(1, x+y)| \leq |x| + |y|$. Also, $N(x^*) = |x^*| = |1-x| = 1 - x = 1 - |x|$. Finally, $N(x) = |x| = 0 \Leftrightarrow x = 0$.
(ii) Let $N : A \longrightarrow [0,1]$ be a homomorphism between MV-algebras, where $[0,1]$ is the standard MV-algebra. Then N is an MV-pseudo norm on A. Moreover, N is a norm if and only if N is one-to-one.

Example 3.3. Let $A = \{0, a, b, c, 1\}$. Define $\oplus$ and $*$ as follows.

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	a	1
b	b	1	b	c	1
c	c	a	c	1	1
1	1	1	1	1	1

$*$	0	a	b	c	1
	1	b	a	c	0

Then $(A, \oplus, *)$ is an MV-algebra such that $0 < a < b < 1$ and $0 < a < c < 1$. The map $N : A \longrightarrow \mathbb{R}$, defined by $N(0) = N(a) = N(c) = 0$ and $N(b) = N(1) = 1$, is an MV-pseudo norm on A. Also $N : A \longrightarrow \mathbb{R}$, defined by $N(0) = 0$ and $N(a) = N(b) = N(c) = 1/2$, is an MV-norm on A.

Theorem 3.2. *Let N_1 and N_2 be MV-pseudo norms on A and $\alpha \geq 0$, then*
(i) the function $N : A \longrightarrow \mathbb{R}$, defined by $N(x) = \alpha N_1(x) + N_2(x)$, is an MV-pseudo norm. Moreover, N is an MV-norm, if N_1 and N_2 are MV-norms.
(ii) the map $N(x) = inf\{N_1(z) : z \in \frac{x}{I}\}$ is an MV-pseudo norm, where I is an ideal in A.

Proof. (i) The proof follows from Definition 4.16.
(ii) Since for any $x, y \in I$ and every $z \in \frac{x \oplus y}{I}$, there exist $a \in \frac{x}{I}$ and $b \in \frac{y}{I}$ such that $z = a \oplus b$, it is easy to prove that $N(x \oplus y) \leq N(x) + N(y)$. The fact that for each $x, z \in A$, $z \in \frac{x^*}{I}$ if and only if $z^* \in \frac{x}{I}$ implies N satisfies $(N2)$. Hence N is an MV-pseudo norm on A. $\square$

Proposition 3.3. *Let $N : A \longrightarrow \mathbb{R}$ be an MV-pseudo norm on A.*
(i) If $* \in \{\odot, \ominus, \vee, \wedge\}$, then $N(x * y) \leq N(x) + N(y)$.
(ii) $N(x \rightarrow y) \geq N(y) - N(x)$ and $N(x \rightsquigarrow y) \geq N(y) - N(x)$.
(iii) $N(x \ominus z) \leq N(x \ominus y) + N(y \ominus z)$.
(iv) $N(x \odot y) + N(x^* \oplus y^*) = N(1), N(x \rightarrow y) + N(x^* \oplus y)^* = N(1)$ and $N(x \rightsquigarrow y) + N(x \oplus y^*) = N(1)$.
(v) $N(x^* \ominus y^*) \leq N(y)$.

Proof. (i) The desired result follows directly from $(M18), (M19)$ and $(N1)$.
(ii) By $(M8), (M9), (N1)$ and $(N2)$,

$$N(x \rightarrow y) = N(1) - N(x \odot y^*) \geq N(1) - (N(x) + N(1) - N(y)) = N(y) - N(x).$$

The proof of the other inequality is similar.
(iii) By $(M21)$ and $(N1)$, $N(x \ominus z) \leq N((x \ominus y) \oplus (y \ominus z)) \leq N(x \ominus y) + N(y \ominus z)$.
(iv) By $(M6)$, $N(x^* \oplus y^*) = N(x \odot y)^* = N(1) - N(x \odot y)$. So, $N(x \odot y) + N(x^* \oplus y^*) = N(1)$. By $(M8)$ and $(M9)$, the proofs of the other equalities are similar.
(v) Since $x^* \ominus y^* \leq y$, by Proposition 3.1, $N(x^* \ominus y^*) \leq N(y)$. $\square$

Theorem 3.4. *Let I be an ideal in an MV-algebra A, and N be an MV-pseudo norm on it. Then,*
(i) the map $n : \frac{A}{I} \longrightarrow \mathbb{R}$ defined by $n(\frac{x}{I}) = \inf\{N(z) : z \in \frac{x}{I}\}$ is an MV-pseudo norm on $\frac{A}{I}$;
(ii) if for every $x \in A$, $\min \frac{x}{I}$ exists and N is an MV-norm on A, then $n(\frac{x}{I})$ is also an MV-norm on $\frac{A}{I}$.

Proof. (i) Since N is an MV-pseudo norm on A, the map n is well-defined because $N(z) \geq 0$ for each $z \in A$. To show that n satisfies $(N1)$, let $x, y \in A, a \in \frac{x}{I}$ and $b \in \frac{y}{I}$. Then $a \oplus b \in \frac{x \oplus y}{I}$. By Proposition 3.1$(iii)$, $n(\frac{x}{I} \oplus \frac{y}{I}) = n(\frac{x \oplus y}{I}) \leq N(a \oplus b) \leq N(a) + N(b)$, which implies that $n(\frac{x}{I} \oplus \frac{y}{I}) \leq n(\frac{x}{I}) + n(\frac{y}{I})$. Now we show that $n(\frac{x}{I})^* \leq n(1) - n(\frac{x}{I})$. If $a \in (\frac{x}{I})^*$, then $a^* \in \frac{x}{I}$ and so $n(\frac{x}{I}) \leq N(a^*) = N(1) - N(a)$. Thus $N(a) \leq 1 - n(\frac{x}{I})$. Hence $n(\frac{x}{I})^* \leq 1 - n(\frac{x}{I})$. Therefore, n is an MV-pseudo norm.
(ii) To claim that n satisfies $(N3)$, suppose that $n(\frac{x}{I}) = 0$ for some $x \in A$. By the hypothesis, there exists $a \in A$ such that $a = \min \frac{x}{I}$. It is easy to see that $N(a) = n(\frac{x}{I}) = 0$.

Since N is an MV-norm, $0 = a \in \frac{x}{I}$. Hence $\frac{x}{I} = \frac{0}{I}$. $\square$

Theorem 3.5. *Let F be a filter in an MV-algebra A, and N be an MV-pseudo norm on it. Then,*
(i) the map $n : \frac{A}{F} \longrightarrow \mathbb{R}$ defined by $n(\frac{x}{F}) = \inf\{N(z) : z \in \frac{x}{F}\}$ is an MV-pseudo norm on $\frac{A}{F}$;
(ii) if for every $x \in A$, $\max \frac{x}{F}$ exists and N is an MV-norm on A, then n is an MV-norm on $\frac{A}{F}$.

Proof. The proof is similar to the proof of Theorem 3.4. $\square$

Theorem 3.6. *Let I be an ideal in an MV-algebra A. Then,*
(i) the set $I_N = \{x \in A : N(x) = 0\}$ is an ideal in A if N is an MV-pseudo norm on A;
(ii) if n is an MV-pseudo norm on $\frac{A}{I}$, then $N(x) = n(\frac{x}{I})$ is an MV-pseudo norm on A. Moreover, n is an MV-norm on $\frac{A}{I}$ if and only if $I = I_N$.

