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Building super-additive manifolds

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Abstract. A new concept of differentiability is introduced and developed, in the framework of super-additive normed linear spaces. This notion is intended to be the first step in rephrasing the grounds of general relativity.

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When we mean to build, We first survey the plot, then draw the model
- William Shakespeare, King Henry IV. Part II., Sc. 3

1. Introduction

The last ten years have seen rapid advances in the understanding of differentiable four-manifolds, not least of which has been the discovery of new "exotic" manifolds.

This recent revolution in differential topology related to the discovery of non-standard smoothness structures on topologically trivial manifolds such as $\mathbb{R}^4$ suggests many exciting opportunities for applications of potentially deep importance for the spacetime models of theoretical physics, especially general relativity. This rich panoply of new differentiable structures lies in the previously unexplored region between topology and geometry. Just as physical geometry was thought to be trivial before Einstein, physicists have continued to work under the tacit - but now shown to be incorrect - assumption that differentiability is uniquely determined by topology for simple four-manifolds. Since diffeomorphisms are the mathematical models for physical coordinate transformations, Einstein's relativity principle requires that these models be physically inequivalent.

We shall mention here some recent advances in theoretical physics developed in the last years.

1.1. Superstring Theory. Superstring theory apparently resolves the most enigmatic problem of the twentieth century theoretical physics: the mathematical incompatibility of the quantum mechanics and the General Theory of Relativity. In doing so, string theory modifies our understanding of spacetime and the gravitational force. One recently discovered consequence of this modification is that spacetime can undergo remarkable rearrangements of its basic structure requiring the fabric of spacetime to tear apart and subsequently reconnect. Such processes are at best unlikely and probably impossible in pre-string theories as they would be accompanied by violent physical effects. In string theory, on the contrary, these processes are physically sensible and thoroughly common.

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The usual domains of general relativity and quantum mechanics are quite different. General relativity describes the force of gravity and hence is usually applied to the largest and most massive structures including stars, galaxies, black holes and even, in cosmology, the universe itself. Quantum mechanics is most relevant in describing the smallest structures in the universe such as electrons and quarks. In most ordinary physical situations, therefore, either general relativity or quantum mechanics is required for a theoretical understanding, but not both. There are, however, extreme physical circumstances which require both of these fundamental theories for a proper theoretical treatment.

Prime examples of such situations are spacetime singularities such as the central point of a black hole or the state of the universe just before the Big Bang. These exotic physical structures involve enormous mass scales (thus requiring general relativity) and extremely small distance scales (thus requiring quantum mechanics). Unfortunately, general relativity and quantum mechanics are mutually incompatible: any calculation which simultaneously uses both of these tools yields nonsensical answers.

String theory solves the deep problem of the incompatibility of the above mentioned two fundamental theories by modifying the properties of general relativity when it is applied to scales on the order of the Planck length. String theory is based on the premise that the elementary constituents of matter are not described correctly when we model them as point-like objects. Rather, according to this theory, the elementary "particles" are actually tiny closed loops of string with radii approximately given by the Planck length. Modern accelerators can only probe down to distance scales around 10 cm and hence these loops of string appear to be point objects. However, the string theoretic hypothesis that they are actually tiny loops, changes drastically the way in which these objects interact on the shortest of distance scales. This modification is what allows gravity and quantum mechanics to form a harmonious union.

There is a price to be paid for this solution, however. It turns out that the equations of string theory are self consistent only if the universe contains, in addition to time, nine spatial dimensions. As this is in gross conflict with the perception of three spatial dimensions, it might seem that string theory must be discarded. This is not true.

1.2. Kaluza-Klein Theory. The idea that our universe might have more than the three familiar spatial dimensions is one which was introduced more than half a century before the advent of string theory by T. Kaluza and O. Klein. The basic premise of such Kaluza-Klein theories is that a dimension can be either large and directly observable or small and essentially invisible. No experiment rules out the possible existence of additional spatial dimensions curled up on scales smaller than 10 cm , the limit of present day accessibility. Although originally introduced in the context of point particle theories, this notion can be applied to strings. String theory, therefore, is physically sensible if the six extra dimensions which it requires curl up in this fashion.

A remarkable property of these theories is that the precise size, shape, number of holes, etc. of these extra dimensions determines properties such as the masses and electric charges of the elementary "particles".

1.3. Gravitational Fluctuations and the Topology of Spacetime. A number of issues, unresolved at present, prevent the application of string theory to the analysis of the kind of spacetime singularities described above. The theory can be successfully applied, though, to another class of singularities which control the topology of the universe (see [17]).

General relativity predicts that the spacetime will smoothly deform its size and shape in response to the presence of matter and energy. A familiar manifestation of this spacetime stretching is the expansion of the universe. The topology of the universe, however, remains fixed.

