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Characterization of wavelets associated with AB-MRA on $L^2(\mathbb{R}^n)$

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Abstract. A wavelet with composite dilations is a function generating an orthonormal basis or a Parseval frame for $L^2(\mathbb{R}^n)$ under the action of lattice translations and dilations by products of elements drawn from non-commuting matrix sets A and B. Typically, the members of B are matrices whose eigenvalues have magnitude one, while the members of A are matrices expanding on a proper subspace of $\mathbb{R}^n$. In this paper, we provide the characterization of composite wavelets based on results of affine and quasi affine frames. Furthermore all the composite wavelets associated with AB-MRA on $L^2(\mathbb{R}^n)$ are also characterized.

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1. Introduction

The concept of wavelet is defined and studied extensively in the Euclidean spaces $\mathbb{R}^n$. The wavelet characterization of $L^2(\mathbb{R})$ was obtained independently by Wang [13] and Gripenberg [7] in terms of two basic equations involving the Fourier transform of the wavelet (see also [6] and [11]). This result was generalized to $L^2(\mathbb{R}^n)$ by Frazier, Garrigos, Wang, and Weiss [13] for dilation by 2 and by Calogero [4] for wavelets associated with a general dilation matrix. Bownik [2] provided a new approach to characterizing multiwavelets in $L^2(\mathbb{R}^n)$. This characterization was obtained by using the results about shift invariant systems and quasi-affine systems.

The notion of multiresolution analysis (MRA) is closely related to wavelets. In fact, it is well known that one can always construct a wavelet from a MRA. But, all wavelets are not obtained in this way. It was proved independently by Gripenberg [7] and Wang [14] that a wavelet arises from a MRA if and only if its dimension function is 1 a.e. Calogero and Garrigos [5] gave a characterization of wavelet families arising from biorthogonal MRAs of multiplicity d. This result was improved by Bownik and Garrigos in [3], where they provided this characterization in terms of the dimension function. Several results in this direction can be found in [1] and the references therein.

Guo, Labate, Lim, Weiss, and Wilson [8, 9, 10] introduced the theory of composite dilation wavelets and detailed the extension of a multiresolution analysis (MRA) to

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this setting. Let $\psi_\ell \in L^2(\mathbb{R}^n)$. Then the *affine systems with composite dilations* are defined by

$$\Psi_{AB} = \{D_a D_b T_k \psi_\ell : k \in \mathbb{Z}^n, b \in B, a \in A, \ell = 1, 2, \ldots, L\},$$

where the Translation operator T_k is defined by $T_k f(x) = f(x-k)$, Dilation operator by $D_a f(x) = |\det a|^{-1/2} f(a^{-1}x)$. $A \subset GL_n(R)$ consist of elements having some expanding properties and $B \subset GL_n(R)$ consist elements having determinant of absolute value one. By choosing ψ_ℓ, A, B, approximately, Ψ_{AB} can be made orthonormal basis or more generally a Parseval frame for $L^2(\mathbb{R}^n)$. Here we call $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ an orthonormal AB-multiwavelet or a Parseval Frame AB-multiwavelet. For $L = 1$, i.e., when we have single generator, we have wavelet instead of multiwavelet.

This paper is organised in the following manner. In Section 2, we recall some basic results and use them to characterize composite wavelets. Here we also give another characterization of these wavelets. In Section 3, we characterize the wavelets associated with the AB-MRA on $L^2(\mathbb{R}^n)$.

2. Characterization of composite wavelets

For any function $f \in L^2(\mathbb{R}^n)$, we define the *dilation operator* D_j and the *translation operator* T_k as follows:

$$D_j f(x) = q^{1/2} f(Ax) \quad \text{and} \quad T_k f(x) = f(x-k).$$

where $j \in \mathbb{Z},\ A \subset GL_n(R)$ and $k \in \mathbb{R}^n$

Definition 2.1. Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ be a finite family of functions in $L^2(\mathbb{R}^n)$. The *affine system* generated by Ψ is the collection

$$X(\Psi) = \left\{\psi^\ell_{m,j,k}(x) = q^{j/2}\psi^\ell\big(A^j B^m x - k\big), j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \le \ell \le L, 1 \le m \le M, \right\}$$

where $M = \min\{r : B^r = I, r \ge 1, r \in \mathbb{Z}\}$, A is an $n \times n$ expansive real matrix with eigenvalues λ satisfying $|\lambda| > 1$, B is a rotation matrix, $AB^m k \in \mathbb{Z}^n (\forall\, k \in \mathbb{Z}^n, 1 \le m \le M)$, whose $AB = BA$ and $q = |\det A|$. It is clear that $X(\Psi) = D_j T_k \psi^\ell(x)$. The *quasi-affine system* generated by Ψ is

$$\widetilde{X}(\Psi) = \left\{\widetilde{\psi}^\ell_{m,j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \le \ell \le L, 1 \le m \le M\right\},$$

where

$$\widetilde{\psi}^\ell_{j,k}(x) = \begin{cases} D_j D_m T_k \psi^\ell(x) = q^{j/2}\psi^\ell\big(A^j B^m x - k\big), & j \ge 0, \\ \\ q^{j/2} T_k D_j D_m \psi^\ell(x) = q^{j/2}\psi^\ell\big(A^j B^m (x-k)\big), & j < 0. \end{cases} \tag{2.1}$$

We say that Ψ is a set of basic wavelets of $L^2(\mathbb{R}^n)$ if the affine system $X(\Psi)$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$.

Definition 2.2. A subset X of $L^2(\mathbb{R}^n)$ is called a Bessel family if there exists a constant $b > 0$ such that

$$\sum_{\eta \in X} |\langle f, \eta\rangle|^2 \le b\big\|f\big\|^2 \quad \text{for all} \quad f \in L^2(\mathbb{R}^n). \tag{2.2}$$

If, in addition, there exists a constant $a > 0, a \leq b$ such that

$$a\|f\|^2 \leq \sum_{\eta\in X} |\langle f, \eta\rangle|^2 \leq b\|f\|^2 \quad \text{for all} \quad f \in L^2(\mathbb{R}^n), \tag{2.3}$$

then X is called a frame. The frame is *tight* if we can choose a and b such that $a = b$. The affine system $X(\Psi)$ is an affine frame if (2.3) holds for $X = X(\Psi)$. Similarly, the quasi-affine system ?$\widetilde{X}(\Psi)$ is a quasi-affine frame if (2.3) holds for $X = \widetilde{X}(\Psi)$.

Theorem 2.1. *[13] Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ be a finite subset of $L^2(\mathbb{R}^n)$. Then*

(a) $X(\Psi)$ is a Bessel family if and only if $\widetilde{X}(\Psi)$ is a Bessel family. Furthermore, their exact upper bounds are equal.

(b) $X(\Psi)$ is an affine frame if and only if $\widetilde{X}(\Psi)$ is a quasi-affine frame. Furthermore, their lower and upper exact bounds are equal.