Proof. (i) Since $N(0) = 0, 0 \in I_N$. If $x, y \in I_N$, then $N(x) = N(y) = 0$, which implies that $N(x \oplus y) = 0$. Hence $x \oplus y \in I_N$. Now suppose $y \in I_N$ and $x \leq y$. Then by Proposition 3.1(iii), $N(x) \leq N(y) = 0$. Therefore, $N(x) = 0$ and $x \in I_N$.
(ii) Let n be an MV-pseudo norm on $\frac{A}{I}$. Then it is easy to prove that $N(x) = n(\frac{x}{I})$ is an MV-pseudo norm on A. Let n be an MV-norm on $\frac{A}{I}$. If $x \in I$, then $N(x) = n(\frac{x}{I}) = n(\frac{0}{I}) = 0$. If $N(x) = 0$ for some $x \in A$, then $n(\frac{x}{I}) = 0$, which implies that $\frac{x}{I} = \frac{0}{I}$ and so $x \in I$. Hence $I = \{x \in A : N(x) = 0\}$.
Conversely, let $I = \{x \in A : N(x) = 0\}$ and $n(\frac{x}{I}) = 0$. Then $N(x) = 0$ and so $x \in I$. Hence $\frac{x}{I} = \frac{0}{I}$. Thus, the MV-pseudo norm n satisfies $(N3)$ and it is, accordingly, an MV-norm. □

Theorem 3.7. *Let f be an isomorphism from an MV-algebra $(A_1, \oplus, 0)$ to an MV-algebra $(A_2, \oplus, 0)$. If N_{A_1} is an MV-pseudo norm on A_1, then $N_{A_2} : A_2 \longrightarrow \mathbb{R}$, defined by $N_{A_2}(y) = N_{A_1} \circ f^{-1}(y)$ for every $y \in A_2$, is an MV-pseudo norm on A_2, and $N_{A_2}(f(x)) = N_{A_1}(x)$.*

Proof. Let $y, y' \in A_2$. Since f is a bijection, there exist $x, x' \in A_1$ such that $f(x) = y$ and $f(x') = y'$. Hence $N_{A_2}(y) = N_{A_1}(x), N_{A_2}(y') = N_{A_1}(x')$ and since f^{-1} is homomorphism,

$$\begin{aligned} N_{A_2}(y \oplus y') = N_{A_1}(f^{-1}(y) \oplus f^{-1}(y')) = N_{A_1}(x \oplus x') &\leq N_{A_1}(x) + N_{A_1}(x') \\ &= N_{A_2}(y) + N_{A_2}(y'). \end{aligned}$$

This means that N_{A_2} satisfies $(N1)$. Now, we show that $N_{A_2}(y^*) \leq N_{A_2}(1) - N_{A_2}(y)$. To see this, let $y = f(x) \in A_2$. Then $N_{A_2}(y^*) = N_{A_1}(f^{-1}(y^*)) = N_{A_1}(x^*) \leq N_{A_1}(1) - N_{A_1}(x) = N_{A_1}(f^{-1}(1)) - N_{A_1} \circ f^{-1}(y) = N_{A_2}(1) - N_{A_2}(y)$. Therefore, N_{A_2} is an MV-pseudo norm on A_2. Clearly, $N_{A_2}(f(x)) = N_{A_1}(x)$ for every $x \in A$. □

Theorem 3.8. *Let A_1 and A_2 be MV-algebras, and N_{A_1} be an MV-pseudo norm on A_1. If $f : A_1 \longrightarrow A_2$ is an epimorphism, then $N_{A_2} : A_2 \longrightarrow \mathbb{R}$ defined by $y \longmapsto \inf\{N_{A_1}(z) : f(z) = y\}$ is an MV-pseudo norm on A_2, and $N_{A_2}(f(x)) \leq N_{A_1}(x)$.*

Proof. Let $y_1, y_2 \in A_2$ and x_1, x_2 be arbitrary elements of A_1 such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is a homomorphism, $f(x_1 \oplus x_2) = y_1 \oplus y_2$. Thus, $N_{A_2}(y_1 \oplus y_2) \leq N_{A_1}(x_1 \oplus x_2) \leq N_{A_1}(x_1) + N_{A_1}(x_2)$. So $N_{A_2}(y_1 \oplus y_2) - N_{A_1}(x_2) \leq N_{A_1}(x_1)$. Since x_1 is an arbitrary element of A which satisfies $f(x_1) = y_1$,

$$N_{A_2}(y_1 \oplus y_2) - N_{A_1}(x_2) \leq \inf\{N(z) : f(z) = y_1\} = N_{A_2}(y_1).$$

Similarly, since x_2 is an arbitrary element of A which satisfies $f(x_2) = y_2$, by the above inequality,

$$N_{A_2}(y_1 \oplus y_2) - N_{A_2}(y_1) \leq \inf\{N(z) : f(z) = y_2\} = N_{A_2}(y_2).$$

Therefore, N_{A_2} satisfies $(N1)$. Now, we show that $N_{A_2}(y^*) \leq N_{A_2}(1) - N_{A_2}(y)$. To see this, let z be an arbitrary element of A such that $f(z) = y^*$. Since $f(z^*) = y$,

$N_{A_2}(y) \leq N_{A_1}(z^*) = N_{A_1}(1) - N_{A_1}(z)$. Since $N_{A_1}(z) \geq N_{A_2}(y^*)$, $N_{A_1}(1) - N_{A_1}(z) \leq N_{A_1}(1) - N_{A_2}(y^*)$. But $N_{A_2}(1) \leq N_{A_1}(1)$, hence

$$N_{A_2}(y) \leq N_{A_1}(z^*) \leq N_{A_1}(1) - N_{A_1}(z) \leq N_{A_2}(1) - N_{A_2}(y^*).$$

Thus, N_{A_2} is an MV-pseudo norm on A_2. The inequality $N_{A_2}(f(x)) \leq N_{A_1}(x)$ can be verified easily. □

Corollary 3.9. *Let $(A_1, \oplus, 0)$ and $(A_2, \oplus, 0)$ be MV-algebras and $A = A_1 \times A_2$. If A has an MV-pseudo norm, then A_1 and A_2 have MV-pseudo norms.*

Proof. Let A has an MV-pseudo norm, say N. Since the map $\pi_i : A \longrightarrow A_i$ defined by $\pi_i(x_1, x_2) = x_i$ is an epimorphism, by Theorem 3.8, $N_{A_1}(x) = \inf\{N(x, z) : z \in A_2\}$ and $N_{A_2}(y) = \inf\{N(z, y) : z \in A_1\}$ are MV-pseudo norms on A_1 and A_2, respectively. □

4. MV-pseudo metrics on MV-algebras

The MV-pseudo metrics are defined in this section and their algebraic properties and their relationship with MV-pseudo norms are presented. Theorem 4.12 shows the method of constructing MV-pseudo metrics and Theorem 4.14 states the relationship between MV-pseudo metric spaces and topological MV-algebras. We also talk about the relationship between uniform MV-algebras and MV-pseudo metric spaces.