In general, a manifold is defined to be a patchwork of coordinate patches, each indistinguishable from a region of a model space, together with their gluing instructions; n-folds are simply n-dimensional manifolds. Such a manifold is said to be modelled on the model space. For example, Euclidean manifolds are modelled on the Euclidean space. For a euclidean 2-fold, each patch is given (local) coordinates, in which it "looks" like a region in the affine (x, y)-plane. The gluing instructions then simply tell how to slightly overlap these patches and specify the transition functions, i.e., how the coordinates of the overlapping regions are to be identified in the overlap.

Taking into consideration the nature of the transition functions, the manifolds are said to be smooth, algebraic, holomorphic (complex-analytic), etc. For a manifold of one category to also belong to another, there is typically an additional object ("structure") to be found on it. For example, for a smooth manifold to be Riemannian, it must admit a global rank-2 covariant, i.e., type-$(0, 2)$ tensor, the metric, the eigenvalues of which are real and positive at every point of the manifold. Pseudo-Riemannian manifolds admit metrics with s positive and t negative eigenvalues, are said to have signature (s, t); metrics in the "real" spacetime have signature $(3, 1)$. The metric may be "read off" from the expression for the square of the line element: where summation over the indices $m, n = 0, 1, 2, 3$ is implied, 0 labeling the temporal axis, and $1, 2, 3$ the three spactial axes, as exemplified after the second equality; the third statement is an alternative representation of the same statement. Complex (holomorphic) manifolds (for which the transition functions are holomorphic) are real even-dimensional smooth manifolds which in addition admit a rank-2 mixed, i.e., type-$(1, 1)$ tensor, the complex structure, precisely a half of the eigenvalues of which are positive and the other half negative. The eigenvectors with positive eigenvalues are the holomorphic (complex) coordinates, and those with negative eigenvalues are the anit-holomorphic (conjugate) coordinates.

Alternatively, instead of specifying a manifold as a patchwork of coordinate patches (each of which is a copy of an affine and well understood "model" space) together with gluing instructions, it may be specified as a subspace of another, presumably more easily describable space.

It is known that (i) a diffeomorphism of manifolds with indefinite metrics preserving degenerate r-plane sections is conformal, (ii) a sectional curvature-preserving diffeomorphism of manifolds with indefinite metrics of dimension 4 is generically an isometry.

Is it possible to isometrically embed a non-Euclidean manifold in a Euclidean manifold of higher dimension? If we limit ourselves to just ONE new dimension the answer is no. This was proved around 1901 by Hilbert, who showed that the original non-Euclidean space (the 2D hyperbolic plane of Lobachevski, Bolyai, et al.) cannot be isometrically embedded in its entirety in 3D Euclidean space. However, it can be embedded in 6D Euclidean space, and, probably, even in 5D Euclidean space. Apparently the question of whether there exists a complete isometric embedding in 4D Euclidean space remains open. In any case, we can always embed a smooth metrical non-Euclidean space in a higher-dimensional Euclidean space, but it usually takes more than just one extra dimension.

2. A possible categorical approach

2.1. Open Problem. Let Cat be the category of all small categories. We need for a (non-trivial) "topology" on $Morph\,(Cat)$ (there is, the class of functors between small categories), with the following property: for each u in $Morph\,(Cat)$ and each D in $Ob\,(Cat)$ there are

$$C \xrightarrow{v} D$$

in $Morph\,(Cat)$, a functor $F : Cat \to Cat$ with $F\,(u) = v$ and "open neighbourhoods" V and W of u and v , respectively, such that F (acting on $Morph\,(Cat)$) is a "homeomorphism" between V and W. Here the topological notions are also to be defined, even in a weaker sense that Giraud-Grothendieck's one.

2.2. Notations. In the sequel, we shall use the following notations:

$E =$ the 4-dimensional Euclidean space $\mathbb{R}^4$.
$Mink_4 =$ the Minkowski space-time $\mathbb{R} \times \mathbb{R}^3$ (see e.g. [10], [11]).
$Mink_2 =$ the Minkowski plane $\mathbb{R} \times \mathbb{R}$.
$Man =$ the category of differentiable manifolds.
$Top =$ the category of topological spaces.
$Cat =$ the category of small categories.
$Ob\,(C) =$ the objects of category C.
If S is in $Ob\,(C)$, then $Isom\,(S)$ will denote the isomorphisms of S.
If X is a linear space, we shall denote by $Norm\,(X)$ the family of all norms defined on X.

The basic cone we shall introduce is the set of functions $K = \{\omega : \mathbb{R}^+ \to \mathbb{R}^+\}$, satisfying the following properties:

$$- \ \omega \text{ is non-decreasing}$$
$$- \lim_{t\to\infty} \frac{\omega(t)}{t} = 0.$$

To work in GR, one must define first pseudo-Riemannian manifolds. To define pseudo-Riemannian manifolds, one must define first smooth manifolds. To define smooth manifolds, one must define first the Euclidean space, endowed with its topology, which is generated by its old-fashioned (positive definite) norm. Of course, general relativity is invariant under diffeomorphisms. But we should raise the following question: Could one explain why

- there are nonlinear f in $Diff\,(Mink_2)$ both preserving/reversing the causal order
- there is no such f in $Diff\,(Mink_4)$.