Definition 2.3. Given $\{t_i : i \in \mathbb{N}\} \subset l^2(\mathbb{Z}^n)$, define the operator $H : l^2(\mathbb{Z}^n) \to l^2(\mathbb{N})$ by

$$H(v) = (\langle v, t_i\rangle)_{i\in\mathbb{N}} \quad \text{for } v = \big(v(k)\big)_{k\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n).$$

If H is bounded then $\widetilde{G} = H^\star H : l^2(\mathbb{Z}^n) \to l^2(\mathbb{Z}^n)$ is called the dual Gramian of $\{t_i : i \in \mathbb{N}\}$. Observe that ?$\widetilde{G}$ is a non negative definite operator on $l^2(\mathbb{Z}^n)$. Also, note that for $k, p \in \mathbb{Z}^n$, we have

$$\left\langle \widetilde{G}e_k, e_p \right\rangle = \langle He_k, He_p\rangle = \sum_{i\in\mathbb{N}} t_i(k)\overline{t_i(p)},$$

where $\{e_i : i \in \mathbb{N}\}$ is the standard basis of $l^2(\mathbb{Z}^n)$.

Theorem 2.2. *[13] Let $\{\varphi_i : i \in \mathbb{N}\} \subset l^2(\mathbb{Z}^n)$ and for a.e. $\xi \in \mathbb{T}^n$, let $\widetilde{G}(\xi)$ denote the dual Gramian of $\{t_i : i \in \mathbb{N}\} \subset l^2(\mathbb{Z}^n)$. The system of translates $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in \mathbb{N}\}$ is a frame for $L^2(\mathbb{R}^n)$ with constants a and b if and only if $\widetilde{G}(\xi)$ is bounded for a.e. $\xi \in \mathbb{T}^n$ and*

$$a\|v\|^2 \leq \left\langle \widetilde{G}(\xi)v, v \right\rangle \leq b\|v\|^2 \quad \text{for } v \in l^2(\mathbb{Z}^n), a.e., \xi \in \mathbb{T}^n$$

that is, the spectrum of ?$\widetilde{G}(\xi)$ is contained in $[a, b]$ for a.e. $\xi \in \mathbb{T}^n$.

Lemma 2.3. *Suppose that $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$. The affine system $X(\Psi)$ is orthonormal in $L^2(\mathbb{R}^n)$ if and only if for $j \geq 0$ and $1 \leq \ell, \ell' \leq L$,*

$$\sum_{m=1}^{M} \sum_{k\in\mathbb{Z}^n} \hat{\psi}^\ell(\xi + k)\overline{\hat{\psi}^{\ell'}\left(A^{*j}B^{*m}(\xi + k)\right)} = \delta_{\ell,\ell'}\delta_{j,0}\delta_{m,0}, \text{ for a.e. } \xi \in \mathbb{R}^n, . \tag{2.4}$$

Proof. By a simple change of variables, one can observe that for $j, j' \in \mathbb{Z}, k, k' \in \mathbb{Z}^n, 1 \leq \ell, \ell' \leq L$ and $1 \leq m, m' \leq M$,

$$\left\langle \psi^\ell_{m,j,k}, \psi^{\ell'}_{m',j',k'} \right\rangle = \delta_{\ell,\ell'}\delta_{j,j'}\delta_{k,k'}\delta_{m,m'}$$

is equivalent to

$$\left\langle \psi^\ell_{m,j,k}, \psi^{\ell'}_{0,0,0} \right\rangle = \delta_{\ell,\ell'}\delta_{j,0}\delta_{k,0}\delta_{m,0}.$$

Taking any $j \geq 0, k \in \mathbb{Z}^n, 1 \leq \ell, \ell' \leq L$ and $1 \leq m \leq M$, we have by Plancherel's formula

$$\begin{aligned}
\left\langle \psi^{\ell}_{m,j,k}, \psi^{\ell'}_{0,0,0} \right\rangle &= \left\langle \hat{\psi}^{\ell}_{m,j,k}, \hat{\psi}^{\ell'}_{0,0,0} \right\rangle \\
&= \int_{\mathbb{R}^n} q^{-j/2} \hat{\psi}^{\ell}\big(A^{*-j}B^{*-m}\xi\big) e^{-2\pi i A^{*-j}B^{*-m}k\xi} \overline{\hat{\psi}^{\ell'}(\xi)} d\xi \\
&= q^{j/2} \int_{\mathbb{R}^n} \hat{\psi}^{\ell}(\xi) e^{-2\pi i k\xi} \overline{\hat{\psi}^{\ell'}\big(B^{*m}A^{*j}\xi\big)} d\xi \\
&= q^{j/2} \sum_{s \in \mathbb{Z}^n} \int_{s+\mathbb{T}^n} \hat{\psi}^{\ell}(\xi) \overline{\hat{\psi}^{\ell'}\big(B^{*m}A^{*j}\xi\big)} e^{-2\pi i k\xi} d\xi \\
&= q^{j/2} \int_{\mathbb{T}^n} \left\{ \sum_{s \in \mathbb{Z}^n} \hat{\psi}^{\ell}(\xi+s) \overline{\hat{\psi}^{\ell'}\big(B^{*m}A^{*j}(\xi+s)\big)} \right\} e^{-2\pi i k\xi} d\xi.
\end{aligned}$$

If $\left\langle \psi^{\ell}_{m,j,k}, \psi^{\ell'}_{0,0,0} \right\rangle = \delta_{\ell,\ell'}\delta_{j,0}\delta_{k,0}\delta_{m,0}$ for all $j \geq 0, k \in \mathbb{Z}^n, 1 \leq \ell, \ell' \leq L, 1 \leq m \leq M$, then the $L^1(\mathbb{T}^n)$ functions

$$K(\xi) = \sum_{s \in \mathbb{Z}^n} \hat{\psi}^{\ell}(\xi+s) \overline{\hat{\psi}^{\ell'}\big(B^{*m}A^{*j}(\xi+s)\big)}$$

has the property that its Fourier coefficients are all zero except for the coefficient corresponding to $k = 0$, which is 1 if $j = 0$ and $\ell = \ell'$. Hence, $K(\xi) = \delta_{\ell,\ell'}\delta_{j,0}$ for a.e. $\xi \in \mathbb{T}^n$. Conversely, if $K(\xi) = \delta_{\ell,\ell'}\delta_{j,0}$, then the same calculation shows that $\left\langle \psi^{\ell}_{m,j,k}, \psi^{\ell'}_{0,0,0} \right\rangle = \delta_{\ell,\ell'}\delta_{j,0}\delta_{k,0}\delta_{m,0}$. This completes the proof of Lemma. □

