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Rotationally symmetrical plane graphs and their Fault-tolerant metric dimension

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Abstract. For a non-trivial connected graph H, a vertex $a \in H$ resolves (recognizes) two vertices x and y in H, if $d(a,x) \neq d(a,y)$. A subset L of distinct ordered vertices in H is called a resolving set for H, if every two distinct vertices of H are recognized by at least one vertex from L. The minimum cardinality of a resolving set L for H is called the metric dimension of H, denoted by $dim(H)$. The subset L of vertices in H is called a fault-tolerant resolving set (FTRS) for H, if L-$\{x\}$ is still the resolving set for all $x \in L$, and the minimum cardinality of such a set L is called the fault-tolerant metric dimension (FTMD) of H. The fault-tolerant metric dimension is an extension of metric dimension in graphs with several intelligent systems applications, for example, robot navigation, network optimization, and sensor networking. The graphs of convex polytopes, which are rotationally symmetric, are essential in intelligent networks due to the uniform rate of data transformation to all vertices. In this article, we consider three well-known rotationally symmetric families of plane graphs and find their minimum fault-tolerant resolving sets.

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1. Introduction

The idea of studying metric dimension for graphs was brought forward by Slater [21], and Harary and Melter in [8]. The applicability of the metric dimension was seen in different fields of science and technology. Metric basis and resolving set have become an integral part of molecular topology and combinatorial chemistry. Applications of metric basis and resolving sets emerge in various fields such as coin weighing problems [22], the connected joints in graphs and chemistry [4], robot navigation [12], network discovery and verification [3], and strategies for the Mastermind game [7]. It is imperative to note that computing metric dimension in graphs is an NP-hard problem [2]. Therefore, it is interesting to study the minimum metric dimension problem for infinite families of graph-theoretic interest.

A set L of elements (vertices or edges) in space is said to be a *generator* of a metric space if each element of the space is uniquely determined (or recognized) by the distances between the elements of L. There are several types of metric generators in networks today, each of which is studied in both applied and theoretical ways, based on its eminence or applications. The metric dimension of graph H is the minimum cardinality of the metric generator L and is denoted by $dim(H)$. The metric generator

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L with minimum cardinality is the metric basis for H. Few recent results concerning the metric dimension are presented in [19, 20, 27].

In an application given in [6], censors were designated as elements of metric basis. A defective sensor will commence to the breakdown in recognizing the thief (intruder, fire, etc.) in the system. Hernando et al. [9] proposed the concept of fault-tolerant metric dimension to address these types of predicaments. That is, if one of the censors is not operating, the fault-tolerant resolving set provides reliable information. As a consequence, the fault-tolerant metric dimension has applications in all of those fields where the metric dimension has them. For a more comprehensive study related to this parameter, we refer to [18, 23].

Fault-tolerant metric dimension (FTMD) is considered in numerous fields of study. Raza et al. considered applications of FTMD in some interconnection networks, and structures of various graphs of convex polytopes [13, 14]. Mithun et al. [24] considered FTMD for the class of circulant graph $C_n(1, 2, 3)$. For, more in-depth review of this particular topic we refer to some of the recent results in [15, 17, 20, 25].

The convex hull of a finite set of points in Euclidean space $\mathbb{R}^d$ is acknowledged as a convex polytope. By preserving the incidence-adjacency relation between vertices, the graphs of convex polytopes emerge from geometric structures of convex polytopes. This class of planar geometric graphs has broadly been analyzed for graph labeling [1], fault-tolerant metric dimension [9, 20], metric dimension [19], locating-dominating sets [10], mixed metric dimension [16, 26].

In this paper, we consider three rotationally symmetric families of planar graphs (viz., flower graph $\digamma_{n\times m}$, S_n, and T_n [11]) and determine their fault-tolerant metric dimension. This article is organized as follows. In Section 2, we recall some existing results related to the metric dimension and fault-tolerant metric dimension of graphs. In Section 3, we set upper and lower bounds of fault-tolerant metric dimension for the flower graph $\digamma_{n\times m}$, when $m = 3$. We compute the fault-tolerant metric dimension for convex polytope graphs S_n, and T_n in Sections 4 and 5 respectively. Finally, the conclusion and future work of this paper is presented in Section 6.

2. Preliminaries

In this section, we discuss some basic preliminary results.

Suppose $H = (V, E)$ be a simple, connected, and undirected graph, with vertex set V and edge set E. The distance $d(a, b)$ between two vertices a and b in a simple connected graph H is the length of the shortest $a - b$ path between the vertices a and b. It equals the minimum number of edges between a and b in that shortest path.