What makes the difference?

As it is widely known, the Einstein-Podolski-Rosen criticism to the Copenhagen Interpretation of Quantum Mechanics resulted in several attempts to improve the foundational bases of QM (Hidden Variable Theories, The Many-Worlds Interpretation, The Transactional Interpretation, ...). After our knowledge, no comparable efforts were made in GR (preserving exactly GR, not string or Kaluza-Klein like theories).

2.3. Space-time decomposition. The 3-metric g_{ij} and extrinsic curvature (second fundamental form) K_{ij} are the fundamental variables describing the geometry in any space-time decomposition of the Einstein equations. (g, K) describe the local geometry of a single space-like hypersurface M, and it is then natural to describe the evolution of the space-time geometry by a 1-parameter family $(g\,(t)\,, K\,(t))$, describing the local geometry of the (space-like) hypersurfaces M_t. In order to piece these hypersurfaces together, however, we must also specify the lapse N and shift vector X^i , which describe the relation between the time evolution vector ∂_t^1 and the space-time vector n normal to the hypersurfaces:

$$\partial_t = Nn + X^i\partial_i,\ n = N^{-1}\left(\partial_t - X^i\partial_i\right).$$

The space-time metric is then fully determined, by

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + X^idt)(dx^j + X^jdt),$$

and the inverse metric in terms of $[g^{ij}] = [g_{ij}]^{-1}$ is

$$\overline{g}^{ab}\partial_a \otimes \partial_b = -N^{-2}\left(\partial_t - X^i\partial_i\right)^2 + g^{ij}\partial_i \otimes \partial_j,$$

where to reduce the risk of confusion we use $\overline{g}_{ab}$ to denote the space-time metric components. Conversely we have

$$\begin{aligned} N^2 &= -\overline{g}_{00} + \overline{g}_{0i}\overline{g}_{0j}g^{ij} \\ &= -1/\overline{g}^{00}, \\ X_i &= g_{ij}X^j = \overline{g}_{0i}, \\ X^i &= -\overline{g}^{0i}/\overline{g}^{00}, \\ g^{ij} &= \overline{g}^{ij} - \overline{g}^{0i}\overline{g}^{0j}/\overline{g}^{00}. \end{aligned}$$

Any numerical formulation brings with it coordinates $\left(t, x^i\right)$- one important challenge then is to develop good choices of the lapse and shift, so that the space-like hypersurfaces (level sets of t) and the spatial coordinates x^i remain as smooth and regular as possible. In geometric terms this amounts to constructing "good" coordinates, where "good" can mean many things. For example, a popular choice for the time coordinate requires that the hyper surfaces are *maximal*, i.e. $trK = 0$. Because this amounts to an elliptic equation (analogous to the minimal surface equation satisfied by soap films), the t coordinate is as smooth as the space-time allows, so coordinate breakdown signals serious geometrical problems, rather than spurious coordinate effects. On the other hand, the resulting elliptic equation on the lapse

$$\Delta_g N = N\left|K\right|^2 = NK_{ij}K^{ij}$$

is expensive to solve numerically, and is "non-local".

The relation between (g, K) and $\left(N, X^i\right)$ is

$$K_{ij} = \frac{1}{2}N^{-1}\left(\partial_t g_{ij} - \nabla_i X_j - \nabla_j X_i\right),$$

where $\nabla_i X_j$ denotes the spatial covariant derivate,

$$\nabla_i X_j = \partial_i\left(X_j\right) - \Gamma_{ijk}X^k.$$

This shows that if (g, K) is given, and $\left(N, X^i\right)$ chosen appropriately, then $\partial_t g_{ij}$ is determined. By analogy with the usual wave equation, it is perhaps not surprising that (g, K) form the geometric initial data for the Einstein equations. However, one point at which the analogy with the wave equation $\Box u = 0$ breaks down, is the matter of constraints.

Whereas the initial data for the wave equation consists of *arbitrary* functions (u_0, u_1) with $u\left(x, 0\right) = u_0\left(x\right),\ \partial_t u\left(x, 0\right) = u_t\left(x\right),$ for the Einstein equations the data (g, K) are not freely specifiable, but must satisfy the constraint equations

$$\begin{aligned} G_{00} &= R\left(g\right) + \left|K\right|^2 - \left(tr_g K\right)^2 \\ G_{0i} &= 2\left(\nabla^j K_{ij} - \nabla_i tr_g K\right), \end{aligned}$$

where G_{0a}, $a = 0, ..., 3^2$ are components of the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$. The Einstein equations

$$G_{ab} = 8\pi k T_{ab}$$

connect the space-time curvature with the stress-energy tensor T_{ab}, which reflects the matter content of any additional fields (perfect fluid, Maxwell, Yang-Mills, dilation etc) present in the simulation. For the vacuum Einstein equations, the stress energy tensor T_{ab} vanishes, so we can consider $G_{0a} = 0$ for simplicity. Note that T_{00} is interpreted physically as describing the local energy density as measured by an observer with world line in the direction e_0 , and likewise T_{0i}, $i = 1, 2, 3$ describes the local momentum density vector.