Suppose $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ be a finite family of functions in $L^2(\mathbb{R}^n)$. For $j \geq 0$ and $1 \leq m \leq M$, let $\mathcal{D}_j$ be a set of q^j representatives of distinct cosets of $\mathbb{Z}^n \setminus A^jB^m\mathbb{Z}^n$, where $q = |\det A|$. For $j < 0$, we define $\mathcal{D}_j = \{0\}$. Since the quasi affine system $\widetilde{X}(\Psi)$ is invariant under integer, we have

$$\widetilde{X}(\Psi) = \Big\{T_k\varphi : k \in \mathbb{Z}^n, \varphi \in \mathcal{A}\Big\},\ \mathcal{A} := \Big\{\widetilde{\psi}^{\ell}_{m,j,d} : j \in \mathbb{Z}, d \in \mathcal{D}_j, 1 \leq \ell \leq L, 1 \leq m \leq M\Big\}. \tag{2.5}$$

The dual Gramian $\widetilde{G}(\xi)$ of the quasi affine system $\widetilde{X}(\Psi)$ at $\xi \in \mathbb{T}^n$ is defined as the dual Gramian of $\big\{\big(\hat{\varphi}(\xi+k)\big)_{k \in \mathbb{Z}^n} : \varphi \in \mathcal{A}\big\} \subset l^2(\mathbb{Z}^n)$, where $\mathcal{A}$ is defined by (2.5). We now compute $\widetilde{G}(\xi)$ in terms of Fourier transforms of functions in Ψ and show that it does not depend upon the choice of representatives $\mathcal{D}_j$.

For $s \in \mathbb{Z}^n \setminus AB\mathbb{Z}^n$, define the function

$$t_s(\xi) = \sum_{\ell=1}^{L} \sum_{m=1}^{M} \sum_{j=0}^{\infty} \hat{\psi}^{\ell}\big(A^{*j}B^{*m}\xi\big) \overline{\hat{\psi}^{\ell}\big(A^{*j}B^{*m}(\xi+s)\big)},\ \xi \in \mathbb{R}^n. \tag{2.6}$$

Lemma 2.4. *Let* $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ *and* $\widetilde{G}(\xi)$ *be the dual Gramian of* $\widetilde{X}(\Psi)$ *at* $\xi \in \mathbb{T}^n$. *Then*

$$\left\langle \widetilde{G}(\xi)e_k, e_k \right\rangle = \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)\right|^2, \quad \textit{for } \xi \in \mathbb{Z}^n, \tag{2.7}$$

$$\left\langle \widetilde{G}(\xi)e_k, e_p \right\rangle = t_{B^{*-m}A^{*-m}(p-k)}\big(B^{*-m}A^{*-m}\xi + B^{*-m}A^{*-m}k\big), \quad \textit{for } k \neq p \in \mathbb{Z}^n, \tag{2.8}$$

where $m = \max\{j \in \mathbb{Z} : B^{*-m}A^{*-j}(p-k) \in \mathbb{Z}^n\}$ *and the functions* $t_s, s \in \mathbb{Z}^n \backslash AB\mathbb{Z}^n$, *are given by (2.6).*

Proof. For $k, p \in \mathbb{Z}^n$, we have

$$\begin{aligned}
\left\langle \widetilde{G}(\xi)e_k, e_p \right\rangle &= \sum_{\varphi\in\mathcal{A}} \hat{\varphi}(\xi+k)\overline{\hat{\varphi}(\xi+p)} \\
&= \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j<0} \hat{\psi}^\ell\big(A^{*-j}B^{*-m}(\xi+k)\big)\overline{\hat{\psi}^\ell\big(A^{*-j}B^{*-m}(\xi+p)\big)} \\
&\quad + \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\geq 0} \hat{\psi}^\ell\big(A^{*-j}B^{*-m}(\xi+k)\big)\overline{\hat{\psi}^\ell\big(A^{*-j}B^{*-m}(\xi+p)\big)} \\
&\qquad\qquad \times \sum_{d\in\mathcal{D}_j} q^{-j}e^{-2\pi i d B^{*-m}A^{*-j}(p-k)} \\
&= \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j=-\infty}^{r} \hat{\psi}^\ell\big(A^{*-j}B^{*-m}(\xi+k)\big)\overline{\hat{\psi}^\ell\big(A^{*-j}B^{*-m}(\xi+p)\big)},
\end{aligned}$$

where $r = \max\{j \in \mathbb{Z} : B^{*-m}A^{*-j}(p-k) \in \mathbb{Z}^n\}$ and $r = \infty$ when $k = p$. The sum over $\mathcal{D}_j$ is equal to 1 if $(k-p) \in A^{*j}B^{*m}\mathbb{Z}^n$ and 0 otherwise. Therefore, if $k = p$, then (2.7) holds. If $k \neq p$, then

$$\begin{aligned}
&\left\langle \widetilde{G}(\xi)e_k, e_p \right\rangle \\
&= \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\geq 0} \hat{\psi}^\ell\big(A^{*-j-r}B^{*-m}(\xi+k)\big)\overline{\hat{\psi}^\ell\big(A^{*-j-r}B^{*-m}(\xi+p)\big)} \\
&= \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\geq 0} \hat{\psi}^\ell\big(A^{*-j}B^{*-m}(A^{*-r}\xi + A^{*-r}k)\big) \\
&\qquad\qquad \times\overline{\hat{\psi}^\ell\big(A^{*-j}B^{*-m}\big(A^{*-r}\xi + A^{*-m}k + A^{*-r}(p-k)\big)\big)} \\
&= t_{B^{*-m}A^{*-r}(p-k)}\big(B^{*-m}A^{*-r}\xi + B^{*-m}A^{*-r}k\big).
\end{aligned}$$

This completes the proof. □

Theorem 2.5. *Suppose that* $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$*. The affine system* $X(\Psi)$ *is tight frame with constant* 1 *for* $L^2(\mathbb{R}^n)$ *i.e.,*

$$\sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\left|f,\psi^{\ell}_{m,j,k}\right|^2 = \|f\|_2^2 \quad \textit{for all } f \in L^2(\mathbb{R}^n)$$

if and only if the functions $\psi^1, \psi^2, \ldots, \psi^L$ *satisfy the following two conditions:*

$$\sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^{\ell}\left(A^{*j}B^{*m}\xi\right)\right|^2 = 1, \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{2.9}$$

and

$$t_m(\xi) = 0, \quad \textit{for a.e. } \xi \in \mathbb{R}^n, m \in \mathbb{Z}^n \setminus AB\mathbb{Z}^n. \tag{2.10}$$

In particular, Ψ *is a set of basic wavelets of* $L^2(\mathbb{R}^n)$ *if and only if* $\left\|\psi^{\ell}\right\|_2 = 1$ *for* $\ell = 1, 2, ..., L$ *and (2.9) and (2.10) hold.*

