Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 48(2), 2021, Pages 319–327
ISSN: 1223-6934

New Hermite-Hadamard inequalities in the framework of generalized fractional integrals

Juan Gabriel Galeano Delgado, Juan E. Nápoles Valdés, and Edgardo Pérez Reyes

Abstract. In this work, we obtain new inequalities of the Hermite-Hadamard type, using generalized fractional integrals. The results obtained contain, as particular cases, several of those reported in the literature.

2010 Mathematics Subject Classification. Primary 26A33; Secondary 26D10, 47A63.
Key words and phrases. Hermite-Hadamard inequality; generalized fractional integral.

1. Introduction

Perhaps one of the most productive mathematical ideas lately, due to its variety of uses and interrelationships with different applications, is that of the convex function.

Definition 1.1. A function $f : I \to \mathbb{R}$ is said to be **convex** on interval $I \subset \mathbb{R}$, if the inequality $f(tx + (1-t)y) \leq t(x) + (1-t)f(y)$, for $x, y \in I$ is fulfilled.

We say that f is concave if $-f$ is convex.

The consequent extensions of this concept, which have appeared lately, have transformed it into an extremely complex concept. To reflect on this, we suggest that the user read the work [21], where a fairly complete classification of most of the known definitions is made.

On the other hand, the average value of an integrable function over a compact interval $[a, b]$ is known to all, which is given by the value $\frac{1}{b-a}\int_a^b f(x)dx$, since it turns out that between of the many important inequalities that involve convex functions, there is one in particular that allows us to limit this mean value $f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$, with $a, b \in I$, the inverse inequalities are maintained if the function f is concave in said interval. This seminal result was proved at [11, 12] and is known as the Hermite-Hadamard inequality (see [5], [7] and [16] for details). Since its discovery, this inequality has received considerable attention, some extensions and generalizations of this inequality, with different fractional and generalized operators and using different convexity operators, can be consulted in [3, 4, 8, 9, 15, 19, 20, 21, 22].

Definition 1.2. $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a quasi-convex function if for every $a_1, a_2 \in I$ with $a_1 < a_2$, and $\tau \in [0, 1]$, we have

$$\varphi\big(a_1\tau + a_2(1-\tau)\big) \leq \max\{\varphi(a_1), \varphi(a_2)\}. \tag{1}$$

Received February 3, 2021. Accepted August 27, 2021.

It is known that fractional calculus, that is, calculus with derivatives and integrals of non-integer order, despite being contemporary with Ordinary Calculus, has been gaining attention in the last 40 years and has become one of the most active areas in Mathematics today. In particular, this has led to the emergence of new comprehensive operators which are natural generalizations of the classical Riemann-Liouville fractional integral. In a previous work (see [10]) the authors define a generalized operator containing, as particular cases, several of the known fractional integral operators.

Definition 1.3. The k-generalized fractional Riemann-Liouville integral of order α with $\alpha \in \mathbb{R}$, and $s \neq -1$ of an integrable function $\varphi(u)$ on $[0, \infty)$, are given as follows (right and left, respectively):

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} \frac{F(\tau,s)\varphi(\tau)d\tau}{[\mathbb{F}(u,\tau)]^{1-\frac{\alpha}{k}}}, \tag{2} $$

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}\varphi(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{u}^{a_2} \frac{F(\tau,s)\varphi(\tau)d\tau}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}, \tag{3} $$

with $F(\tau, 0) = 1$, $\mathbb{F}(u,\tau) = \int_\tau^u F(\theta, s)d\theta$ and $\mathbb{F}(\tau, u) = \int_u^\tau F(\theta, s)d\theta$.

With the functions Γ (see [23, 24, 25, 27, 28]) and Γ_k defined by (cf. [5]):

$$ \Gamma(z) = \int_0^\infty \tau^{z-1}e^{-\tau}\, d\tau, \quad \Re(z) > 0, \tag{4} $$

$$ \Gamma_k(z) = \int_0^\infty \tau^{z-1}e^{-\tau^k/k}\, d\tau, k > 0. \tag{5} $$

It is clear that if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z + k) = z\Gamma_k(z)$. As well, we define the k-beta function as follows

$$ B_k(u, v) = \frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1}d\tau, $$

notice that $B_k(u, v) = \frac{1}{k}B(\frac{u}{k}, \frac{v}{k})$ and $B_k(u, v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

The main purpose of this paper, using the generalized fractional integral operator of the Riemann- Liouville type from Definition 1.3, is to establish several integral inequalities of Hermite-Hadamard type for quasi-convex functions, as we shall see, our results boil down to several known theorems and results.

2. Main results

Let $\varphi : I^o \to \mathbb{R}$ be a given function, where $a_1, a_2 \in I^o$ with $0 < a_1 < a_2 < \infty$. We assume that $\varphi \in L_\infty[a_1, a_2]$ such that ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(u)$ and ${}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}\varphi(u)$ are well defined. We define

$$ \tilde{\varphi}(u) := \varphi(a_1 + a_2 - u),\ u \in [a_1, a_2] $$

and

$$ G(u) := \varphi(u) + \tilde{\varphi}(u),\ u \in [a_1, a_2]. $$

Notice that by using the change of variables $w = \frac{\tau - a_1}{u - a_1}$ and $v = \frac{a_2 - \tau}{a_2 - u}$, we have that (2) and (3) becomes in