We think this brilliant non-Euclidean construction (GR) is built on a non-adequate Euclidean foundation (more precisely, using the Euclidean math apparatus existing at the time when General Relativity was born). The question is: can we preserve the building and change its foundation? In other words, should we seek for new foundations of GR ?

2.4. Connections to general relativity (GR). It is well-known that topology is not well-adapted to the study of relativity. Indeed, there exist points A, B, C in the spacetime such that the absolute distances AB and BC are both less than the Planck length, and yet the distance AC is the radius of the observable universe. Also, the huge homeomorphism group of $\mathbb{R}^4$ is of no physical significance. Actually, the natural spacetime "metric" is super-additive with respect to the causal order of $Mink_4$. In order to formalize this fact, we shall introduce a new category, called X, a non-topological analogue of Top. For example, $Mink_4$ belongs to $Ob\,(X)$ (of course, $Mink_4$ is also in $Ob\,(Top)$, but it is not the same $Mink_4$). The self-morphisms of $Mink_4$ in X are just the Lorentz transformations (see [1], [2], [4], [6], [15], [18]). Moreover, one could prove that Top and X are non-isomorphic categories. Unfortunately, the next step of our approach to relativity, namely introducing in the above framework a category of "differentiable manifolds", say $X - Man$, seems to be an extremely difficult task, because we don't know what a differentiable structure should be in this non-topological context. Therefore, we are trying to use for the above purpose the Category Theory. More precisely, we are trying to introduce "differentiable manifolds" without introducing differentiability. From this viewpoint (remark that we have a natural forgetful functor $Man \to Top$), our problem can be reformulated as:

"$X - Man$ should be to X, as Man is to Top".

Of course, in order to be able to work in Cat, we should restrict our discussion to subcategories of small size. We would like to remind that our original problem was a purely categorical one. In free translation: request for a "Local Inversion Theorem" in Cat, the category of small categories.

Let us observe that it's not the metric, but the concept of "locality" in Relativity Theory which we are attacking. On the other hand, at least for the sake of physical measurements, we must assume metrisability. Also, the group of diffeomorphisms of $\mathbb{R}^4$ is of no relativistic relevance. A general nonlinear diffeomorphism of $\mathbb{R}^4$ has nothing to do with causality. Despite of the fact that there are some exotic structures on the Minkowskian space-time $Mink_4$, which were developed around 1970 (see e.g. [12], [13], [17], [19], [20]), one must remark that they essentially reflect the anisotropic structure of spacetime, hence they are NOT locally Euclidean, hence NOT manifolds.

For the sake of simplicity, in what follows we will describe only the "metrisable" version of some newly introduced category X . Super-additivity does not arise only in Relativity. This notion may be encountered also in the theory of classical Banach spaces (L^p spaces, and $C\,(S)$), and also in game theory. Furthermore, whenever someone divides the proof of a "heavy" theorem into a (finite) sequence of easier-to-prove lemmas, then super-additivity is being used. From this viewpoint, the objects of X are triples (W, K, d), where (W, K) is an ordered set, and $d : K \to [0, \infty)$ satisfies:

a) $d\,(x, y) = 0$ if $x = y$ and

b) $x \leq y$ and $y \leq z$ imply $d\,(x, y) + d\,(y, z) \leq d\,(x, z)$, i.e. d is *super-additive* on K.

If x is an element of W and $\varepsilon > 0$, then every subset A of W satisfying $d\,(x, a) \geq \varepsilon$, for all a in A will be called a *slice of future* of x. A function $f : (W_1, K_1, d_1) \to (W_2, K_2, d_2)$ will be called *inflationnary* if for each x in W_1 and A *slice of future* of x, $f\,(A)$ is a *slice of future* of $f\,(x)$ in W_2. The inflationnary functions are the morphisms of X. In fact, X is much more bigger than this: it contains all the classical Banach spaces, sets of ordinals, and lexicographically ordered spaces.

Maybe one could use, alternatively, something like sheaf theory? I.e., maybe one could define a "manifold" as a pair (Y, F), where Y is in $Ob\,(X)$ and F is a "sheaf" of morphisms over Y where (Y, F), "locally" looks like Minkowski spacetime? We are wondering if this idea could be combined with that one of considering the category of manifolds with pseudo-Riemannian metric, the morphisms being metric preserving maps), but, unfortunately, "locally" is not a natural concept in X.