Recall the map $d : X \times X \longrightarrow \mathbb{R}^+$ is called a *pseudo metric* on X if the following conditions hold for all $x, y, z \in X$.
$(D1)$ $d(x, y) \geq 0$ and $d(x, x) = 0$.
$(D2)$ $d(x, y) = d(y, x)$.
$(D3)$ $d(x, z) \leq d(x, y) + d(y, z)$.
A pseudo metric d on A is a *metric* if it satisfies the following condition.
$(D4)$ $d(x, y) = 0 \Longleftrightarrow x = y$.[8]

Definition 4.1. A pseudo metric d on an MV-algebra A is called an *MV-pseudo metric* if for every $x, y, a, b \in A$,
$(D5)$ $d(x \oplus y, a \oplus b) \leq d(x, a) + d(y, b)$, and
$(D6)$ $d(x^*, y^*) \leq d(x, y)$.
An *MV-metric* on A is an MV-pseudo metric that satisfies $(D4)$.

Proposition 4.1. *If d is an MV-pseudo metric on an MV-algebra A and $\star \in \{\odot, \ominus, \rightarrow, \rightsquigarrow, \wedge, \vee\}$, then for every $x, y, a, b \in A$,*
(i) $d(x \star y, a \star b) \leq d(x, a) + d(y, b)$,
(ii) $x \leq y \Longrightarrow d(x, 0) \leq d(y, 0)$,
(iii) $d(x, y) = d(x^*, y^*)$, and
(iv) $d(x \oplus a, y \oplus a) \leq d(x, y)$.

Proof. (i) Let $x, y, a, b \in A$. Then by $(M6)$, $(D5)$ and $(D6)$,

$$d(x \odot y, a \odot b) \leq d(x^* \oplus y^*, a^* \oplus b^*) \leq d(x^*, a^*) + d(y^*, b^*) \leq d(x, a) + d(y, b).$$

The proofs of the other cases are similar.
(ii) Let $x \leq y$. Then $x^* \oplus y = 1$. Hence $d(x, 0) = d((x^*)^*, (x^* \oplus y)^*) \leq d(x^*, x^* \oplus y) \leq d(0, y)$.
The proofs (iii), (iv) are straightforward. □

Proposition 4.2. *Let d be an MV-pseudo metric on an MV-algebra A. Then $d(0,1) = d(0,x) + d(x,1)$ if and only if $N(x) = d(x,0)$ is an MV-pseudo norm on A. Furthermore, N is an MV-norm if d is an MV-metric.*

Proof. Let for any $x \in A$, $d(0,1) = d(0,x) + d(x,1)$. We prove that $N(x) = d(x,0)$ is an MV-pseudo norm on A. For do this, suppose x and y are in A. By $(D5)$, $N(x \oplus y) = d(x \oplus y, 0) \leq d(x,0) + d(y,0) = N(x) + N(y)$. By $(D6)$, $N(x^*) = d(x^*,0) = d(x,1) = d(1,0) - d(0,x) = N(1) - N(x)$. Therefore, N is an MV-pseudo norm on A. Conversely, let $N(x) = d(x,0)$ be an MV-pseudo norm. By Proposition 4.1(iii),

$$d(0,x) + d(1,x) = d(0,x) + d(0,x^*) = N(x) + N(x^*) = N(1) = d(1,0).$$

If d is an MV-metric, then N is an MV-norm clearly. □

Theorem 4.3. *If N is an MV-pseudo norm on an MV-algebra A, then $d_N(x,y) = N(x \ominus y) + N(y \ominus x)$ is an MV-pseudo metric on A.*

Proof. Let x, y, z, a and b be in A. Obviously, $d_N(x,y) \geq 0$, $d_N(x,x) = 0$ and $d_N(x,y) = d_N(y,x)$. By $(M21)$, $N(x \ominus z) \leq N(x \ominus y) + N(y \ominus z)$ and $N(z \ominus x) \leq N(z \ominus y) + N(y \ominus z)$. Hence $d_N(x,z) \leq d_N(x,y) + d_N(y,z)$. Thus d_N is a pseudo metric. Now we show that d_N satisfies $(D5)$ and $(D6)$. By $(M31)$, $(x \oplus y) \ominus (a \oplus b) \leq (x \ominus a) \oplus (y \ominus b)$, and so $N((x \oplus y) \ominus (a \oplus b)) \leq N(x \ominus a) + N(y \ominus b)$. Similarly, $N((a \oplus b) \ominus (x \oplus y)) \leq N(a \ominus x) + N(b \ominus y)$. Hence $(D5)$ holds for d_N. The map d_N satisfies $(D6)$ because by $(M7)$, $d_N(x^*, y^*) = N(x^* \odot y) + N(y^* \odot x) = N(y \ominus x) + N(x \ominus y) = d_N(x,y)$. □

Corollary 4.4. *MV-pseudo metric d_N of Theorem 4.3, satisfies the following properties.*
(i) For every x, $d_N(0,x) + d_N(1,x) = N(1)$.
(ii) The mapping d_N is an MV-metric if and only if N is an MV-norm.
(iii) For every x, $d_N(x,x^*) \leq N(1)$.

Proof. (i) $d_N(0,x) + d_N(1,x) = N(x) + N(x^*) = N(1)$.
(ii) Suppose N is an MV-norm and $d_N(x,y) = 0$. Then $N(x \ominus y) = N(y \ominus x) = 0$, which implies that $x \ominus y = y \ominus x = 0$. Hence $x = y$. Therefore, d_N is an MV-metric. Conversely, suppose d_N is an MV-metric and $N(x) = 0$. Then $d_N(x,0) = N(x \ominus 0) + N(0 \ominus x) = N(x) = 0$. So $x = 0$. Hence N is an MV-norm on A.
(iii) By $(M30)$, $x \ominus x^* \leq x$ and $x^* \ominus x \leq x^*$. So $d_N(x,x^*) = N(x \ominus x^*) + N(x^* \ominus x) \leq N(x) + N(x^*) = N(1)$. □

Remark. From now on, if N is an MV-pseudo norm on an MV-algebra, then d_N is the MV-pseudo metric induced by N in Theorem 4.3.

Proposition 4.5. *Let $f : A_1 \longrightarrow A_2$ be an isomorphism between MV-algebras. If N_1 is an MV-pseudo norm on A_1, then there exist MV-pseudo metrics d_1 and d_2 on A_1 and A_2, respectively, such that f is isometry, i.e, $d_1(x,y) = d_2(f(x), f(y))$.*

Proof. By Theorem 3.7, the map $N_2 : A_2 \longrightarrow \mathbb{R}$ defined by $N_2(y) = N_1 \circ f^{-1}(y)$ is an MV-pseudo norm on A_2. By Theorem 4.3, d_{N_1} and d_{N_2} are MV-pseudo metrics on A_1 and A_2, respectively. If $x, y \in A$, then by Theorem 3.7, $d_{N_2}(f(x), f(y)) = N_2(f(x \ominus y)) + N_2(f(y \ominus x)) = N_1(x \ominus y) + N_1(y \ominus x) = d_{N_1}(x,y)$. □

Proposition 4.6. *Let $f : A_1 \longrightarrow A_2$ be an epimorphism from an MV-algebra A_1 to an MV-algebra A_2. If N_1 is an MV-pseudo norm on A_1, then there exist MV-pseudo metrics d_1 and d_2 on A_1 and A_2, respectively, such that $d_2(f(x), f(y)) \leq d_1(x, y)$.*

Proof. By Theorem 3.8, the map $N_2 : A_2 \longrightarrow \mathbb{R}$, given by $N_2(y) = \inf\{N(z) : f(z) = y\}$, is an MV-pseudo norm on A_2 such that $N_2(f(x)) \leq N_1(x)$ for every $x \in A$. By Theorem 4.3, d_{N_1} and d_{N_2} are MV-pseudo metrics on A_1 and A_2, respectively. For every $x, y \in A_1$,

$$d_{N_2}(f(x), f(y)) = N_2(f(x \ominus y)) + N_2(f(y \ominus x)) \leq N_1(x \ominus y) + N_1(y \ominus x) = d_{N_1}(x, y).$$