Proof. It follows from Theorem 2.1 that $X(\Psi)$ is a tight frame with constant 1 if and only if $\widetilde{X}(\Psi)$ is a tight frame with constant 1. By Theorem 2.5, this is equivalent to the spectrum of $\widetilde{G}(\xi)$ consisting of a single point 1, i.e., $\widetilde{G}(\xi)$ is identity on $l^2(\mathbb{Z}^n)$ for a.e. $\xi \in \mathbb{T}^n$. By Lemma 2.4, this is equivalent to the fact that Eqs. (2.9) and (2.10) hold. The second assertion follows since a tight frame $X(\Psi)$ with constant 1 is an orthonormal basis if and only if $\left\|\psi^{\ell}\right\|_2 = 1$ for $\ell = 1, 2, ..., L$ (see Theorem 1.8, section 7.1 in [12]). This completes the proof. $\square$

Theorem 2.6. *Suppose that* $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$*. Assume that* $X(\Psi)$ *is a Bessel family with constant* 1*. Then the following are equivalent:*

(a) $X(\Psi)$ *is a tight frame with constant* 1*.*
(b) Ψ *satisfies equality (2.9).*
(c) Ψ *satisfies*

$$\sum_{\ell=1}^{L}\int_{\mathbb{R}^n}\left|\hat{\psi}^{\ell}(\xi)\right|^2\frac{d\xi}{\rho(\xi)} = 1, \tag{2.11}$$

for some quasi-norm ρ *associated with* B^*A^**.*

Proof. It is obvious from Theorem 2.5 that (a) $\Rightarrow$ (b). To show (b) implies (c), assume that (2.10) holds. Then, since $\{A^{*j}B^{*m}S : 1 \le \ell \le L, j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^n$ (modulo sets of measure zero), for any $S \subset \mathbb{R}^n$, we have

$$\begin{aligned}
\sum_{\ell=1}^{L}\int_{\mathbb{R}^n}\left|\hat{\psi}^{\ell}(\xi)\right|^2\frac{d\xi}{\rho(\xi)} &= \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\in\mathbb{Z}}\int_{A^{*j}B^{*m}S}\left|\hat{\psi}^{\ell}(\xi)\right|^2\frac{d\xi}{\rho(\xi)}\\
&= \sum_{\ell=1}^{L}\int_{S}\sum_{m=1}^{M}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^{\ell}\left(A^{*j}B^{*m}\xi\right)\right|^2\frac{d\xi}{\rho(\xi)}\\
&= 1.
\end{aligned}$$

To prove (c) $\Rightarrow$ (a), we assume that (2.11) holds. Since $X(\Psi)$ is a Bessel family with constant 1, so is $\widetilde{X}(\Psi)$, by condition (a) of Theorem 2.1. Let $\widetilde{G}(\xi)$ be the dual

Gramian of $\widetilde{X}(\Psi)$ at $\xi \in \mathbb{T}^n$. By Theorem 2.2, we have $\|\widetilde{G}(\xi)\| \leq 1$ for a.e. $\xi \in \mathbb{T}^n$. In particular, $\|\widetilde{G}(\xi)e_k\| \leq 1$. Hence,

$$1 \geq \left\|\widetilde{G}(\xi)\right\|^2 = \sum_{p\in\mathbb{Z}^n} \left|\left\langle \widetilde{G}(\xi)e_k, e_p \right\rangle\right|^2 = \left|\left\langle \widetilde{G}(\xi)e_k, e_p \right\rangle\right|^2 + \sum_{p\in\mathbb{Z}^n, p\neq k} \left|\left\langle \widetilde{G}(\xi)e_k, e_p \right\rangle\right|^2. \tag{2.12}$$

By Lemma 2.4, we have

$$\sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}^\ell\left(A^{*j}B^{*m}(\xi+k)\right)\right|^2 \leq 1, \quad \text{for } k \in \mathbb{Z}^n,\ \xi \in \mathbb{T}^n.$$

Hence,

$$1 = \int_S \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}^\ell\left(A^{*j}B^{*m}(\xi)\right)\right|^2 \frac{d\xi}{\rho(\xi)} \leq \int_D \frac{d\xi}{\rho(\xi)} = 1,$$

From this it follows that $\sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}^\ell\left(A^{*j}B^{*m}\xi\right)\right|^2 = 1$ for a.e. $\xi \in D$ and hence for a.e. $\xi \in \mathbb{R}^n$. This means that equation (2.9) holds. By Lemma 2.4 and equality (2.9), $\left|\left\langle \widetilde{G}(\xi)e_k, e_p\right\rangle\right|^2 = 1$ for all $k \in \mathbb{Z}^n$. Thus, by (2.12), it follows that $\left\langle \widetilde{G}(\xi)e_k, e_p\right\rangle = 0$ for $k \neq p$ so that $\widetilde{G}(\xi)$ is the identity operator on $l^2(\mathbb{Z}^n)$. Hence, by Theorem 2.2, $\widetilde{X}(\Psi)$ is a tight frame with constant 1. Therefore, $X(\Psi)$ is also a tight frame with constant 1, by Theorem 2.1 This completes the proof. □

In the consequence of above theorem, we provide a new characterization of wavelets.

Theorem 2.7. *Suppose $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$. Then the following are equivalent:*

(a) *Ψ is a set of basic wavelets of $L^2(\mathbb{R}^n)$.*
(b) *satisfies (2.4) and (2.9).*
(c) *satisfies (2.4) and (2.11).*

Proof. It follows from Theorem 2.5 and Lemma 2.4 that (a)$\Rightarrow$(b)$\Rightarrow$(c).We now prove that (c) implies (a). Assume that Ψ satisfies (2.4) and (2.11). The equation (2.4) implies that $X(\Psi)$ is an orthonormal system, hence it is a Bessel family with constant 1. By Theorem 2.5 and (2.11), $X(\Psi)$ is a tight frame with constant 1. Since each ψ^ℓ has L^2 norm 1, it follows that $X(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. That is, Ψ is a set of basic wavelets of $L^2(\mathbb{R}^n)$. □

3. Characterization of composite MRA wavelets

As usual, we construct wavelets from multiresolution analysis(MRA).

Definition 3.1. A closed subspaces sequence $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ is called an AB-multiresolution analysis or Composite multiresolution analysis with A and B same as in Section 2, if the following conditions are satisfied:

(1) $V_j \subset V_{j+1}$, $\forall j \in \mathbb{Z}$;
(2) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;
(3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(4) $f(x) \in V_j$ if and only if $f(Ax) \in V_{j+1}$;

(5) there exists a function $\varphi(x) \in V_0$, such that $\{\varphi_{0,\ell,k}\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis of $V_{0,\ell}$, in addition, $V_0 = \oplus_{\ell=1}^{L} V_{0,\ell}$, where $\{V_{0,\ell}\}_{1\le\ell\le L}$ are mutually orthogonal. Here function $\varphi(x)$ is called the *scaling function (or generator).*