2.5 In order to explain what's wrong with Man from the relativistic viewpoint, one cannot resist to the temptation of raising the following

METACONJECTURE. One cannot define, using the formal language of Man, a subcategory C of Man with the following properties:

1) if Y, Z are in $Ob\,(C)$, then $C\,(Y, Z)$ is nonempty;

2) for each $d > 1$ there exists M_d in $Ob(C)$ such that $Isom\,(M_d)$ is isomorphic to the special Lorentz group of the spacetime $\mathbb{R} \times \mathbb{R}^{d-1}$;

3) C has direct products;

4) $\mathbb{R}$ belongs to $Ob\,(C)$.

Here are a few miscellaneous open questions:

Let M be a smooth n-dimensional manifold and let $Diff\,(M)$ denote the collection of all diffeomorphisms of M.

(1) Is there a topology τ on the set M such that the homeomorphism group of (M, τ) is exactly (or, at least, isomorphic to) $Diff\,(M)$? In other words (alternatively, "vaguely speaking"), can a new (exotic) topology on M incorporate (alternatively, contain intrinsically) its (old) differential structure?

(2) If M is connected, is $Diff\,(M)$ a simple group? If S is connected, is *Autohomeo*(S) a simple group?

(3) What are the right topologies for $Diff\,(M)$ and *Autohomeo*(S)? For *Autohomeo*(S), our wild guess would be to take the compact open topology on the space $C\,(S, S)$, of continuous maps $S \to S$ and give *Autohomeo*(S) the relative topology from $C\,(S, S) \times C\,(S, S)$, after identifying an autohomeomorphism f with the pair (f, g), where g is the inverse of f.

(4) Are the aforementioned "right" topologies on $Diff\,(M)$ and *Autohomeo*(S) determined by the group structures?

(5) Doesn't $Diff\,(M)$ admit a structure of Frechet manifold with some kind of differentiable structure of its own? (let us remark that J.A. Leslie, in [7], considered the differential structure on the group of diffeomorphism of compact connected manifold

M. The conclusion of that paper is as follows: $Diff\,(M)$ is a locally Frechet C^∞ group. The author used the notion of quasi-topological space which has been elaborately studied by A. Bastiani). Does $Autohomeo(S)$ ever admit such a structure?

Unfortunately, there is no "Local Inversion Theorem" in categories known until now.

3. Super-additive normed spaces

3.1. Super-additivity generated by probability measures [9]. Let $g^0 : I \to \mathbb{R}^+$, be a strictly positive potential which belongs to BV and admits a conformal measure m. By L_0 we designate the usual Perron-Frobenius operator (or transfer operator) associated to the dynamic and g^0. The operator L_0 acts on $L^1(m)$ and BV:

$$L_0 f(x) = \sum_{Ty=x} f(y) g^0(y).$$

Here, an useful tool is the transfer operator L defined by

$$L(f) = L_0(f\ \mathbf{1}_{X_0}).$$

We can now define the functional

$$\wedge(f) := \lim_{n\to\infty} \inf_{x\in D_n} \tfrac{L^n f(x)}{L^n 1(x)}.$$

The relevant properties of the above functional are the following:

- $\wedge(1) = 1$;
- $\wedge$ is continuous in the L^∞ norm;
- $f \geq g$ implies $\wedge(f) \geq \wedge(g)$ (monotonicity);
- $\wedge(\lambda f) = \lambda \wedge (f)$ (homogeneity);
- $\wedge(f+g) \geq \wedge(f) + \wedge(g)$ (super-additivity);
- $(\forall)\, b \in \mathbb{R}$, $\wedge(f+b) = \wedge(f) + b$;
- if for $p \subset I$ there exists $n \in \mathbb{N}$ such that $p \cap X_n = \phi$, then $\wedge(\mathbf{1}_p) = 0$.

3.2. Trace functions and super-additivity (according to [8]). The convexity of the function $x \to Tr(f(x))$, when f is a convex function of one variable and x is a self-adjoint operator, was known to von Neumann, cf. **J. von Neumann** [14, p.390]. An early proof for $f(x) = \exp(x)$ can be found, e.g. in **D. Ruelle** [16].

More generally, we replace the trace Tr in a Hilbert space setting by τ, a densely defined, lower semi-continuous trace on a C^*- algebra A; i.e. a functional defined on the set A_+ of positive elements with values in $[0,\infty]$, such that $\tau(x^*x) = \tau(xx^*)$ for all x in A.

Then, in particular, we can easily see that

$$\left(\tau\left((x+y)^{1/p}\right)\right)^p \geq \left(\tau\left(x^{1/p}\right)\right)^p + \left(\tau\left(y^{1/p}\right)\right)^p,$$

for all x, y in A_+, so that the Schatten p-norms are super-additive for $p < 1$.

As a consequence, the *Kadison-Fuglede determinant* Δ associated with a tracial state τ on a C^*- algebra A:

$$\Delta(x) = \exp(\tau(\log|x|)) \qquad \text{whenever } x \in A^{-1}$$

is a concave map on the set of positive invertible elements.