Metric Dimension: [21] If for any three vertices a, b, $c \in V(H)$, we have $d(a, b) \neq d(a, c)$, then the vertex a is said to resolve the pair of vertices b, c ($b \neq c$) in $V(H)$. If this condition of resolvability is fulfilled by some vertices comprising a subset $L \subseteq V(H)$ i.e., every pair of different vertices in the given undirected graph H is resolved by at least one element of L, then L is said to be a *metric generator* (or *resolving set*) of H. The *metric dimension* of the given graph H is the minimum cardinality of the resolving set L, and is usually denoted by $dim(H)$. The metric generator L with minimum cardinality is the metric basis for H. For an ordered subset of vertices $L = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_k\}$, (by L^* for fault-tolerant metric dimension) the k-code (coordinate or representation) of vertex c in $V(H)$ is:

$$\varphi(c|L) \quad = \quad (d(\varepsilon_1, c), d(\varepsilon_2, c), ..., d(\varepsilon_k, c))$$

In this respect, the set L is a resolving set for H, if $\varphi(q|L) \neq \varphi(p|L)$, for any pair of distinct vertices $p, q \in V(H)$.

Example 2.1. Consider a graph H on 8 vertices as shown in Fig. 1. The set $L_1 = \{a_1, a_2, a_3\}$ is a resolving set for the graph H, since the metric codes for the vertices of the graph H with respect to L_1 are: $\varphi(a_1|L_1) = (0, 1, 1)$; $\varphi(a_2|L_1) = (1, 0, 2)$; $\varphi(a_3|L_1) = (1, 2, 0)$; $\varphi(a_4|L_1) = (2, 1, 1)$; $\varphi(a_5|L_1) = (1, 2, 1)$; $\varphi(a_6|L_1) = (2, 1, 2)$; $\varphi(a_7|L_1) = (3, 2, 2)$; $\varphi(a_8|L_1) = (4, 3, 3)$. However, L_1 is not a minimum resolving set, as $L_2 = \{a_1, a_3\}$ is likewise a resolving set with smaller cardinality. Then again, the set $L_3 = \{a_1\}$ is not a resolving set, as $\varphi(a_2|L_3) = \varphi(a_3|L_3) = 1$. Utilizing a comparable contention it is anything but difficult to watch that none of singleton vertex forms a resolving set for H, and hence $dim(H) = 2$.

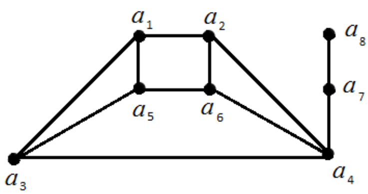

FIGURE 1. The graph H

Independent resolving set: [5] A subset L consisting of distinct vertices of the graph H is said to be an independent resolving set for H if L is both resolving and independent set.

Fault-Tolerant Metric Dimension: [9] A *fault-tolerant resolving set* is a resolving set in which the removal of an arbitrary vertex keeps up the resolvability i.e., a resolving set L^* is said to be fault-tolerant if L^*-$\{a\}$ is also a resolving set for every $a \in L^*$. For the sake of simplicity, we can write fault-tolerant resolving set, fault-tolerant metric codes (i.e., $\varphi_F(c|L^*)$), and fault-tolerant metric dimensions as FTRS, FTMC, and FTMD respectively. The fault-tolerant metric basis, fault-tolerant metric codes and fault-tolerant metric dimension are characterized correspondingly as metric dimensions. We represent the FTMD of graph H with $fdim(H)$. By the definition of FTRS, it is clear that for every graph H, we have;

$$dim(H) + 1 \leq fdim(H) \tag{1}$$

Now, for an arbitrary graph H the following lemma represents a connection between a resolving set and a FTRS. Suppose that $\mathbb{N}(a)$ represents an open neighborhood of a vertex $a \in V(H)$ where $\mathbb{N}(a) = \{b \in V(H)|ab \in E(H)\}$, and the close neighborhood of a vertex a is given as $\mathbb{N}[a] = \mathbb{N}(a) \cup \{a\}$.

Lemma 2.1. [9] *Let L represents a resolving set for a connected graph H. Then, for any $a \in L$, let $\mathbb{T}(a) = \{c \in V(H) : \mathbb{N}(a) \subseteq \mathbb{N}(c)\}$. Then $L^* = \cup_{a \in L}(\mathbb{N}[a] \cup \mathbb{T}(a))$ is a FTRS of the graph H.*

Imran et al. [11], studied the metric dimension for three rotationally symmetric families of plane graphs viz., the flower graph $\digamma_{n\times m}$, S_n, and T_n. For these three rotationally symmetric graphs, they obtained the following results

Theorem 2.2. *For $n \geqslant 6$, we have*

$$dim(\digamma_{n\times 3}) = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.3. *$dim(S_n) = 3$, for every $n \geq 6$.*

Theorem 2.4. *$dim(T_n) = 3$, for every $n \geq 6$.*

3. Bounds on FTMD for the flower graph $\digamma_{n\times m}$

The plane graph $\digamma$ is known as a $(n\times m)$-flower graph if it has n vertices that structure an n-cycle and n sets of $m-2$ vertices that structure m-cycles around the n cycle so every m-cycle uniquely intersects with the n-cycle on a solitary edge. This plane graph will be indicated by $\digamma_{n\times m}$. Unmistakably $\digamma_{n\times m}$ has nm number of edges and $n(m-1)$ vertices. The m-cycles are known as the petals of the graph $\digamma_{n\times m}$ and the n-cycle is known as the center of the graph $\digamma_{n\times m}$. The n vertices which structure the center are all of valency four and the rest of the vertices have valency two. Figure 2 shows some examples of the flower graph.