□

Proposition 4.7. *Let A_1, A_2 and $A = A_1 \times A_2$ be MV-algebras. Then:*
(i) If N_1, N_2 are MV-pseudo norms on A_1 and A_2, respectively, then there is an MV-pseudo norm N on A such that $d_N(x, y, a, b) = d_{N_1}(x, a) + d_{N_2}(y, b)$;
(ii) if N is an MV-pseudo norm on A, then there exist MV-pseudo norms N_1 and N_2 on A_1 and A_2, respectively, such that $d_N(x, y, a, b) \geq d_{N_1}(x, a)$ and *$d_{N_2}(y, b)$.*

Proof. (i) The map $N : A \to \mathbb{R}$ defined by $N(x, y) = N_1(x) + N_2(y)$ is an MV-pseudo norm on A. Now $d_N(x, y, a, b) = N(x \ominus a, y \ominus b) + N(a \ominus x, b \ominus y) = N_1(x \ominus a) + N_2(y \ominus b) + N_1(a \ominus x) + N_2(b \ominus y) = d_{N_1}(x, a) + d_{N_2}(y, b)$.

(ii) By Corollary 3.9, $N_1(x) = \inf\{N(x, z) : z \in A_2\}$ and $N_2(x) = \inf\{N(z, x) : z \in A_1\}$ are MV-pseudo norms on A_1 and A_2, respectively. Let $x, a \in A_1$ and $y, b \in A_2$. Then

$$d_N(x, y, a, b) = N(x \ominus a, y \ominus b) + N(a \ominus x, b \ominus y) \geq N_1(x \ominus a) + N_1(a \ominus x) = d_{N_1}(x, a).$$

In a similar way, we can show that $d_N(x, y, a, b) \geq d_{N_2}(y, b)$. □

Let $\{A_i : i \in I\}$ be a family of MV-algebras and N_i be an MV-pseudo norm on A_i, for any $i \in I$. If the family $\{N_i(1_i) : i \in I\}$ is bounded, then it is easy to verify that $N(\{x_i\}_{i \in I}) = \sum_{i=1}^{\infty} \frac{N_i(x_i)}{2^i}$ is an MV-pseudo norm on $A = \prod_{i=1}^{\infty} A_i$ such that $d_N(\{x_i\}_{i \in I}, \{y_i\}_{i \in I}) = \sum_{i=1}^{\infty} \frac{d_{N_i}(x_i, y_i)}{2^i}$. Also, if N is an MV-pseudo norm on MV-algebra $A = \prod_{i=1}^{\infty} A_i$, then for each $k \in I$, the map $N_k : A_k \to \mathbb{R}$ defined by $N_k(x) = inf\{N(\{x_i\}_{i \in I}) : x_k = x, x_i \in A_i\}$ is an MV-pseudo norm on A_k such that

$$d_N(\{x_i\}_{i \in I}, \{y_i\}_{i \in I}) \geq d_{N_k}(x_k, y_k).$$

In continue we are going to talk about the relation between MV-pseudo metrics and uniform MV-algebras. To do this, we first recall the definition of uniform MV-algebras.

Let A be an MV-algebra and $\mathcal{U}$ be a uniformity on A. By Definition 2.2,
(i) the operation $\oplus : (A \times A, \mathcal{U} \times \mathcal{U}) \to (A, \mathcal{U})$ is *uniformly continuous* if for every $W \in \mathcal{U}$, there exist $U, V \in \mathcal{U}$ such that $U \oplus V \subseteq W$ or equivalently, for every $(x, x') \in U$ and $(y, y') \in V$, $(x \oplus y, x' \oplus y') \in W$;
(ii) the map $* : (A, \mathcal{U}) \to (A, \mathcal{U})$ is *uniformly continuous* if for every $W \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that if $(x, y) \in V$, then $(*(x), *(y)) \in W$.
The pair $(A, \mathcal{U})$ is called a *uniform MV-algebra* if $\oplus$ and $*$ are uniformly continuous.

Let d be an MV-pseudo metric on an MV-algebra A. Then, it is easy to prove that the set $\mathcal{B} = \{U_\epsilon : \varepsilon > 0\}$ is a base for a uniformity $\mathcal{U}_d$ on A, where $U_\epsilon = \{(x, y) : d(x, y) < \epsilon\}$. Thus, by Definition 2.2 and $(D5)$ and $(D6)$, the operations $\oplus$ and $*$ are uniformly continuous.

Example 4.1. (i) Let $([0,1],\oplus,*,0)$ be the standard MV-algebra and $d(x,y)=|x-y|$, for any $x,y\in[0,1]$. Then for every $x,y\in[0,1]$, $d(x^*,y^*)=|x^*-y^*|=|1-x-1+y|=|y-x|=d(x,y)$. Hence d satisfies $(D6)$. The following steps show that d satisfies $(D5)$. Let x,x',y and y' be arbitrary element of $[0,1]$. Then:

Step 1. If $x+x'<1$ and $y+y'<1$, then

$$d(x\oplus x',y\oplus y')=|x+x'-y-y'|\leq|x-y|+|x'-y'|=d(x,y)+d(x',y').$$

Step 2. If $x+x'<1$ and $y+y'\geq 1$, then

$$d(x\oplus x',y\oplus y')=|x+x'-1|\leq|y+y'-x-x'|\leq|y-x|+|y'-x'|=d(x,y)+d(x',y').$$

Step 3. If $x+x'\geq 1$ and $y+y'<1$, the proof is similar to that of step 2.

Step 4. If $x+x'\geq 1$ and $y+y'\geq 1$, then

$$d(x\oplus x',y\oplus y')=|x\oplus x'-y\oplus y'|=|1-1|=0<d(x,y)+d(x',y').$$

Therefore, $([0,1],\mathcal{U}_d)$ is a uniform MV-algebra.

Proposition 4.8. *Let $\mathcal{U}$ be a uniformity on an MV-algebra A. If $*$ is uniformly continuous, then:*
(i) uniformly continuity $\oplus,\odot,\ominus,\to$, and $\rightsquigarrow$ are equivalent;
(ii) the map $f:(A,\mathcal{U})\times(A,\mathcal{U})\longrightarrow(A,\mathcal{U})$ given by $f(x,y)=x\oplus y^*$ is uniformly continuous if and only if the map $\oplus$ is uniformly continuous.

Proof. (i) Since the composition of two uniformly continuous functions is uniformly continuous, the conditions $(M6),(M7),(M8)$ and $(M9)$ show that uniformly continuity $\oplus,\odot,\ominus,\to$, and $\rightsquigarrow$ are equivalent.