3.3. Bayesian theory and super-additivity (see [3]). Quasi-Bayesian theory, like Bayesian theory, assumes the existence of a utility function. A decision is a function f that assigns a utility value for each possible state of the world. The key problem is how to compare decisions. A preference pattern must be defined so that the decision-maker can compare functions.

The axioms below are valid for a preference relation $\geq$ defined for pairs of functions. The statement $f \succeq g$ means f *is at least as preferred as* g. To simplify notation, define the strict preference relation $\succ$ by: $f \succ g$ if and only if $f \succeq g$ and not $g \succeq f$.

Giron and Rios' axioms are:

1. If $f \succeq g$ and $g \succeq h$ then $f \succeq h$. (*transitivity*)
2. If $f > g$ then $f \succ g$. (*dominance*)
3. For $\lambda \in (0,1]$, $f \succeq g$ if and only if $\lambda f + (1-\lambda)h \succeq \lambda g + (1-\lambda)h$. (*convex combination*)
4. If $f_i \to f$ and $g \succeq f_i \succeq h$ for all i, then $g \succeq f \succeq h$. (*convergence*)

These axioms are similar to axioms proposed by Walley. Actually, Walley indicates that his axioms are "apparently equivalent" to Giron and Rios' axioms. The following theorem proves the equivalence of the Giron-Rios and Walley systems.

Theorem 3.1. *Giron and Rios' axioms are equivalent to the following axioms:*

1. *If $f = -1$ and $g = 0$, it is not the case that $f \succeq g$. (sure gain)*
2. *If $f \succeq g$ and $g \succeq h$ then $f \succeq h$. (transitivity)*
3. *If $f \geq g$ then $f \succeq g$. (monotonicity)*
4. *If $f \succeq g$ and $\lambda > 0$ then $\lambda f \succeq \lambda g$. (positive homogeneity)*
5. *If $f + \lambda \succeq g$ for all $\lambda > 0$ then $f \succeq g$. (continuity)*
6. *$f \succeq g$ if and only if $f - g \succeq 0$. (cancellation)*

To investigate the consequences of the axioms, define a functional $\underline{E}[f]$, called the lower expectation of function f:

$$\underline{E}[f] = \max_{\mu} [f \succeq \mu].$$

Theorem 3.2. *Lower expectations have the following properties:*

1. *$\underline{E}[f] \geq \inf f$.*
2. *$\underline{E}[\lambda f] = \lambda \underline{E}[f]$ for $\lambda > 0$. (positive homogeneity)*
3. *$\underline{E}[f+g] \geq \underline{E}[f] + \underline{E}[g]$. (super-additivity)*

The above mentioned three concepts of super-additivity arising from various mathematical fields lead us to introduce the following natural analogue of normed linear spaces.

3.4. Definition.

Definition 3.1. *Let L be a real linear space ordered by some cone K of positive vectors.*

The functional $\|\cdot\| : K \to \mathbb{R}^+$ shall be named a ***super-additive norm*** *if it satisfies the following three axioms:*

(1) $\|x\| = 0 \Longleftrightarrow x = 0$;
(2) $\|\lambda \cdot x\| = \lambda \cdot \|x\|$, *for all* $\lambda \in \mathbb{R}^+$ *and* $x \in K$;
(3) $\|x+y\| \geq \|x\| + \|y\|$, *for each* $x, y \in K$.

The triplet $(L, K, \|\cdot\|)$ will be called ***a super-additive normed space****.*

In what follows, we will denote by **SAN** the category of super-additive normed spaces, and by $D(E,F)$ the family of all inflationnary morphisms acting between two super-additive normed spaces E and F. Recall that a (positive) linear operator T from E to F is called inflationnary iff there is some $a > 0$ such that $\|Tx\| \geq a\|x\| \quad (x \in E^+)$. In this case, we shall write $T \in D(E,F)$. Also, we shall denote by $Iso(E,F)$ the family of all isotone (not necessarily linear) operators from E to F. Conversely, for a detailed study of nonlinear contractions on spaces endowed with indefinite metrics, one may take a look on [5].

4. The construction of super-additive differentiability

It is important to observe that a bilinear (or equivalently, quadratic) form cannot be inflationnary (using the terminology of P.Taylor). Therefore, we are lead to the following methodological principle, which we shall call, by analogy to complex analysis, **The Super-Additive Morera Principle** (briefly, **SAMP**).