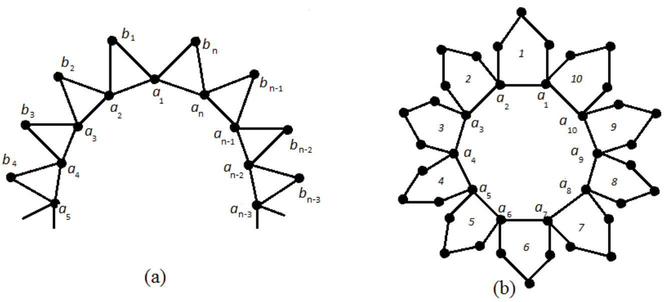

FIGURE 2. The graphs $\digamma_{n\times 3}$ and $\digamma_{10\times 5}$

For $m=3$, the graph $\digamma_{n\times 3}$ [11] consists of n triangular faces, a faces having n sides, and a face having $2n$ sides. It has a $2n$ number of vertices and a $3n$ number of edges (see Fig. 2(a)). For $\digamma_{n\times 3}$, the set of edges and vertices are denoted by $E(\digamma_{n\times 3})$ and $V(\digamma_{n\times 3})$, respectively. Therefore, we have $V(\digamma_{n\times 3}) = \{a_j, b_j : 1 \leq j \leq n\}$ and $E(\digamma_{n\times 3}) = \{a_jb_j, a_ja_{j+1}, b_ja_{j+1} : 1 \leq j \leq n\}$.

We call the cycle induced by the vertices $\{a_j : 1 \leq j \leq n\}$ in the graph, $\digamma_{n\times 3}$ as the a-cycle, and the vertices $\{b_j : 1 \leq j \leq n\}$ in the graph, $\digamma_{n\times 3}$ as the b-vertices. For our purpose, we consider $a_1 = a_{n+1}$ and $b_1 = b_{n+1}$. Imran et al. [11], proved that the flower graph $\digamma_{n\times 3}$ consists of a minimum resolving set with cardinality three and it constitutes the family of the plane graph with constant metric dimension. In the next result, we determine lower and upper bounds on the FTMD for the rotationally symmetrical plane graph $\digamma_{n\times 3}$.

Theorem 3.1. *For $n = 7$, we have $4 \le fdim(F_{n\times3}) \le 11$ and for $n \ge 8$, the lower and upper bounds for the FTMD of the graph $F_{n\times3}$ are*

$$fdim(F_{n\times3}) \ge \begin{cases} 3, & \text{if } n \text{ is even;} \\ 4, & \text{if } n \text{ is odd.} \end{cases}$$

and

$$fdim(F_{n\times3}) \le \begin{cases} 6, & \text{if } n \text{ is even;} \\ 12, & \text{if } n \text{ is odd.} \end{cases}$$

respectively.

Proof. From Theorem 2.2, we find that the metric dimension of the plane graph $F_{n\times3}$ is two when n is even and three when n is odd. In [11], it was proved that the sets $L = \{b_1, b_w\}$ and $L = \{a_1, a_2, a_{w+1}\}$ are the basis sets for the plane graph $F_{n\times3}$, when n is even and odd respectively. Then by using equation (1), we obtain the lower bound for the FTMD of $F_{n\times3}$ as

$$fdim(F_{n\times3}) \ge \begin{cases} 3, & \text{if } n \text{ is even;} \\ 4, & \text{if } n \text{ is odd.} \end{cases}$$

Next, one can easily find that, for $n = 7$, the upper bound for the FTMD of $F_{n\times3}$ is 11 (see graph $F_{n\times3}$ and use Lemma 2.1). Now, in order to finish the proof, we have to set the upper bound for the FTMD of $F_{n\times3}$, for $n \ge 8$.

Claim: $F_{n\times3}$ has a FTRS of cardinality 6 and 12, for $n \ge 8$.

To explain this, we look more closely at the two situations that arise by using the positive integer n i.e., $n \equiv 0 (mod\ 2)$ and $n \equiv 1 (mod\ 2)$.