(ii) Let $\oplus$ be uniformly continuous and $W\in\mathcal{U}$. Then there exist U_1,U_2 and V in $\mathcal{U}$ such that $U_1\oplus U_2\subseteq W$ and $V^*\subseteq U_2$. Hence $U_1\oplus V^*\subseteq U_1\oplus U_2\subseteq W$ and so f is uniformly continuous. Conversely, let f be uniformly continuous and W be in $\mathcal{U}$. Then for some $U,V,V_1\in\mathcal{U}_d$, $U\oplus {V_1}^*\subseteq W$ and $V^*\subseteq V_1$. Hence $U\oplus V=U\oplus(V^*)^*\subseteq U\oplus {V_1}^*\subseteq W$, which shows that $\oplus$ is uniformly continuous. □

Proposition 4.9. *Let N be an MV-pseudo norm on an MV-algebra A. Then:*
(i) there are pseudo metrics d'_N and d''_N on A such that $\oplus$ and $*$ are uniformly continuous in the uniform spaces (A,d'_N) and (A,d''_N), respectively.
(ii) there is a uniformity $\mathcal{U}_N$ on A such that $(A,\mathcal{U}_N)$ is a uniform MV-algebra and $\mathcal{U}_{d_N}\subseteq\mathcal{U}_N$, where d_N is the MV-pseudo metric in Theorem 4.3.

Proof. (i) The maps d'_N and d''_N from $A\times A$ to $\mathbb{R}^+$ defined by $d'_N(x,y)=|N(x)-N(y)|$ and

$$d''_N(x,y)=\begin{cases}0 & \text{if } x=y\\ N(x)+N(y) & \text{if } x\neq y\end{cases}$$

are pseudo metrics on A, which satisfy the inequalities $d'_N(x^*,y^*)\leq d'_N(x,y)$ and $d''_N(x\oplus y,a\oplus b)\leq d'_N(x,a)+d'_N(y,b)$. Hence $*$ and $\oplus$ are uniformly continuous in (A,d'_N) and (A,d''_N), respectively.

(ii) It is easy to prove that $x\overset{I_N}{\equiv}y$ if and only if $d_N(x,y)=0$, where $I_N=\{x\in A:N(x)=0\}$. If $U_N=\{(x,y):x\overset{I_N}{\equiv}y\}$, then the set $\{U_N\}$ satisfies $(B1),(B2)$ and $(B3)$ of Lemma 2.4. Hence $\{U_N\}$ is a base for a uniformity $\mathcal{U}_N$ on A. By Theorem 4.3 and Corollary 4.4, $U_N\oplus U_N\subseteq U_N$ and $U_N^*=U_N$. Hence $(A,\mathcal{U}_N)$ is a uniform

MV-algebra. Since $\{U_\varepsilon\}_{\varepsilon>0}$, where $U_\varepsilon = \{(x,y) : d_N(x,y) < \varepsilon\}$, is a base for the uniformity $\mathcal{U}_{d_N}$, obviously, $\mathcal{U}_{d_N} \subseteq \mathcal{U}_N$. □

Proposition 4.10. *If N is an MV-pseudo norm on an MV-algebra A, then N and d_N are uniformly continuous in uniform MV-algebra $(A, \mathcal{U}_{d_N})$, where d_N is the MV-pseudo metric in Theorem 4.3.*

Proof. For every $x, y \in A$, $y \leq (y \ominus x) \oplus x$ because $y^* \oplus (y \odot x^*) \oplus x = (x \oplus y^*) \oplus (y \odot x^*) = 1$. By $(N2)$, $N(y) \leq N(x) + N(y \ominus x)$ and so $N(y) - N(x) \leq N(y \ominus x) \leq N(y \ominus x) + N(x \ominus y)$. Similarly, $N(x) - N(y) \leq N(x \ominus y) \leq N(x \ominus y) + N(y \ominus x)$. Hence $|N(x) - N(y)| \leq d_N(x,y)$, which implies that N is uniformly continuous. Since the composition of uniformly continuous functions is uniformly continuous, by Proposition 4.8, the MV-pseudo metric d_N is uniformly continuous. □

A subset S of an MV-algebra A is said to be *convex* if for any $x, y, z \in A$, $x \leq z \leq y$, and $x, y \in S$ imply that $z \in S$.

Proposition 4.11. *Let A be an MV-algebra, $S \subseteq A$ and $\widehat{S} = \{x \in A : \exists\, y \in$ S such that $x \leq y\}$. Then,*
(i) if $0 \in S$, then S is convex if and only if for any $x, y \in A$, if $x \leq y$ and $y \in S$, then $x \in S$;
(ii) $0 \in \widehat{S}$ and $\widehat{S}$ is the smallest convex set of A containing S;
(iii) if $S \subseteq T$, then $\widehat{S} \subseteq \widehat{T}$;
(iv) $\widehat{S} \oplus \widehat{T} \subseteq \widehat{S \oplus T}$.

Proof. By the definition of convex set, the proofs of (i), (ii) and (iii) are obvious. We only prove (iv). Let $z \in \widehat{S} \oplus \widehat{T}$. Then for some $x \in \widehat{S}$ and $y \in \widehat{T}$, $z = x \oplus y$. Since $x \in \widehat{S}$ and $y \in \widehat{T}$, there are $x_1 \in S$ and $y_1 \in T$ such that $x \leq x_1$ and $y \leq y_1$. Now $z = x \oplus y \leq x_1 \oplus y_1 \in S \oplus T$. So $z \in \widehat{S \oplus T}$. □

Remark. Let d be a pseudo metric on MV-algebra A. We denote the set $\{x : d(x,0) < r\}$ by $B(r)$ i.e $B(r) = \{x : d(x,0) < r\}$. Also, we recall that the first part of the proof of the following theorem is from [1].

Theorem 4.12. *Let $\{U_n\}_{n \geqslant 0}$ be a family of subsets of an MV-algebra A such that $0 \in U_n$ and $U_{n+1} \oplus U_{n+1} \subseteq U_n$ for any $n \geq 0$. Then there is an MV-pseudo metric d on A such that the operations $\oplus$ and $*$ are uniformly continuous on $(A, \mathcal{U}_d)$ and for any $n \geq 0$,*

$$\{x : d(x,0) < 1/2^n\} \subseteq \widehat{U_n} \subseteq \{x : d(x,0) < 2/2^n\}.$$

Moreover, d is an MV-metric if and only if $\bigcap_{n \geq 0} \widehat{U}_n = 0$.

Proof. Let $V(1) = U_0$, $n \geq 0$ and assume that $V(\frac{m}{2^n})$ are defined for each $m = 1, 2, 3, ..., 2^n$ such that $0 \in V(\frac{m}{2^n})$. Put then $V(\frac{1}{2^{n+1}}) = U_{n+1}$, $V(\frac{2m}{2^{n+1}}) = V(\frac{m}{2^n})$ for $m = 1, 2, 3, ..., 2^n$ and for each $m = 1, 2, 3, ..., 2^n - 1$, $V(\frac{2m+1}{2^{n+1}}) = V(\frac{m}{2^n}) \oplus U_{n+1} = V(\frac{m}{2^n}) \oplus V(\frac{1}{2^{n+1}})$. We also define $V(\frac{m}{2^n}) = A$, when $m > 2^n$. By induction on n we prove that for any $m > 0$ and $n \geq 0$,

$$(*) \qquad V(\frac{m}{2^n}) \oplus V(\frac{1}{2^n}) \subseteq V(\frac{m+1}{2^n}).$$

First notice that if $m + 1 > 2^n$, then $(*)$ is obviously true. Let $m < 2^n$. If $n = 1$, then m is also 1, so $V(\frac{1}{2}) \oplus V(\frac{1}{2}) = U_1 \oplus U_1 \subseteq U_0 = V(1)$. Assume that $(*)$ holds for some n. We verify it for $n+1$. If $m=2k$, then by the definition of $V(\frac{2m+1}{2^{n+1}})$, $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{k}{2^n}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k+1}{2^n})$. Suppose now that $m = 2k+1 < 2^{n+1}$ for some $x \geq 0$. Then

$$\begin{aligned} V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k+1}{2^{n+1}}) \oplus U_{n+1} &= V(\frac{k}{2^n}) \oplus U_{n+1} \oplus U_{n+1} \\ &\subseteq V(\frac{k}{2^n}) \oplus U_n = V(\frac{k}{2^n}) \oplus V(\frac{1}{2^n}). \end{aligned}$$

But by the inductive assumption, $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) \subseteq V(\frac{k+1}{2^n}) = V(\frac{m+1}{2^{n+1}})$.