Theorem 4.1. *(The Super-Additive Morera Principle) If $E, F \in Ob(SAN)$ and $T \in D(E,F)$ is "differentiable" in any suitable sense, then T must be "C^∞" (infinitely) differentiable.*

Obstructions arising when trying to define super-additive manifolds:
- *the linear obstruction*: a linear inflationnary operator acting between two SAN spaces must be differentiable. But, if we try to define the differentiability along the traditional lines, then the derivative of a linear operator would be constant at each point, hence not inflationnary (recall that a inflationnary map is strictly isotone). Therefore, a linear inflationnary operator will be differentiable, but not of class "C^1" , which is hard to imagine and, on the other hand, does not permit us to define "C^∞" manifolds.
- *the dimensional obstruction*: there is no natural number $n = Dim(E)$, with $E \in Ob(SAN)$, E finite-dimensional, such that

a) $Dim(E) = Dim(F)$ implies E is isomorphic to F.
b) $Dim(E \times F) = Dim(E) + Dim(F)$.

For, suppose that $Dim(\mathbb{R}) = s \in \mathbb{N}$. Let $E = \mathbb{R}^2$, endowed with the super-additive norm $|(x,y)| = \min\{x,y\}$, $x, y \geq 0$. Then, $Dim(E) = 2s$. Since $\mathbb{R}$ is linearly ordered, and E is not, it follows that $s > 0$. Now, let $d = Dim(Mink_4) > 0$. Then $Dim\left(E^d\right) = Dim\left(Mink_4^{2s}\right)$, and, consequently, $E^d \approx Mink_4^{2s}$ in the category SAN. But E^d $(= \mathbb{R}^{2d})$ is a lattice, and $Mink_4^{2s}$ is not, contradiction.

Moreover, let us observe that, even on $\mathbb{R}^2$, there are uncountable many non-isomorphic super-additive structures generated by super-additive norms. Therefore, the only naturally dimension of a SAN is its isomorphism type.

An useful hint: If $E, F \in Ob(SAN)$, and $T \in D(E,F)$ then, if we want T to be differentiable in any reasonable sense, we must impose

$$Ty - Tx \geq L(y-x) \qquad (x < y)$$

for some $L \in L_d(E,F)$.

Of course, this is a necessary condition only.

Definition 4.1. *Let E, F be two real linear spaces. A cone of nonlinear operators (quickly, **CNO**) with respect to (E,F) is a set $C = \{T : E \to F\} \subset F^E$ with the following four properties:*
1) $T, S \in C \Longrightarrow T + S \in C$.

2) $T \in C,\ \lambda > 0 \Longrightarrow \lambda T \in C.$
3) constants $\subset C$.
4) $(C - C) \cap L(E, F) = \{0\}$.

(Note that C is not a cone in the classical sense, due to 3)).

Definition 4.2. *Let X, Y be two real linear spaces. A nonlinear mapping $T : X \to Y$ is said to have sub-linear growth (shortly,* ***SLG****) if it satisfies the following condition:*
For each $p \in Norm(Y)$ there is q in $Norm(X)$ and $\omega \in K$ such that

$$p(Tx) \leq \omega(q(x)) \qquad (x \in X).$$

We shall denote by $SLG(X, Y)$ the above introduced class of mappings.

Example 4.1. *Let $X = Y = \mathbb{R}^N$. Then each asymptotically zero operator $T : X \to Y$, i.e. satisfying*

$$\lim_{\|x\| \to \infty} \frac{\|Tx\|}{\|x\|} = 0,$$

where $\|\cdot\|$ denotes the Euclidean norm, belongs to SLG, due to the equivalence of all finite-dimensional norms.

Example 4.2. *In the case $\dim(X) = \dim(Y) = \infty$, where X, Y are two Banach spaces, let $y \in Y$, $y \neq 0$, $\alpha \in (0, 1)$ and define $T : X \to Y$ by*

$$Tx = \|x\|^{\alpha} y.$$

Then it is easily shown that T belongs to $SLG(X, Y)$.

Remark 4.1. *Warning ! There is no topology here; we use only the elementary axioms of a norm (and a lot of norms!). In fact, a norm p can be non-topologically viewed also as $-q$, with q super-additive and negative.*

Remark 4.2. *Another useful class (for our purposes) of nonlinear operators, which can be defined without using norms, but is perhaps more restrictive than SLG, is the following:*

$\{T : E \to F,\ (\exists)\, \alpha \in (0, 1),\ f \in E_d^*$ *and* $w \in F^+$ *such that*
$Tu - Tv \leq [1 + f(u - v)]^{\alpha} w$ *wherever* $u, v \in X,\ v \leq u\}$
where $E, F \in Ob(SAN)$.

But here we must demand on each super-additive normed space X that

$-X$ *is directed* $(X = X^+ - X^-)$
$-X_d^*$ *is separating* ($(\forall)\ x \in X,\ x \neq 0,\ \exists f \in X_d^*,\ f(x) \neq 0)$

Now, let $E, F \in Ob(SAN)$.

Proposition 4.1. ***(The Splitting Scheme)***
1. $SLG(X, Y)$ is a ***CNO****.*
2. (linear inflationnary) $\circ$ (SLG) = (SLG).
3. (SLG) $\circ$ (linear inflationnary + SLG) = (SLG).
4. A mapping T can be written in at most one way as (linear inflationnary) + (SLG).