Case 1 When $n \equiv 0\ (mod\ 2)$.
For this, the integer n can be written as $n = 2w$, $w \in \mathbb{N}$, and $w \ge 3$ (for $n = 6$ the result is also true). Then, $L = \{b_1, b_w\} \subset V(F_{n\times3})$ is a minimum resolving set for the rotationally symmetric graph $F_{n\times3}$ [11]. Next, we will show that the plane graph $F_{n\times3}$ has a FTRS of cardinality 6. From Fig. 2(a), one can find that $N[b_1] = \{b_1, a_1, a_2\}$, and $N[b_w] = \{b_w, a_w, a_{w+1}\}$. Also, we find that $\lambda(N(b_1)) = \lambda(N(b_w)) = \phi$. From this fact and Lemma 2.1, we obtain that $L^* = \{b_1, a_1, a_2, b_w, a_w, a_{w+1}\}$ is a FTRS of $F_{n\times3}$. Thus, we find that there exists a FTRS for the rotationally symmetrical plane graph $F_{n\times3}$ of cardinality 6, if n is even.

Case 2 When $n \equiv 1\ (mod\ 2)$.
For this, the integer n can be written as $n = 2w + 1$, $w \in \mathbb{N}$, and $w \ge 4$. Then, $L = \{a_1, a_2, a_{w+1}\} \subset V(F_{n\times3})$ is a minimum resolving set for the rotationally symmetric graph $F_{n\times3}$ [11]. Next, we will show that the plane graph $F_{n\times3}$ has a FTRS of cardinality 12. From Fig. 2(a), one can find that $N[a_1] = \{a_1, a_2, a_n, b_1, b_n\}$, $N[a_2] = \{a_1, a_2, a_3, b_1, b_2\}$, and $N[a_{w+1}] = \{a_w, a_{w+1}, a_{w+2}, b_w, b_{w+1}\}$. Also, we find that $\lambda(N(a_1)) = \lambda(N(a_2)) = \lambda(N(a_{w+1})) = \phi$. From this fact and Lemma 2.1, we obtain that $L^* = \{a_1, a_2, a_3, a_w, a_{w+1}, a_{w+2}, a_n, b_1, b_2, b_w, b_{w+1}, b_n\}$ is a FTRS of $F_{n\times3}$. Thus, we find that there exists a FTRS for the rotationally symmetrical plane graph $F_{n\times3}$ of cardinality 12, if n is odd.

Hence, we can obtain that there exists a FTRS of cardinality 6 and 12 for $\digamma_{n\times 3}$, and thus, the claim. $\square$

The immediate conclusion of Theorem 3.1 is the following corollary:

Corollary 3.2. *The FTMD of the flower graph $\digamma_{n\times 3}$ is constant.*

4. Fault-tolerant metric dimension for the plane graph S_n

The plane graph S_n [11] consists of n triangular faces, n pentagonal faces, n hexagonal faces, and a pair of faces each having n sides. It has a $5n$ number of vertices and a $8n$ number of edges (see Fig. 3). For S_n, the set of edges and vertices are denoted by $E(S_n)$ and $V(S_n)$, respectively. Therefore, we have $V(S_n) = \{a_j, b_j, c_j, d_j, e_j : 1 \leq j \leq n\}$ and $E(S_n) = \{a_jb_j, b_jc_j, c_jd_j, d_je_j : 1 \leq j \leq n\} \cup \{a_ja_{j+1}, b_ja_{j+1}, d_jc_{j+1}, e_je_{j+1} : 1 \leq j \leq n\}$.