By Proposition 4.11, for any $r \geq 0$, $\widehat{V(r)}$ is a convex set containing 0, it is easy to derive that the map $f : A \longrightarrow \mathbb{R}$ defined by $f(x) = inf\{r : x \in \widehat{V(r)}\}$ is increasing bounded function. Define the map $N : A \longrightarrow \mathbb{R}$ by $N(x) = sup\{f(x \oplus z) - f(z) : z \in A\}$. The function N is obviously well defined and increasing. In a similar method with the proof of Theorem 4.3, we can show that $d_N(x,y) = N(x \ominus y) + N(y \ominus x)$ is an MV-pseudo metric. By $(D5)$ and $(D6)$, we can prove that the operations $\oplus$ and $*$ are uniformly continuous on $(A, \mathcal{U}_{d_N})$. Let us prove that d_N satisfies

$$\{x : d_N(x,0) < \frac{1}{2^n}\} \subseteq \widehat{U_n} \subseteq \{x : d_N(x,0) \leq \frac{2}{2^n}\}.$$

Notice that $f(0) = 0$, hence if $d_N(x,0) < \frac{1}{2^n}$, then $f(x) = f(x \oplus 0) - f(0) \leq N(x) = d_N(x,0) < \frac{1}{2^n}$. Hence for some $0 \leq r < \frac{1}{2^n}$, $x \in \widehat{V(r)}$. Since $V(r) \subseteq V(\frac{1}{2^n}) = U_n$, $x \in \widehat{V(r)} \subseteq \widehat{V(\frac{1}{2^n})} = \widehat{U_n}$. Now let $x \in \widehat{U_n}$. Then there is a $x' \in U_n$ such that $x \leq x'$. Clearly for any $z \in A$, there exists a $k \geq 0$ such that $\frac{k-1}{2^n} \leq f(z) \leq \frac{k}{2^n}$. Since $z \in \widehat{V(\frac{k}{2^n})}$, there is a $z' \in V(\frac{k}{2^n})$ such that $z \leq z'$. From condition $(*)$ it follows that $z' \oplus x' \in V(\frac{k}{2^n}) \oplus V(\frac{1}{2^n}) \subseteq V(\frac{k+1}{2^n})$ and from $z \oplus x \leq z' \oplus x'$ deduces that $z \oplus x \in \widehat{V(\frac{k+1}{2^n})}$. Hence $f(x \oplus z) - f(z) \leq \frac{k+1}{2^n} - \frac{k-1}{2^n} = \frac{2}{2^n}$.

In the end of proof, let us prove that d_N is an MV-metric if and only if $\bigcap_{n\geq 0} \widehat{U_n} = 0$. Let $\bigcap_{n\geq 0} \widehat{U_n} = 0$ and $d_N(x,y) = 0$. Then $N(x \ominus y) = N(y \ominus x) = 0$. Hence for any $n \geq 0$, $x \ominus y$ and $y \ominus x$ are in $\widehat{U_n}$. This concludes that $x \ominus y = y \ominus x = 0$ and so $x = y$. Therefore d_N is metric.

Conversely let d_N be metric and $x \in \bigcap_{n\geq 0} \widehat{U_n}$. Since $\widehat{U_n} \subseteq \{x : d_N(x,0) \leq \frac{2}{2^n}\}$ for every $n \geq 0$, we derive that $d_N(x,0) = 0$. This implies that $x = 0$. □

Corollary 4.13. *In Theorem 4.12, if $g : A \longrightarrow A$ is an isomorphism such that $g(U_n) = U_n$ for any $n \geq 0$, then $d_N(g(x), g(y)) = d_N(x,y)$ for any $x, y \in A$, i.e. $g : (A, \mathcal{U}_{d_N}) \longrightarrow (A, \mathcal{U}_{d_N})$ is uniformly continuous.*

Proof. Since g is an isomorphism map and $g(U_n) = U_n$ for any $n \geq 0$, $g(V(r)) = V(r)$ for every $r \geq 0$. Let $x \in A$. Then $f \circ g(x) = inf\{r : g(x) \in V(r)\} = inf\{r : x \in g^{-1}(V(r))\} = inf\{r : x \in V(r)\} = f(x)$. From this, we deduce that $N(g(x)) = sup\{f(g(x) \oplus z) - f(z) : z \in A\} = sup\{f(g(x) \oplus g(z')) - f(g(z')) : z' \in A\} = sup\{f \circ g(x \oplus z') - f \circ g(z') : z' \in A\} = sup\{f(x \oplus z') - f(z') : z' \in A\} = N(x)$. Thus it is obvious that $d_N(g(x), g(y)) = d_N(x,y)$ for any $x, y \in A$. This implies that $g : (A, \mathcal{U}_{d_N}) \longrightarrow (A, \mathcal{U}_{d_N})$ is uniformly continuous. □

Theorem 4.14. *Let A be a MV-algebra. Then, there is an MV-pseudo metric d on A such that $(A, \mathcal{U}_d)$ is a uniform MV-algebra if and only if there is a topology τ on*

A such that (A,τ) is a topological MV-algebra and τ has a countable local base at 0. Moreover, d is continuous in (A,τ).

Proof. The necessity is obvious. We prove the sufficiency. Fix a countable base $\{W_n : n \geq 0\}$ of a topological MV-algebra (A,τ) at 0. Since $\oplus$ is continuous, by induction, we obtain a sequence $\{U_n : n \geq 0\}$ of open neighborhoods of 0 such that $U_n \subseteq W_n$ and $U_{n+1} \oplus U_{n+1} \subseteq U_n$ for each $n \geq 0$. This sequence is also a base of (A,τ) at 0. By Theorem 4.12, there exists an MV-pseudo metric d_N on A such that $(A,\mathcal{U}_{d_N})$ is a uniform MV-algebra and $B(\frac{1}{2^n}) \subseteq \widehat{U_n} \subseteq \{x : d_N(x,0) \leq \frac{2}{2^n}\}$, where N is the map defined in Theorem 4.12.