Let $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ be a set of basic wavelets of $L^2(\mathbb{R}^n)$. We define the spaces $W_j, j \in \mathbb{Z}$, by $W_j = \overline{\operatorname{span}}\{\psi^{\ell}_{m,j,k} : 1 \le \ell \le L, 1 \le m \le M, k \in \mathbb{Z}^n\}$. We also define $V_j = \oplus_{m<j} W_m, j \in \mathbb{Z}$. Then it follows that $\{V_j : j \in \mathbb{Z}\}$ satisfies the properties (a)-(d) in the definition of a MRA. Hence, $\{V_j : j \in \mathbb{Z}\}$ will form a MRA of $L^2(\mathbb{R}^n)$ if we can find a function $\varphi \in L^2(\mathbb{R}^n)$ such that the system $\{\varphi(x-k) : k \in \mathbb{Z}^n\}$ is an orthonormal basis for V_0. In this case, we say that Ψ is associated with a MRA, or simply that Ψ is a MRA-wavelet.
Now suppose that $\{\psi^1, \psi^2, \ldots, \psi^{q-1}\}$ is a set of basic wavelets for $L^2(\mathbb{R}^n)$ associated with a MRA $\{V_j : j \in \mathbb{Z}\}$. Let $\varphi \in L^2(\mathbb{R}^n)$ be the corresponding scaling function. Then in view of [1], we have

$$\varphi(A^{-1}x) = \sum_{m=1}^{M} \sum_{k\in\mathbb{Z}^n} d_{1,m,k}\,\varphi\big(B^m x - k\big), \tag{3.1}$$

for any $\{d_{1,m,k}\}_{1\le m\le M, k\in\mathbb{Z}^n} \in l^2(\mathbb{N}_0)$. Taking Fourier transform of equation (3.1), we get

$$\hat{\varphi}\big(A^*\xi\big) = \sum_{m=1}^{M} h_0^{(m)}(\xi)\hat{\varphi}\big(B^{*-m}\xi\big), \tag{3.2}$$

where

$$h_0^{(m)}(\xi) = \sum_{k\in\mathbb{Z}^n} d_{1,m,k} e^{-2\pi i k\xi}$$

is an integral periodic function in $L^\infty(\mathbb{T}^n)$. Also, since $\{\psi^1, \psi^2, \ldots, \psi^{q-1}\}$ are the wavelets associated with a MRA corresponding to the scaling function φ, there exist integral-periodic functions $h_{1,\ell}^{(m)}, 1 \le m \le M, 1 \le \ell \le q-1$, such that the matrix

$$\mathcal{M}^{(m)}(\xi) = \Big[h_{1,\ell_1}^{(m)}\big(\xi + \ell_2\big)\Big]_{\ell_1,\ell_2=0}^{q-1}$$

is unitary for a.e. $\xi \in [0, 2\pi]$ and

$$\hat{\psi}^{\ell}\big(A^*\xi\big) = \sum_{m=1}^{M} h_{1,\ell}^{(m)}(\xi)\hat{\varphi}\big(B^{*-m}\xi\big), \tag{3.3}$$

where

$$h_{1,\ell}^{(m)}(\xi) = \sum_{k\in\mathbb{Z}^n} c_{\ell,m,k} e^{-2\pi i k\xi}.$$

Hence, by (3.2), we have

$$\begin{aligned}\big|\hat{\varphi}\big(A^*\xi\big)\big|^2 + \sum_{\ell=1}^{q-1}\Big|\hat{\psi}\big(A^*\xi\big)\Big|^2 &= \left|\sum_{m=1}^{M} h_0^{(m)}(\xi)\hat{\varphi}\big(B^{*-m}\xi\big)\right|^2 + \sum_{\ell=1}^{q-1}\left|\sum_{m=1}^{M} h_{1,\ell}^{(m)}(\xi)\hat{\varphi}\big(B^{*-m}\xi\big)\right|^2\\ &= \sum_{m=1}^{M}\big|\varphi\big(B^{*-m}\xi\big)\big|^2\left(\sum_{\ell=0}^{q-1}\Big|h_{1,\ell}^{(m)}(\xi)\Big|^2\right).\end{aligned}$$

Since $\mathcal{M}^{(m)}(\xi)$ is unitary for each $m, 1 \leq m \leq M$, we have

$$\left|\hat{\varphi}\big(A^*\xi\big)\right|^2 + \sum_{\ell=1}^{q-1}\left|\hat{\psi}\big(A^*\xi\big)\right|^2 = \sum_{m=1}^{M}\left|\varphi\big(B^{*-m}\xi\big)\right|^2.$$

Thus equality holds for for a.e, $\xi \in \mathbb{R}^n$. Hence, we have

$$|\hat{\varphi}(\xi)|^2 = \sum_{m=1}^{M}\left(\left|\hat{\varphi}\big(A^*B^{*m}\xi\big)\right|^2 + \sum_{\ell=1}^{q-1}\left|\psi^\ell\big(A^*B^{*m}\xi\big)\right|^2\right)$$

Iterating for any integer $N \geq 1$, we get,

$$|\hat{\varphi}(\xi)|^2 = \sum_{m=1}^{M}\left(\left|\hat{\varphi}\big(A^{*N}B^{*m}\xi\big)\right|^2 + \sum_{\ell=1}^{q-1}\sum_{j=1}^{N}\psi^\ell\big(A^{*j}B^{*m}\xi\big)\right).$$

Since $|\hat{\varphi}(\xi)|^2 \leq 1$, the sequence $\left\{\sum_{\ell=1}^{q-1}\sum_{m=1}^{M}\sum_{j=1}^{N}\psi^\ell\big(A^{*j}B^{*m}\xi\big) : N \geq 1\right\}$ of real numbers is increasing and is bounded by 1, hence it converges.

Therefore $\lim_{N\to\infty}\sum_{m=1}^{M}\left|\hat{\varphi}\big(A^{*N}B^{*m}\xi\big)\right|^2$ also exists. Now

$$\int_{\mathbb{R}^N}\sum_{m=1}^{M}\left|\hat{\varphi}\big(A^{*N}B^{*m}\xi\big)\right|^2\xi = q^{-N}\int_{\mathbb{R}^n}|\hat{\varphi}(\xi)|^2\,d\xi \to 0 \text{ as } N \to \infty.$$

Hence, by Fatou's Lemma, we have

$$\int_{\mathbb{R}^n}\lim_{N\to\infty}\sum_{m=1}^{M}\left|\hat{\varphi}\big(A^{*N}B^{*m}\xi\big)\right|^2 d\xi \leq \lim_{N\to\infty}\int_{\mathbb{R}^n}\sum_{m=1}^{M}\left|\hat{\varphi}\big(A^{*N}B^{*m}\xi\big)\right|^2 d\xi = 0.$$

This shows that $\lim_{N\to\infty}\sum_{m=1}^{M}\left|\hat{\varphi}\big(A^{*N}B^{*m}\xi\big)\right|^2 = 0$. Hence, we get

$$|\hat{\varphi}(\xi)|^2 = \sum_{\ell=1}^{q-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\left|\hat{\psi}^\ell\big(A^{*j}B^{*m}\xi\big)\right|^2.$$