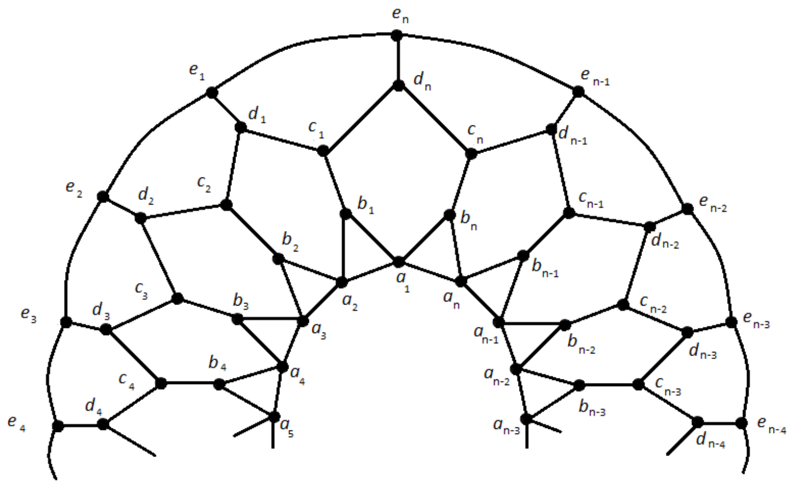

FIGURE 3. The graph S_n

We call the cycle induced by the vertices $\{a_j : 1 \leq j \leq n\}$ in the graph, S_n as the a-cycle, the vertices $\{b_j : 1 \leq j \leq n\}$ in the graph, S_n as the b-vertices, the cycle induced by the vertices $\{c_j, d_j : 1 \leq j \leq n\}$ in the graph, S_n as the cd-cycle, and the cycle induced by the vertices $\{e_j : 1 \leq j \leq n\}$ in the graph, S_n as the e-cycle. For our purpose, we consider $a_1 = a_{n+1}$, $b_1 = b_{n+1}$, $c_1 = c_{n+1}$, $d_1 = d_{n+1}$, and $e_1 = e_{n+1}$. Imran et al. [11], proved that the plane graph S_n consists of a minimum resolving set with cardinality three and it constitutes the family of the plane graph with constant metric dimension. In the next result, we determine the FTMD for the rotationally symmetrical plane graph S_n.

Theorem 4.1. *$fdim(S_n) = 4$, for every positive integer $n \geq 6$.*

Proof. From Theorem 2.3, we find that the metric dimension of the plane graph S_n is three, that is, $dim(S_n) = 3$ for every $n \geq 6$.

Claim: Convex polytope graph S_n has a minimum FTRS L^* of cardinality four.

To explain this, we look more closely at the two situations that arise by using the positive integer n i.e., $n \equiv 0\ (mod\ 2)$ and $n \equiv 1\ (mod\ 2)$.

Case 1 When $n \equiv 0\ (mod\ 2)$.

From this, we have $n = 2w$, $w \in \mathbb{N}$, and $w \geq 3$. Suppose $L^* = \{a_1, a_2, a_{w+1}, a_{w+2}\} \subset V(S_n)$. Next, we give fault-tolerant metric codes to every vertex of S_n with respect to the set L^*.

For the vertices of a-cycle $\{a_j : 1 \leq j \leq n\}$, the FTMC are as follows

$\varphi_F(a_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+2}
$j = 1$	$j-1$	1	$w-j+1$	$w-1$
$2 \leq j \leq w+1$	$j-1$	$j-2$	$w-j+1$	$w-j+2$
$j = w+2$	$2w-j+1$	$j-2$	$j-w-1$	$w-j+2$
$w+3 \leq j \leq 2w$	$2w-j+1$	$2w-j+2$	$j-w-1$	$j-w-2$

For the inward vertices $\{b_j : 1 \leq j \leq n\}$, the FTMC are as follows

$\varphi_F(b_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+2}
$j = 1$	j	1	$w-j+1$	w
$2 \leq j \leq w$	j	$j-1$	$w-j+1$	$w-j+2$
$j = w+1$	$2w-j+1$	$j-1$	$j-w$	$w-j+2$
$w+2 \leq j \leq 2w$	$2w-j+1$	$2w-j+2$	$j-w$	$j-w-1$

For the vertices of cd-cycle $\{c_j, d_j : 1 \leq j \leq n\}$, the FTMC are $\varphi_F(c_j|L^*) = \varphi_F(b_j|L^*) + (1,1,1,1)$ for $1 \leq j \leq 2w$. Next, the FTMC for the outward vertices $\{d_j : 1 \leq j \leq n\}$ are as follows

$\varphi_F(d_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+2}
$j = 1$	$j+2$	3	$w-j+2$	$w-j+3$
$2 \leq j \leq w-1$	$j+2$	$j+1$	$w-j+2$	$w-j+3$
$j = w$	$2w-j+2$	$j+1$	3	$w-j+3$
$j = w+1$	$2w-j+2$	$2w-j+3$	$j-w+2$	3
$w+2 \leq j \leq 2w-1$	$2w-j+2$	$2w-j+3$	$j-w+2$	$j-w+1$
$j = 2w$	3	$2w-j+3$	$j-w+2$	$j-w+1$

Finally, for e-cycle $\{e_j : 1 \leq j \leq n\}$, the FTMC are $\varphi_F(e_j|L^*) = \varphi_F(d_j|L^*) + (1,1,1,1)$ for $1 \leq j \leq 2w$. From these FTMC, we see that no two elements in $V(S_n)$ have the same fault-tolerant metric codes, suggesting L^* to be resolving set for S_n. Since, by definition of FTRS, the subsets $L^* \setminus \{a\}$, $\forall\, a \in L^*$ are $L_1 = \{a_1, a_2, a_{w+1}\}$, $L_2 = \{a_1, a_2, a_{w+2}\}$, $L_3 = \{a_1, a_{w+1}, a_{w+2}\}$, and $L_4 = \{a_2, a_{w+1}, a_{w+2}\}$. Now, to unveil that the set L^* is the FTRS for the graph S_n, we have to prove that the sets L_1, L_2, L_3, and L_4 are the resolving sets for S_n. Then, effortlessly one can find from the fault-tolerant metric codes, as shown above, that the sets L_1, L_2, L_3, and L_4 are also resolving sets for S_n, as the metric representation for every different pair of vertices of S_n are distinct with respect to the sets L_1, L_2, L_3, and L_4. Then, for

FTMD, we have $fdim(S_n) \leq 4$. Thus, from these lines, Theorem 2.3, and equation (1), we have $fdim(S_n) = 4$, in this case.