We are now going to show that d_N is continuous in (A,τ). To do this, we first prove that N is continuous at 0 and then derive that N is continuous on A. Let $\epsilon > 0$ be arbitrary. Then there exists an $n \geq 1$ such that $\frac{2}{2^n} < \epsilon$. Now the relation, $U_n \subseteq \widehat{U_n} \subseteq \{x : d_N(x,0) \leq \frac{2}{2^n}\}$, deduces that N is continuous at 0. To continuity N on A, take a $b \in A$ and assume that $\epsilon > 0$ is arbitrary. It is easy to verify that for arbitrary x in A, $N(x \oplus b) \leq N(x) + N(b)$. By $(M33)$, $(b \ominus x) \oplus x = (x \ominus b) \oplus b \geq b$, so $N(b) \leq N(b \ominus x) + N(x)$. This inequality implies that $\mid N(x) - N(b) \mid \leq max\{N(b \ominus x), N(x \ominus b)\}$. Since N is continuous at 0, there is an $n \geq 0$ such that $N(x) < \epsilon$, for any $x \in U_n$. Since $\ominus$ is continuous and $b \ominus b = 0 \in U_n$, there is an open neighborhood V of b such that $b \ominus V \subseteq U_n$ and $V \ominus b \subseteq U_n$. Thus for each $x \in V$, $\mid N(x) - N(b) \mid \leq max\{N(b \ominus x), N(x \ominus b)\} < \epsilon$, which implies that N is continuous at b. Finally, since N and $\ominus$ are continuous, it follows that d_N is also continuous. □

If d is an MV-pseudo metric on MV-algebra A, then it is easy to derive that $B(r) = \{x : d(x,0) < r\}$ is a convex set. Hence $\{B(\frac{1}{2^n}) : n \geq 0\}$ is a countable local base at 0 which every element of it is convex. In Theorem 4.14, for any $n \geq 0$, there is a $k \geq 0$ such that $0 \in U_k \subseteq \widehat{U_k} \subseteq \{x : d(x,0) \leq \frac{2}{2^k}\} \subseteq \{x : d(x,0) < \frac{1}{2^n}\}$. Hence the sequence $\{B(\frac{1}{2^n}) : n \geq 0\}$ is a local base at 0 of τ such that for any $n \geq 0$ the set $B(\frac{1}{2^n})$ is a convex set. Moreover, if transfer map $x \to x \oplus b$ is open in (A,d) and (A,τ), for every $b \in A$, then continuity $\oplus$ implies that τ is a uniform topology.

Corollary 4.15. *Let A be a MV-algebra. Then, there is a continuous MV-pseudo metric d on A such that $(A,\mathcal{U}_d)$ is a uniform MV-algebra if and only if there is a uniformity $\mathcal{U}$ on A such that $(A,\mathcal{U})$ is a uniform MV-algebra and $\mathcal{U}$ has a countable base.*

Proof. The necessity is obvious. Let $\mathcal{U}$ be a uniformity on A such that $(A,\mathcal{U})$ is a uniform MV-algebra and $\{U_n : n \geq 0\}$ be a countable base of it. Then the sequence $\{U_n[0] : n \geq 0\}$ is a local base at 0. Now by Theorem 4.14, the proof is clear. □

Proposition 4.16. *Let $S = \{N_i : i \in I\}$ be a chain of MV-pseudo norms on an MV-algebra A. Then, there exists a uniformity $\mathcal{U}$ on A such that $(A,\mathcal{U})$ is a uniform MV-algebra.*

Proof. It is easy to prove that if $N_i, N_j \in S$, then $I_{N_j} \subseteq I_{N_i}$ if $N_j \geq N_i$, where $I_{N_i} = \{x : N_i(x) = 0\}$. If $\mathcal{I} = \{I_{N_i} : i \in I\}$, then $\mathcal{I}$ is, clearly, a family of ideals that is closed under finite intersection. Let $U_i = \{(x,y) : x \overset{I_{N_i}}{\equiv} y\}$ and $H = \{U_i : i \in I\}$. Then it is easy to verify that H satisfies $(B_1), (B_2), (B_3)$ and (B_4) of Lemma 2.4. So, it is a base for the uniformity $\mathcal{U} = \{U \subseteq A \times A : U_i \subseteq U, \text{for some } U_i \in H\}$.

Let $(x, y), (a, b) \in U_i \in H$. Then $x \overset{I_{N_i}}{\equiv} y$ and $a \overset{I_{N_i}}{\equiv} b$. By Proposition 2.3, $\overset{I_{N_i}}{\equiv}$ is a congruence relation. So, $a \oplus x \overset{I_{N_i}}{\equiv} b \oplus y$ and $a^* \overset{I_{N_i}}{\equiv} b^*$. Hence $U_i \oplus U_i \subseteq U_i$ and $U_i^* \subseteq U_i$. This implies that $(A, \mathcal{U})$ is a uniform MV-algebra. $\square$

Proposition 4.17. *Suppose A is an MV-algebra, I is an ideal and $q : A \longrightarrow \frac{A}{I}$, given by $q(x) = \frac{x}{I}$, is the quotient map. Then there are uniformities η_I and ε_I on A and $\frac{A}{I}$ such that (A, η_I) and $(\frac{A}{I}, \varepsilon_I)$ are uniform MV-algebras and $q : (A, \eta_I) \to (\frac{A}{I}, \varepsilon_I)$ is uniformly continuous.*

Proof. Before proving the theorem, we first show that the set $S \subseteq \frac{A}{I}$ is an ideal in $\frac{A}{I}$ if and only if $S = \frac{J}{I}$ for some ideal J of A containing I. Suppose S is an ideal of $\frac{A}{I}$ and $J = \{x \in A : \frac{x}{I} \in S\}$. Since S is an ideal, $\frac{0}{I} \in S$, and so $0 \in J$. If $y, x \ominus y \in J$, then $\frac{y}{I}, \frac{x}{I} \ominus \frac{y}{I} \in S$. Since S is an ideal, $\frac{x}{I} \in S$ and so $x \in J$. Thus J is an ideal of A such that $I \subseteq J$. To prove $S = \frac{J}{I}$, let $\frac{x}{I} \in \frac{J}{I}$. Then for some $z \in J$, $\frac{z}{I} = \frac{x}{I}$. Hence $x \ominus z \in I \subseteq J$. Since $z \in J$, by Proposition 2.2, $x \in J$. This implies that $\frac{J}{I} \subseteq S$. It is easy to see that $S \subseteq \frac{J}{I}$. Hence $\frac{J}{I} = S$. Conversely, suppose J is an ideal of A such that $I \subseteq J$ and $S = \{\frac{x}{I} | x \in J\}$. Then $0 \in J$ implies that $\frac{0}{I} \in S$. If $\frac{y}{I}, \frac{x \ominus y}{I} \in S$, then $y, x \ominus y \in J$. By Proposition 2.2, $x \in J$. Hence $\frac{x}{I} \in S$. Consequently, S is an ideal of $\frac{A}{I}$.

Now to prove theorem let H be the family of ideals J in A such that $I \subseteq J$ and let $H' = \{\frac{J}{I} : J \in H\}$. Put $U_J = \{(x, y) : x \overset{J}{\equiv} y\}$ and $U_{\frac{J}{I}} = \{(\frac{x}{I}, \frac{y}{I}) : \frac{x}{I} \overset{\frac{J}{I}}{\equiv} \frac{y}{I}\}$, for every $J \in H$. Since H and H' are closed under finite intersection, the sets $B = \{U_J : J \in H\}$ and $B' = \{U_{\frac{J}{I}} : J \in H\}$ satisfy $(B_1), (B_2), (B_3)$ and (B_4) of Lemma 2.4. Hence they are bases for uniformities η_I and ε_I on A and $\frac{A}{I}$, respectively. In a similar way to the proof of Proposition 4.16, we can obtain that (A, η_I) and $(\frac{A}{I}, \varepsilon_I)$ are uniform MV-algebras. If J is an ideal of A containing I, then $U_{\frac{J}{I}} = q^{(2)}(U_J)$, because for every $x, y \in A$,

$$(\frac{x}{I}, \frac{y}{I}) \in U_{\frac{J}{I}} \Leftrightarrow \frac{x \ominus y}{I}, \frac{y \ominus x}{I} \in \frac{J}{I} \Leftrightarrow x \ominus y, y \ominus x \in J \Leftrightarrow (x, y) \in U_J.$$