Since $\{\varphi(x-k) : k \in \mathbb{Z}^n\}$ is an orthonormal system, we get for a.e. $\xi \in \mathbb{R}^n$,

$$1 = \sum_{k\in\mathbb{Z}^n}|\hat{\varphi}(\xi+k)|^2 = \sum_{\ell=1}^{q-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)\right|^2.$$

Definition 3.2. Suppose $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ is a set of basic wavelets for $L^2(\mathbb{R}^n)$. The *dimension function* of Ψ is defined as

$$D_\Psi(\xi) = \sum_{\ell=1}^{q-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)\right|^2. \tag{3.4}$$

Note that if $\psi^1, \psi^2, \ldots, \psi^L \in L^2(\mathbb{R}^n)$, then

$$\int_{[0,2\pi]}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)\right|^2 d\xi = \sum_{j=1}^{\infty}\int_{\mathbb{R}}\left|\hat{\psi}^\ell(\xi)\right|^2 d\xi < \infty. \tag{3.5}$$

Then D_Ψ is well defined for a.e. $\xi \in \mathbb{R}^n$. In particular, $\sum_{k\in\mathbb{Z}^n}\left|\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)\right|^2 < \infty$ for a.e. $\xi \in \mathbb{R}^n$. Thus for all $j \geq 1, 1 \leq \ell \leq L, 1 \leq m \leq M$, and a.e. $\xi \in \mathbb{R}^n$, we can define the vector $\omega^\ell_{j,m}(\xi) \in l^2(\mathbb{Z}^n)$, where

$$\omega^\ell_{j,m}(\xi) = \left\{\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big) : k \in \mathbb{Z}^n\right\}.$$

Hence, D_Ψ can also be written as

$$D_\Psi(\xi) = \sum_{\ell=1}^{L}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\left\|\omega^\ell_{j,m}(\xi)\right\|^2_{l^2(\mathbb{Z}^n)}. \tag{3.6}$$

We have thus proved that if $\Psi = \{\psi^1, \psi^2, \dots, \psi^L\}$ is a set of basic wavelets associated with a MRA of $L^2(\mathbb{R}^n)$, then it is necessary that $D_\Psi(\xi) = 1$ a.e. Our aim is to show that this condition is also sufficient. We will show that if $\Psi = \{\psi^1, \psi^2, \dots, \psi^L\}$ is a set of basic wavelets of $L^2(\mathbb{R}^n)$ and $D_\Psi(\xi) = 1$ a.e., then Ψ is an AB-MRA wavelet. To prove this we need the following lemma.

Lemma 3.1. *For all $j \geq 1, 1 \leq \ell \leq q-1$, and a.e. $\xi \in \mathbb{R}^n$, we have*

$$\omega^\ell_{j,m}(\xi) = \sum_{h=1}^{q-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty}\left\langle \omega^\ell_{j,m}(\xi), \omega^h_{i,m}(\xi)\right\rangle \omega^h_{i,m}(\xi). \tag{3.7}$$

Proof. The series appearing in the lemma converges absolutely by (3.5) for a.e. $\xi \in \mathbb{R}^n$. We first show that

$$\hat{\psi}^\ell\big(A^{*j}B^{*m}\xi\big) = \sum_{h=1}^{q-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)\overline{\hat{\psi}^h(A^{*i}B^{*m}(\xi+k))}\hat{\psi}^h\big(A^{*j}B^{*m}\xi\big). \tag{3.8}$$

Let us denote the series on the right of (3.8) by $G^\ell_{j,m}(\xi)$. Then by using Lemma 2.3 and equation (2.6), we have

$$\begin{aligned}
G^\ell_{j,m}(\xi) &= \sum_{k\in\mathbb{Z}^n}\sum_{m=1}^{M}\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)\sum_{h=1}^{q-1}\sum_{i=1}^{\infty}\overline{\hat{\psi}^h(A^{*i}B^{*m}(\xi+k))}\hat{\psi}^h\big(A^{*j}B^{*m}\xi\big)\\
&= \sum_{k\in\mathbb{Z}^n}\sum_{m=1}^{M}\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)\left\{t_k(\xi) - \sum_{h=1}^{q-1}\sum_{i=1}^{\infty}\overline{\hat{\psi}^h((\xi+k))}\hat{\psi}^h(\xi)\right\}\\
&= \sum_{k\in AB\mathbb{Z}^n}\sum_{m=1}^{M}\hat{\psi}^\ell\big(A^{*j}B^{*m}(\xi+k)\big)t_k(\xi)\\
&= \sum_{h=1}^{q-1}\sum_{k\in\mathbb{Z}^n}\sum_{m=1}^{M}\sum_{i=0}^{\infty}\hat{\psi}^\ell(A^{*j}B^{*m}(\xi+B^*A^*k))\overline{\hat{\psi}^h(A^{*i}B^{*m}(\xi+B^*A^*k))}\hat{\psi}^h\big(A^{*j}B^{*m}\xi\big)\\
&= \sum_{h=1}^{q-1}\sum_{k\in\mathbb{Z}^n}\sum_{m=1}^{M}\sum_{i=1}^{\infty}\hat{\psi}^\ell(A^{*j+1}B^{*m+1}(A^{*-1}B^{*-1}\xi+k))\\
&\qquad\times\overline{\hat{\psi}^h(A^{*i}B^{*m}(A^{*-1}B^{*-1}\xi+k))}\hat{\psi}^h(A^{*j}B^{*m}A^{*-1}B^{*-1}\xi)\\
&= G^\ell_{j+1,m+1}(A^{*-1}B^{*-1}\xi).
\end{aligned}$$

This is equivalent to $G^{\ell}_{j,m}(\xi) = G^{\ell}_{j-1,m-1}(A^*B^*\xi)$. Iterating this equation, we obtain, $G^{\ell}_{j,m}(\xi) = G^{\ell}_{1,m}(A^{*j-1}B^{*m-1}\xi)$. We now calculate $G^{\ell}_{1,m}(\xi)$. We have

$$\begin{aligned}
G^{\ell}_{1,m}(\xi) &= \sum_{k\in\mathbb{Z}^n} \hat{\psi}^{\ell}(A^*B^*(\xi+k)) \sum_{h=1}^{q-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty} \overline{\hat{\psi}^h(A^{*i}B^{*m}(\xi+k))}\hat{\psi}^h(A^{*i}B^{*m}\xi) \\
&= \sum_{k\in\mathbb{Z}^n} \hat{\psi}^{\ell}(A^*B^*\xi + A^*B^*k)) \sum_{h=1}^{q-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty} \overline{\hat{\psi}^h(A^{*i}B^{*m}(A^*B^*\xi + A^*B^*k))} \\
&\qquad \times \hat{\psi}^h(A^{*i}B^{*m}A^*B^*\xi) \\
&= \sum_{k\in AB\mathbb{Z}^n} \hat{\psi}^{\ell}(A^*B^*\xi + k) \sum_{h=1}^{q-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty} \overline{\hat{\psi}^h(A^{*i}B^{*m}(A^*B^*\xi + k))} \\
&\qquad \times \hat{\psi}^h(A^{*i}B^{*m}A^*B^*\xi) \\
&= \sum_{h=1}^{q-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty} \hat{\psi}^h(A^{*i}B^{*m}A^*B^*\xi)\delta_{i,0}\delta_{m,0}\delta_{\ell,h} \\
&= \hat{\psi}^{\ell}(A^*B^*\xi).
\end{aligned}$$