Case 2 When $n \equiv 1\ (mod\ 2)$.
From this, we have $n = 2w+1$, $w \in \mathbb{N}$, and $w \geq 3$. Suppose $L^* = \{a_1, a_2, a_{w+1}, a_{w+3}\} \subset V(S_n)$. Next, we give fault-tolerant metric codes to every vertex of $S_n \setminus L^*$ with respect to the set L^*.

For the vertices of a-cycle $\{a_j : 1 \leq j \leq n\}$, the FTMC are as follows

$\varphi_F(a_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+3}
$j = 1$	$j-1$	1	$w-j+1$	$w-1$
$j = 2$	$j-1$	$j-2$	$w-j+1$	w
$3 \leq j \leq w+1$	$j-1$	$j-2$	$w-j+1$	$w-j+3$
$j = w+2$	$2w-j+2$	$j-2$	$j-w-1$	$w-j+3$
$j = w+3$	$2w-j+2$	$2w-j+3$	$j-w-1$	$w-j+3$
$w+4 \leq j \leq 2w+1$	$2w-j+2$	$2w-j+3$	$j-w-1$	$j-w-3$

For the inward vertices $\{b_j : 1 \leq j \leq n\}$, the FTMC are as follows

$\varphi_F(b_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+3}
$j = 1$	j	1	$w-j+1$	w
$2 \leq j \leq w$	j	$j-1$	$w-j+1$	$w-j+3$
$j = w+1$	$2w-j+2$	$j-1$	$j-w$	$w-j+3$
$j = w+2$	$2w-j+2$	$2w-j+3$	$j-w$	$w-j+3$
$w+3 \leq j \leq 2w+1$	$2w-j+2$	$2w-j+3$	$j-w$	$j-w-2$

For the vertices of cd-cycle $\{c_j, d_j : 1 \leq j \leq n\}$, the FTMC are $\varphi_F(c_j|L^*) = \varphi_F(b_j|L^*) + (1,1,1,1)$ for $1 \leq j \leq 2w+1$. Next, the FTMC for the vertices $\{d_j : 1 \leq j \leq n\}$ are as follows

$\varphi_F(d_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+3}
$j = 1$	$j+2$	3	$w-j+2$	$w+2$
$2 \leq j \leq w-1$	$j+2$	$j+1$	$w-j+2$	$w-j+4$
$j = w$	$j+2$	$j+1$	3	$w-j+4$
$j = w+1$	$2w-j+3$	$j+1$	$j-w+2$	$w-j+4$
$j = w+2$	$2w-j+3$	$2w-j+4$	$j-w+2$	3
$w+2 \leq j \leq 2w$	$2w-j+3$	$2w-j+4$	$j-w+2$	$j-w$
$j = 2w+1$	3	$2w-j+4$	$j-w+2$	$j-w$

Finally, for e-cycle $\{e_j : 1 \leq j \leq n\}$, the FTMC are $\varphi_F(e_j|L^*) = \varphi_F(d_j|L^*) + (1,1,1,1)$ for $1 \leq j \leq 2w+1$. From these FTMC, we see that no two elements in $V(S_n)$ have the same fault-tolerant metric codes, suggesting L^* to be resolving set for S_n. Since, by definition of FTRS, the subsets $L^* \setminus \{a\}$, $\forall\ a \in L^*$ are $L_1 = \{a_1, a_2, a_{w+1}\}$, $L_2 = \{a_1, a_2, a_{w+3}\}$, $L_3 = \{a_1, a_{w+1}, a_{w+3}\}$, and $L_4 = \{a_2, a_{w+1}, a_{w+3}\}$. Now, to unveil that the set L^* is the FTRS for the graph S_n, we have to prove that the sets L_1, L_2, L_3, and L_4 are the resolving sets for S_n. Then, effortlessly one can find from the fault-tolerant metric codes, as shown above, that the sets L_1, L_2, L_3, and L_4 are also resolving sets for S_n, as the metric representation for every different pair of vertices of S_n are distinct with respect to the sets L_1, L_2, L_3, and L_4. Then, for FTMD, we have $fdim(S_n) \leq 4$. Thus, from these lines, Theorem 2.3, and equation (1), we have $fdim(S_n) = 4$, in this case also and hence the theorem. □

The immediate conclusion of Theorem 4.1 is the following corollary.