Hence $q : (A, \eta_I) \to (\frac{A}{I}, \varepsilon_I)$ is uniformly continuous. $\square$

Proposition 4.18. *Let N be an MV-pseudo norm and I be an ideal in an MV-algebra A. Then there exists an MV-pseudo metric D_n on $\frac{A}{I}$ such that $(\frac{A}{I}, \mathcal{U}_{D_n})$ is a uniform MV-algebra and the quotient map $q : (A, \mathcal{U}_{d_N}) \longrightarrow (\frac{A}{I}, \mathcal{U}_{D_n})$, given by $q(x) = \frac{x}{I}$, is uniformly continuous.*

Proof. By Proposition 3.4, the mapping $n(\frac{x}{I}) = \inf\{N(z) : z \in \frac{x}{I}\}$ is an MV-pseudo norm on $\frac{A}{I}$, and by Theorems 4.3, the map $D_n(\frac{x}{I}, \frac{y}{I}) = n(\frac{x}{I} \ominus \frac{y}{I}) + n(\frac{y}{I} \ominus \frac{x}{I})$ is an MV-pseudo metric on $\frac{A}{I}$ such that $(\frac{A}{I}, \mathcal{U}_{D_n})$ is a uniform MV-algebra. It is easy to prove that $D_n(\frac{x}{I}, \frac{y}{I}) \leq d_N(x, y)$. Hence, the quotient map $q : A \longrightarrow \frac{A}{I}$ is uniformly continuous. $\square$

Proposition 4.19. *Let N be an MV-pseudo norm on an MV-algebra A. Then:*
(i) for every $e \in A$, the set $I_e = \{x \in A : N(e \oplus x) = 0\} \cup \{0\}$ is an ideal contained in I_N. Moreover, there exists a uniformity $\mathcal{U}$ on A such that $(A, \mathcal{U})$ is a uniform MV-algebra;

(iii) there exists an MV-pseudo metric D on $\frac{A}{I_e}$ such that $(\frac{A}{I_e},\mathcal{U}_D)$ is a uniform MV-algebra;

Proof. (i) Let $x, y \in I_e$. Then $N(e \oplus (x \oplus y)) \leq N(e \oplus x) + N(e \oplus y) = 0$. Hence $x \oplus y \in I_e$. If $x \leq y$ and $y \in I_e$, then the inequality $e \oplus x \leq e \oplus y$ implies that $N(e \oplus x) = 0$. So $x \in I_e$. Thus I_e is an ideal in A. By $(M19)$, it is obvious that $I_e \subseteq I_N$.

The set $B = \{I_e : e \in A\}$ is a family of ideals which is closed under finite intersection because $I_e \cap I_c = I_{e \oplus c}$, for every $e, c \in A$,. If for any $e \in A$, $U_e = \{(x, y) : x \overset{I_e}{\equiv} y\}$, then it is easy to show that the set $B = \{I_e : e \in A\}$ satisfies $(B1), (B2)$ and $(B3)$ of Lemma 2.4. Hence it is a base for a uniformity $\mathcal{U}$ on A. In a similar way to the proof of Proposition 4.16, we can prove that operations $\oplus$ and $*$ are uniformly continuous in $(A,\mathcal{U})$.

(ii) Define the map $D : \frac{A}{I_e} \times \frac{A}{I_e} \longrightarrow \mathbb{R}$ by $D(\frac{x}{I_e}, \frac{y}{I_e}) = d_N(x, y)$. If $\frac{x}{I_e} = \frac{a}{I_e}$ and $\frac{y}{I_e} = \frac{b}{I_e}$, then

$$N(e \oplus (x \ominus a)) = N(e \oplus (a \ominus x)) = N(e \oplus (y \ominus b)) = N(e \oplus (b \ominus y)) = N(e \oplus (b \ominus y)) = 0.$$

By $(M21)$ and $(M7)$,

$$x \ominus y \leq (x \ominus a) \oplus (a \ominus b) \oplus (b \ominus y) \leq [e \oplus (x \ominus a)] \oplus (a \ominus b) \oplus [e \oplus (b \ominus y)].$$

Hence $N(x \ominus y) \leq N(a \ominus b)$. In a similar way, $N(a \ominus b) \leq N(x \ominus y)$, and so $N(x \ominus y) = N(a \ominus b)$. Similarly $N(y \ominus x) = N(b \ominus a)$. So, $D(\frac{x}{I_e}, \frac{y}{I_e}) = D(\frac{a}{I_e}, \frac{b}{I_e})$. Since d_N is an MV-pseudo metric, it is easy to prove that D is an MV-pseudo metric on $\frac{A}{I_e}$. Hence the operations $\oplus$ and $*$ are uniformly continuous in $(\frac{A}{I_e},\mathcal{U}_D)$.

□

Proposition 4.20. *Let N be an MV-pseudo norm and E be the set of all idempotents in an MV-algebra A. Then:*
(i) for every $e \in E$, the set $I^e = \{x \in A : N(x \ominus e) = 0\}$ is an ideal in A containing I_N;
(ii) there exist uniformities $\mathcal{U}$ and $\mathcal{V}$ on A such that $(A,\mathcal{U})$ and $(A, \mathcal{V})$ are uniform MV-algebras and $\mathcal{V} \subseteq \mathcal{U}$.

Proof. (i) Let $e \in E$. Clearly, $e, 0 \in I^e$. If x and y are in I^e, then by $e \oplus e = e$ and $(M21)$, we get $(x \oplus y) \ominus e \leq (x \ominus e) \oplus (y \ominus e)$. Hence $N((x \oplus y) \ominus e) = 0$, which implies that $x \oplus y \in I^e$. If $x \leq y$ and $y \in I^e$, since $x \ominus e \leq y \ominus e$, $N(x \ominus e) = 0$. So $x \in I^e$. Thus I^e is an ideal. By $(M19)$, it is easy to see that $I_N \subseteq I^e$.

(ii) The sets $\mathcal{I} = \{I_e : e \in E\}$ and $\mathcal{I}' = \{I^e : e \in E\}$ are closed under finite intersections because for every $e_1, e_2 \in E$, $I_{e_1 \oplus e_2} = I_{e_1} \cap I_{e_2}$ and $I^{e_1 \wedge e_2} \subseteq I^{e_1} \cap I^{e_2}$, where I_e is the ideal defined in Theorem 4.19. Now, the sets $B = \{U_{I_e} : e \in E\}$ and $B' = \{U_{I^e} : e \in E\}$ are bases for uniformities $\mathcal{U}$ and $\mathcal{V}$, respectively, such that $(A,\mathcal{U})$ and $(A, \mathcal{V})$ are uniform MV-algebras. For each $e \in E$, the ideal I_e is a subset of I^e. Hence $U_{I_e} \subseteq U_{I^e}$, which implies that $\mathcal{V} \subseteq \mathcal{U}$. □

5. Conclusion

In this article, MV-pseudo norms and MV-pseudo metrics are defined on MV-algebras. The relation between them and uniform MV-algebras has been studied. In future, researchers can search some conditions under which an MV-algebra endowed to an MV-norm becomes a Tychonoff space. They can also find continuous homomorphisms between these spaces.

Acknowledgments

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the editor and referees.

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