Thus $G^{\ell}_j(\xi) = \hat{\psi}^{\ell}(A^{*-j}B^{*-m}\xi)$ a.e. $\xi \in \mathbb{R}^n$. Since $\left\langle \omega^{\ell}_j(\xi), \omega^h_i(\xi)\right\rangle$ is integral periodic, (3.7) follows. This completes the proof. □

Lemma 3.2. *Let $\{\nu_j : j \geq 1\}$ be a family of vectors in a Hilbert space H such that*

(i) $\sum_{n=1}^{\infty} \|\nu_n\|^2 = C < \infty,$

(ii) $\nu_n = \sum_{n=1}^{\infty} \langle \nu_n, \nu_m\rangle \nu_m$ *for all $n \geq 1$. Let $\mathbb{F} = \overline{span}\{\nu_j : j \geq 1\}$. Then*

$$\dim \mathbb{F} = \sum_{j=1}^{\infty} \|\nu_j\|^2 = C.$$

Theorem 3.3. *A wavelet $\Psi = \left\{\psi^1, \psi^2, \ldots, \psi^L\right\} \subset L^2(\mathbb{R}^n)$ is an AB-MRA wavelet if only if $D_\Psi(\xi) = 1$ for almost every $\xi \in \mathbb{R}^n$.*

Proof. We have already observed that $D_\Psi(\xi) = 1$ for almost every $\xi \in \mathbb{R}^n$ when Ψ is an AB-MRA wavelet. We now prove the converse. Assume that $D_\Psi(\xi) = 1$ for almost every $\xi \in \mathbb{R}^n$. Let E be the subset of $\mathbb{T}^n$ on which $D_\Psi(\xi)$ is finite and (3.7) is satisfied. Then $\omega^{\ell}_{j,m}$ are well-defined on E. For $\xi \in E$, we define the space

$$\mathcal{F}(\xi) = \overline{\text{span}}\left\{\omega^{\ell}_{j,m}(\xi) : 1 \leq \ell \leq q-1, 1 \leq m \leq M, j \geq 1\right\}.$$

Then, by Lemmas 3.1 and 3.2, we have

$$\dim \mathcal{F}(\xi) = \sum_{\ell=1}^{q-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty} \left\|\omega^{\ell}_{j,m}(\xi)\right\|^2_2 = D_\Psi(\xi) = 1. \tag{3.9}$$

That is, for each $\xi \in E, \mathcal{F}(\xi)$ is generated by a single unit vector $U(\xi)$. We now choose a suitable vector. For $j \geq 1$, let us define

$$X_j = \left\{\xi \in E : \omega^{\ell}_{j,m}(\xi) \neq 0 \text{ and } \omega^{\ell}_{m,m}(\xi) = 0, \forall\, m < j \text{ and } 1 \leq \ell \leq q-1, 1 \leq m \leq M\right\}$$

and

$$X_0 = \Big\{\xi \in \mathbb{T}^n : \omega^{\ell}_{j,m}(\xi) \neq 0, \forall\, j \geq 1, \text{ and } 1 \leq \ell \leq q-1, 1 \leq m \leq M\Big\}.$$

Then $\{X_j : j = 0,1,2,\dots\}$ forms a partition of E. Note that $X_0 = \{\xi \in \mathbb{T}^n : D_\Psi(\xi) = 0\}$. So for a.e. $\xi \in E \setminus X_0$, there exists $j \geq 1$ such that $\xi \in X_j$. Hence, there exists at least one $\ell, 1 \leq \ell \leq q-1$, and one $m, 1 \leq m \leq M$ such that $\omega^{\ell}_{j,m}(\xi) \neq 0$. Choose the smallest such ℓ and m define

$$U(\xi) = \frac{\omega^{\ell}_{j,m}(\xi)}{\big\|\omega^{\ell}_{j,m}(\xi)\big\|_{l^2}}.$$

Thus, $U(\xi)$ is well defined and $\big\|U(\xi)\big\|_{l^2} = 1$ for a.e. $\xi \in \mathbb{T}^n$. We write $U(\xi) = \{u_k(\xi) : \xi \in \mathbb{Z}^n\}$. Now, define $\hat{\varphi}(\xi) = u_k(\xi - k)$, where k is the unique integer in $\mathbb{Z}^n$ such that $\xi \in \mathbb{T}^n + k$. This defines $\hat{\varphi}$ on $\mathbb{R}^n$. We first show that $\varphi \in L^2(\mathbb{R}^n)$ and $\{\varphi(x-k) : k \in \mathbb{Z}^n\}$ is an orthonormal system in $L^2(\mathbb{R}^n)$. We have

$$\begin{aligned}
\big\|\hat{\varphi}\big\|_2^2 &= \int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2\, d\xi \;=\; \int_{\mathbb{T}^n} \sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi+k)|^2\, d\xi \\
&= \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{T}^n} |u_k(\xi)|^2\, d\xi \;=\; \int_{\mathbb{T}^n} \big\|U(\xi)\big\|^2_{l^2} d\xi \\
&= 1.
\end{aligned}$$

Thus $\varphi \in L^2(\mathbb{R}^n)$. Also,

$$\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi+k)|^2 = \sum_{k\in\mathbb{Z}^n} |u_k(\xi)|^2 = \big\|U(\xi)\big\|^2_{l^2} = 1. \tag{3.10}$$

This is equivalent to the fact that $\{\varphi(x-k) : k \in \mathbb{Z}^n\}$ is an orthonormal system. We now define $V_0^{\#} = \overline{\operatorname{span}}\{\varphi(x-k) : k \in \mathbb{Z}^n\}$. Let $W_j = \overline{\operatorname{span}}\{\psi^{\ell}_{m,j,k} : 1 \leq \ell \leq q-1, 1 \leq m \leq M, k \in \mathbb{Z}^n\}$ and $V_0 = \oplus_{j<0} W_j$. If we can show that $V_0^{\#} = V_0$, then it will follow that $\{V_j : j \in \mathbb{Z}\}$ is the required MRA.