Corollary 4.2. *The FTMD for the convex polytope graph S_n is constant.*

5. Fault-tolerant metric dimension for the plane graph T_n

The plane graph T_n [11] consists of n triangular faces, n pentagonal faces, and a pair of faces each having n sides. It has a $3n$ number of vertices and a $5n$ number of edges (see Fig. 4). For T_n, the set of edges and vertices are denoted by $E(T_n)$ and $V(T_n)$, respectively. Therefore, we have $V(T_n) = \{a_j, b_j, c_j : 1 \leq j \leq n\}$ and $E(T_n) = \{a_jb_j, b_jc_j, a_ja_{j+1}, b_ja_{j+1}, c_jc_{j+1} : 1 \leq j \leq n\}$.

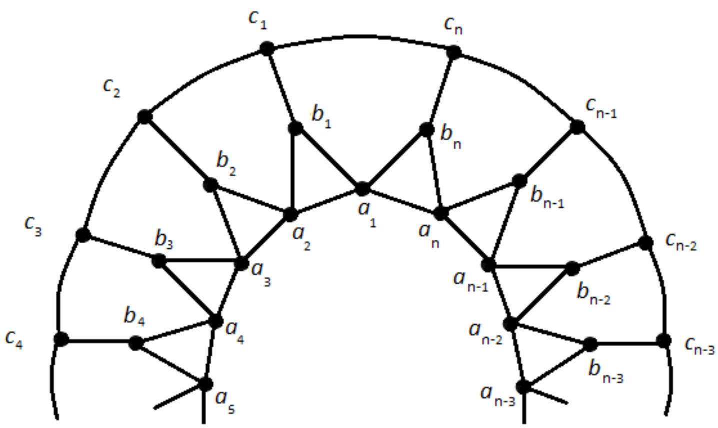

FIGURE 4. The graph T_n

We call the cycle induced by the vertices $\{a_j : 1 \leq j \leq n\}$ in the graph, T_n as the a-cycle, the vertices $\{b_j : 1 \leq j \leq n\}$ in the graph, T_n as the b-vertices, and the cycle induced by the vertices $\{c_j : 1 \leq j \leq n\}$ in the graph, T_n as the c-cycle. For our purpose, we consider $a_1 = a_{n+1}$, $b_1 = b_{n+1}$, and $c_1 = c_{n+1}$. Imran et al. [11], proved that the plane graph T_n consists of a minimum resolving set with cardinality three and it constitutes the family of the plane graph with constant metric dimension. In the next result, we determine the FTMD for the rotationally symmetrical plane graph T_n.

Theorem 5.1. *$fdim(T_n) = 4$, for every positive integer $n \geq 6$.*

Proof. From Theorem 2.4, we find that the metric dimension of the plane graph T_n is three, that is, $dim(T_n) = 3$ for every $n \geq 6$.

Claim: Convex polytope graph T_n has a minimum FTRS L^* of cardinality four. To explain this, we look more closely at the two situations that arise by using the positive integer n i.e., $n \equiv 0$ $(mod\ 2)$ and $n \equiv 1$ $(mod\ 2)$.

Case 1 When $n \equiv 0$ $(mod\ 2)$.
From this, we have $n = 2w$, $w \in \mathbb{N}$, and $w \geq 3$. Suppose $L^* = \{a_1, a_2, a_{w+1}, a_{w+2}\} \subset V(T_n)$. Next, we give fault-tolerant metric codes to every vertex of T_n with respect to the set L^*.

For the vertices of a-cycle $\{a_j : 1 \leq j \leq n\}$, the FTMC are as follows

$\varphi_F(a_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+2}
$j = 1$	$j-1$	1	$w-j+1$	$w-1$
$2 \leq j \leq w+1$	$j-1$	$j-2$	$w-j+1$	$w-j+2$
$j = w+2$	$2w-j+1$	$j-2$	$j-w-1$	$w-j+2$
$w+3 \leq j \leq 2w$	$2w-j+1$	$2w-j+2$	$j-w-1$	$j-w-2$

For the inward vertices $\{b_j : 1 \leq j \leq n\}$, the FTMC are as follows

$\varphi_F(b_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+2}
$j = 1$	j	1	$w-j+1$	w
$2 \leq j \leq w$	j	$j-1$	$w-j+1$	$w-j+2$
$j = w+1$	$2w-j+1$	$j-1$	$j-w$	$w-j+2$
$w+2 \leq j \leq 2w$	$2w-j+1$	$2w-j+2$	$j-w$	$j-w-1$