Proof. The only nontrivial aspects are:

$$(SLG(E, F) - SLG(E, F)) \cap L(E, F) = \{0\},\ \ (2)\ \text{and}\ \ (3).$$

Let $T, S \in SLG(E,F)$ such that $T - S = L \in L(E,F)$, $L \neq 0$. Take $x \in E$ with $Lx \neq 0$ and $p \in Norm(F)$. For each $t > 0$ we get

$$tp(Lx) \leq p(T(tx)) + p(S(tx)) \leq \omega_1(tq_1(x)) + \omega_2(tq_2(x)),$$

hence, after dividing by t and letting $t \to \infty$,

$$p(Lx) \leq 0,$$

contradiction.

Proof of 2. Let $T \in SLG(E,F)$ and $L \in L_d(F,G)$ with $E, F, G \in Ob(SAN)$. Take $p \in Norm(G)$ and an arbitrary $p^* \in Norm(F)$. Define $p^{**} = p \circ L + p^* \in Norm(F)$.

Then there is $q \in Norm(E)$ and $\omega \in K$ such that

$$p^{**}(Tx) \leq \omega(q(x)),$$

and consequently $p(LTx) \leq \omega(q(x))$, thus $LT \in SLG(E,G)$.

Proof of 3. Let $T \in SLG(F,G)$, $L \in L_d(E,F)$ and $S \in SLG(E,F)$, with E, F, G as above. Put $H = T(L+S)$ and let $p \in Norm(G)$.

Let $p \in Norm(G)$. There are $\omega_1 \in K$ and $q \in Norm(F)$ such that

$$p(Tx) \leq \omega_1(q(x)) \qquad (x \in F).$$

There also exist $\omega_2 \in K$ and $q^* \in Norm(E)$ such that

$$q(Sx) \leq \omega_2(q^*(x)) \qquad (x \in E).$$

Then let us define $q^{**} \in Norm(E)$,

$$q^{**} = q \circ L + q^*.$$

Take $t_0 > 0$ such that

$$\omega_2(t) \leq t \qquad (t \geq t_0).$$

and define $\omega_3 \in K$ expressed by

$$\omega_3(t) = \omega_1(t + \omega_2(t_0)) \qquad (t \geq 0).$$

We obtain

$$\begin{aligned} p(Hx) &\leq \omega_1(q(Lx + Sx)) \leq \omega_1(q(Lx) + \omega_2(q^*(x))) \leq \\ &\leq \omega_1(q(Lx) + q^*(x) + \omega_2(t_0)) = \\ &= \omega_3(q^{**}(x)), \qquad (x \in E) \end{aligned}$$

thus $H \in SLG(E,G)$.

The **SAMP** and **The Splitting Scheme** will be the two foundational pillars of our construction of super-additive differentiability.

In what follows, $E, F \in Ob(SAN)$. We define

$$D^\infty(E,F) = L_d(E,F) + (SLG(E,F) \cap Iso(E,F)).$$

If $T \in D^\infty(E,F)$, $T = L + U$, with $L \in L_d(E,F)$ and $U \in SLG(E,F)$, then we introduce the "*super-additive derivative*" $DT : E \to F^E$, expressed by

$$(DT)(x) = (p \to Lx + Up) \qquad (x, p \in E).$$

We shall use also the notation

$$(DT)_p(x) = Lx + Up.$$

Let us observe that DT is affine and inflationnary, thus also "*super-additively differentiable*", i.e. there also exist D^2T, D^3T, ...

Here is perhaps the most striking aspect of super-additivity. This concept has only a separating role; it cannot analyze nonlinear structures (due probably to the simple fact that the "tangent space" must be the same at each point, which in turn, is the effect of denying locality).

Einstein stated this problem as follows: "But on one point we should, in my opinion, absolutely hold fast: the real factual situation of system $S1$ is independent of what is done with system $S2$, which is spatially separated from the former" (in Albert Einstein, *Philosopher Scientist*, ed. P.A. Schilpp, Library of Living Philosophers, Evanston, IL, 1949).

Theorem 4.2. *Let $T, S \in D^{\infty}(E, F)$ and $\lambda > 0$. Then we have:*
1. $D(\lambda T) = \lambda DT$.
2. $D(T + S) = DT + DS$.
3. *For every $p \in E$,*

$$D(TS)_p = (DT)_{S_p} \circ (DS)_p .$$

The proof is straightforward.

Example 4.3. *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + \log(2 + |x|)$ sgn x. Then $f \in D^{\infty}(\mathbb{R},\mathbb{R})$ and $Df : \mathbb{R} \to \mathbb{R}^{\mathbb{R}}$,*

$$(Df)_p(x) = 2x + \log(2 + |p|) \ \text{sgn}\ p \qquad (x, p \in \mathbb{R}).$$

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