We first show that $V_0^{\#} \subset V_0$. It is sufficient to verify that $\psi^{\ell}_{m,j,k} \in V_0^{\#}, k \in \mathbb{Z}^n, j < 0, 1 \leq \ell \leq q-1, 1 \leq m \leq M$. For each $j \geq 1$, there exists a measurable function $\nu^{\ell}_{j,m}$ on $\mathbb{T}^n$ such that $\omega^{\ell}_{j,m}(\xi) = \nu^{\ell}_{j,m}(\xi) U(\xi)$ for a.e. $\xi \in \mathbb{T}^n$. That is,

$$\hat{\psi}^{\ell}\big(A^{*j}B^{*m}(\xi+k)\big) = \nu^{\ell}_{j,m}(\xi)\hat{\varphi}(\xi+k) \quad \text{for all } \xi \in \mathbb{T}^n, k \in \mathbb{Z}^n.$$

Therefore, by (3.10), for a.e. $\xi \in \mathbb{T}^n$, we have

$$\sum_{k\in\mathbb{Z}^n} \Big|\hat{\psi}^{\ell}\big(A^{*j}B^{*m}(\xi+k)\big)\Big|^2 = \sum_{k\in\mathbb{Z}^n} \big|\nu^{\ell}_{j,m}(\xi)\big|^2 |\hat{\varphi}(\xi+k)|^2 = \big|\nu^{\ell}_{j,m}(\xi)\big|^2. \tag{3.11}$$

This shows that $\nu^{\ell}_{j,m} \in L^2(\mathbb{T}^n)$ so that we can write its Fourier series expansion. Thus, for $j \geq 1$, there exists $\{a^{\ell}_{m,j,k} : k \in \mathbb{Z}^n\} \in l^2(\mathbb{Z}^n)$ such that $\nu^{\ell}_{j,m}(\xi) = \sum_{k\in\mathbb{Z}^n} a^{\ell}_{m,j,k} e^{-2\pi i k\xi}$, with convergence in $L^2(\mathbb{T}^n)$. Extending $\nu^{\ell}_{j,m}$ integer periodically, we have

$$\hat{\psi}^{\ell}\big(A^{*j}B^{*m}\xi\big) = \nu^{\ell}_{j,m}(\xi)\hat{\varphi}(\xi), \quad \text{for a. e. } \xi \in \mathbb{Z}^n, j \geq 1. \tag{3.12}$$

Taking inverse Fourier transform, we get

$$\psi^{\ell}_{-j,-m,0}(x) = q^{j/2} \sum_{k\in\mathbb{Z}^n} a^{\ell}_{m,j,k} \varphi(\xi - k), \quad j \geq 1.$$

Hence, $\psi^{\ell}_{-j,-m,0} \in V_0^{\#}$ for $j \geq 1$. Moreover, since $V_0^{\#}$ is invariant under translations by $k, k \in \mathbb{Z}^n$, we have $\psi^{\ell}_{m,j,k} \in V_0^{\#}, j < 0, k \in \mathbb{Z}^n, 1 \leq \ell \leq q-1, 1 \leq m \leq M$.
To show the reverse inclusion, it suffices to show that $V_0^{\#} \perp W_j$, for $j \geq 0$. For $j \geq 0, k \in \mathbb{Z}^n, 1 \leq \ell \leq q-1, 1 \leq m \leq M$, we have

$$\begin{aligned}\left\langle \varphi, \psi^{\ell}_{m,j,k} \right\rangle &= \left\langle \hat{\varphi}, \hat{\psi}^{\ell}_{m,j,k} \right\rangle \\ &= \int_{\mathbb{R}^n} \hat{\varphi}(\xi) q^{-j/2} \overline{\hat{\psi}^{\ell}\big(A^{*j}B^{*m}\xi\big)} e^{-2\pi i A^{*j}B^{*m}k\xi} d\xi \\ &= q^{j/2} \int_{\mathbb{R}^n} \hat{\varphi}(B^{*-m}A^{*-j}\xi) \overline{\hat{\psi}^{\ell}(\xi)} e^{-2\pi i k\xi} d\xi \\ &= q^{j/2} \int_{\mathbb{T}^n} \sum_{n \in \mathbb{Z}^n} \hat{\varphi}(B^{*-m}A^{*-j}(\xi+n)) \overline{\hat{\psi}^{\ell}(\xi+n)} e^{-2\pi i k\xi} d\xi. \quad (3.12)\end{aligned}$$

Using Equation (3.11), we get

$$\sum_{\ell=1}^{q-1} \sum_{m=1}^{M} \sum_{j=1}^{\infty} \left|\nu^{\ell}_{j,m}(\xi)\right|^2 = \sum_{\ell=1}^{q-1} \sum_{m=1}^{M} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \left|\hat{\psi}^{\ell}\big(A^{*j}B^{*m}(\xi+k)\big)\right|^2 = 1 \quad \text{for a. e. } \xi \in \mathbb{Z}^n.$$

Hence, for such ξ and for all $j \geq 0$, there exists $j_0 \geq 1$ such that $\nu^{\ell}_{j,m}\big(A^{*j}B^{*m}\xi\big) \neq 0$. Thus, (3.12) implies that $\hat{\psi}^{\ell}\big(A^{*j+j_0}B^{*m}\xi\big) = \nu^{\ell}_{j_0,m}\big(B^{*-m}A^{*-j}\xi\big)\hat{\varphi}\big(B^{*-m}A^{*-j}\xi\big)$. Therefore, for $k \in \mathbb{Z}^n$, we get

$$\hat{\psi}^{\ell}\big(A^{*j+j_0}B^{*m}(\xi+k)\big) = \nu^{\ell}_{j_0,m}\big(B^{*-m}A^{*-j}(\xi+k)\big)\hat{\varphi}\big(B^{*-m}A^{*-j}(\xi+k)\big)$$

Using integral periodicity of $\nu^{\ell}_{j_0}$, we get

$$\hat{\varphi}\big(B^{*-m}A^{*-j}(\xi+k)\big) = \frac{1}{\nu^{\ell}_{j_0,m}\big(B^{*-m}A^{*-j}\xi\big)} \hat{\psi}^{\ell}\big(A^{*j+j_0}B^{*m}(\xi+k)\big).$$

Therefore, using Lemma 2.3, for any h with $1 \leq h \leq q-1$ and for $1 \leq m \leq M$, we have

$$\begin{aligned}&\sum_{k \in \mathbb{Z}^n} \hat{\varphi}\big(B^{*-m}A^{*-j}(\xi+k)\big)\overline{\hat{\psi}(\xi+k)} \\ &\quad = \frac{1}{\nu^{\ell}_{j_0,m}\big(B^{*-m}A^{*-j}\xi\big)} \sum_{k \in \mathbb{Z}^n} \hat{\psi}^{\ell}\big(A^{*j+j_0}B^{*m}(\xi+k)\big)\overline{\hat{\psi}(\xi+k)} \;=\; 0,\end{aligned}$$

since $j + j_0 \geq 1$. Substituting this in (3.12), we get $\left\langle \varphi, \psi^{\ell}_{m,j,k} \right\rangle = 0$ for $j \geq 0, k \in \mathbb{Z}^n, 1 \leq \ell \leq q-1, 1 \leq m \leq M$. From this we conclude that $V_0^{\#} \subset V_0$. This completes the proof of theorem. □

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