Finally, for the vertices of c-cycle $\{c_j : 1 \leq j \leq n\}$, the FTMC are $\varphi_F(c_j|L^*) = \varphi_F(b_j|L^*) + (1,1,1,1)$ for $1 \leq j \leq 2w$. From these FTMC, we see that no two elements in $V(T_n)$ have the same fault-tolerant metric codes, suggesting L^* to be resolving set for T_n. Since, by definition of FTRS, the subsets $L^* \setminus \{a\}$, $\forall\ a \in L^*$ are $L_1 = \{a_1, a_2, a_{w+1}\}$, $L_2 = \{a_1, a_2, a_{w+2}\}$, $L_3 = \{a_1, a_{w+1}, a_{w+2}\}$, and $L_4 = \{a_2, a_{w+1}, a_{w+2}\}$. Now, to unveil that the set L^* is the FTRS for the graph T_n, we have to prove that the sets L_1, L_2, L_3, and L_4 are the resolving sets for T_n. Then, effortlessly one can find from the fault-tolerant metric codes, as shown above, that the sets L_1, L_2, L_3, and L_4 are also resolving sets for T_n, as the metric representation for every different pair of vertices of T_n are distinct with respect to the sets L_1, L_2, L_3, and L_4. Then, for FTMD, we have $fdim(T_n) \leq 4$. Thus, from these lines, Theorem 2.4, and equation (1), we have $fdim(T_n) = 4$, in this case.

Case 2 When $n \equiv 1$ $(mod\ 2)$.
From this, we have $n = 2w+1$, $w \in \mathbb{N}$, and $w \geq 3$. Suppose $L^* = \{a_1, a_2, a_{w+1}, a_{w+3}\} \subset V(T_n)$. Next, we give fault-tolerant metric codes to every vertex of $T_n \setminus L^*$ with respect to the set L^*.

For the vertices of a-cycle $\{a_j : 1 \leq j \leq n\}$, the FTMC are as follows

$\varphi_F(a_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+3}
$j=1$	$j-1$	1	$w-j+1$	$w-1$
$j=2$	$j-1$	$j-2$	$w-j+1$	w
$3\leq j\leq w+1$	$j-1$	$j-2$	$w-j+1$	$w-j+3$
$j=w+2$	$2w-j+2$	$j-2$	$j-w-1$	$w-j+3$
$j=w+3$	$2w-j+2$	$2w-j+3$	$j-w-1$	$w-j+3$
$w+4\leq j\leq 2w+1$	$2w-j+2$	$2w-j+3$	$j-w-1$	$j-w-3$

For the inward vertices $\{b_j : 1 \leq j \leq n\}$, the FTMC are as follows

$\varphi_F(b_j\|L^*)$	a_1	a_2	a_{w+1}	a_{w+3}
$j=1$	j	1	$w-j+1$	w
$2\leq j\leq w$	j	$j-1$	$w-j+1$	$w-j+3$
$j=w+1$	$2w-j+2$	$j-1$	$j-w$	$w-j+3$
$j=w+2$	$2w-j+2$	$2w-j+3$	$j-w$	$w-j+3$
$w+3\leq j\leq 2w+1$	$2w-j+2$	$2w-j+3$	$j-w$	$j-w-2$

Finally, for the vertices of c-cycle $\{c_j : 1 \leq j \leq n\}$, the FTMC are $\varphi_F(c_j|L^*) = \varphi_F(b_j|L^*) + (1,1,1,1)$ for $1 \leq j \leq 2w$. From these FTMC, we see that no two elements in $V(T_n)$ have the same fault-tolerant metric codes, suggesting L^* to be resolving set for T_n. Since, by definition of FTRS, the subsets $L^* \setminus \{a\}$, $\forall\ a \in L^*$ are $L_1 = \{a_1, a_2, a_{w+1}\}$, $L_2 = \{a_1, a_2, a_{w+3}\}$, $L_3 = \{a_1, a_{w+1}, a_{w+3}\}$, and $L_4 = \{a_2, a_{w+1}, a_{w+3}\}$. Now, to unveil that the set L^* is the FTRS for the graph T_n, we have to prove that the sets L_1, L_2, L_3, and L_4 are the resolving sets for T_n. Then, effortlessly one can find from the fault-tolerant metric codes, as shown above, that the sets L_1, L_2, L_3, and L_4 are also resolving sets for T_n, as the metric representation for every different pair of vertices of T_n are distinct with respect to the sets L_1, L_2, L_3, and L_4. Then, for FTMD, we have $fdim(T_n) \leq 4$. Thus, from these lines, Theorem 2.4, and equation (1), we have $fdim(T_n) = 4$, in this case also and hence the theorem. □

The immediate conclusion of Theorem 6 is the following corollary.

Corollary 5.2. *The FTMD for the convex polytope graph T_n is constant.*

6. Conclusion

In this article, we studied the fault-tolerant metric dimension for three rotationally symmetrical families of the plane graphs (viz., the flower graph $\digamma_{n\times 3}$, S_n, and T_n). For S_n and T_n, we proved that $fdim(S_n) = fdim(T_n) = 4$, and for the flower graph $\digamma_{n\times 3}$ we set upper and lower bounds for its FTMD. In the future, we will try to obtain the other variants of metric dimension (for instance, edge metric dimension, fault-tolerant edge metric dimension, mixed metric dimension, etc) for these rotationally symmetric plane graphs.

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