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(u) = \frac{(u-a_1)}{k\Gamma_k(\alpha)} \int_0^1 \frac{F(wu+a_1(1-w),s)\varphi\big(wu+a_1(1-w)\big)dw}{\big[\mathbb{F}\big(u,wu+a_1(1-w)\big)\big]^{1-\frac{\alpha}{k}}}, \tag{6} $$

where $u > a_1$,

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}\varphi(u) = \frac{(a_2-u)}{k\Gamma_k(\alpha)} \int_0^1 \frac{F(vu+a_2(1-v),s)\varphi\big(vu+a_2(1-v)\big)dv}{\big[\mathbb{F}\big(u,vu+a_2(1-v)\big)\big]^{1-\frac{\alpha}{k}}}, \tag{7} $$

where $u < a_2$.
Applying the identity $\tilde{\varphi}\big((1-w)a_1+a_2w\big) = \varphi(a_1w+(1-w)a_2)$, from (6) we get

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\tilde{\varphi}(a_2) = \frac{(a_2-a_1)}{k\Gamma_k(\alpha)} \int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi(a_1w+(1-w)a_2)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \tag{8} $$

and

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2) = \frac{(a_2-a_1)}{k\Gamma_k(\alpha)} \int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi((1-w)a_1+a_2w)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}. \tag{9} $$

If in (7) we use the identity $\tilde{\varphi}\big((1-v)a_2+a_1v\big) = \varphi(a_2v+(1-v)a_1)$, we deduce

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}\tilde{\varphi}(a_1) = \frac{(a_2-a_1)}{k\Gamma_k(\alpha)} \int_0^1 \frac{F(va_1+(1-v)a_2,s)\varphi(a_1v+(1-v)a_2)dv}{[\mathbb{F}\big(a_1,va_1+a_2(1-v)\big)]^{1-\frac{\alpha}{k}}} \tag{10} $$

and

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}\varphi(a_1) = \frac{(a_2-a_1)}{k\Gamma_k(\alpha)} \int_0^1 \frac{F(va_1+(1-v)a_2,s)\varphi((1-v)a_2+a_1v)dv}{[\mathbb{F}\big(a_1,va_1+a_2(1-v)\big)]^{1-\frac{\alpha}{k}}}. \tag{11} $$

Now we are in a position to propose our first result.

Theorem 2.1. *Let $\varphi : I \to \mathbb{R}$ be a positive function, where $[a_1,a_2] \subset I^o$ with $0 < a_1 < a_2 < \infty$. If we also consider $\varphi \in L_\infty[a_1,a_2]$ and quasi-convex function on $[a_1,a_2]$, then we get*

$$ \frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\big[\, {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) + \, {}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\big] \le \max\{\varphi(a_1),\varphi(a_2)\}. \tag{12} $$

Proof. For $w \in [0,1]$, let $\eta_1 = a_1w+(1-w)a_2$, $\eta_2 = (1-w)a_1+a_2w$ and $m = \max\{\varphi(a_1),\varphi(a_2)\}$. As φ is quasi-convex on $[a_1,a_2]$, we get

$$ \varphi(\eta_1), \varphi(\eta_2) \le m. $$

That is,

$$ \varphi(a_1w+(1-w)a_2) + \varphi((1-w)a_1+a_2w) \le 2m. \tag{13} $$

Now, multiplying both sides of (13) by

$$ \frac{(a_2-a_1)}{k\Gamma_k(\alpha)} \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} $$

and integrating over $(0,1)$ with respect to w, we have

$$\begin{aligned}
&\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi(a_1w+(1-w)a_2)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&+\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi((1-w)a_1+a_2w)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&\le \frac{2m(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}.
\end{aligned}$$

Thus, from the equalities (8), (9) and

$$\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}=\frac{k[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}{\alpha(a_2-a_1)}, \tag{14}$$

we obtain

$${}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}\tilde{\varphi}(a_2)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2)\le \frac{2m[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}, \tag{15}$$

which implies that

$${}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)\le \frac{2m[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}. \tag{16}$$

Analogously, multiplying both sides of (13) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\frac{F(wa_1+a_2(1-w),s)}{[\mathbb{F}\big(wa_1+a_2(1-w),a_1\big)]^{1-\frac{\alpha}{k}}},$$

integrating over $(0,1)$ with respect to w, and from the equalities (10), (11), (14), we also have

$${}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\le \frac{2m[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}. \tag{17}$$

Finally, adding the inequalities (16) and (17), we get

$${}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\le \frac{4m[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}.$$

From the above inequality we deduce the result of the theorem. □

Remark 2.1. If we choose $F\equiv 1$ and $\alpha=k=1$ in the previous result, we obtain Theorem 3.3 of [6]. If we only take $F\equiv 1$ then the previously proved result contains as a particular case Lemma 3 of [13].

The proof of the following result is found in [9] and it will be useful hereafter.

Lemma 2.2. *If φ is a differentiable function on I^o such that $\varphi'\in L[a_1,a_2]$, then we have*

$$\begin{aligned}
&\frac{\varphi(a_1)+\varphi(a_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\big]\\
&=\frac{(a_2-a_1)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\int_0^1 \chi_{\alpha,s}(\tau)\varphi'(\tau a_1+(1-\tau)a_2)d\tau,
\end{aligned} \tag{18}$$

where

$$\chi_{\alpha,s}(\tau) = [\mathbb{F}(\tau a_1 + (1-\tau)a_2, a_1)]^{\frac{\alpha}{k}} - [\mathbb{F}(\tau a_2 + (1-\tau)a_1, a_1)]^{\frac{\alpha}{k}} + [\mathbb{F}(a_2, \tau a_2 + (1-\tau)a_1)]^{\frac{\alpha}{k}} - [\mathbb{F}(a_2, \tau a_1 + (1-\tau)a_2)]^{\frac{\alpha}{k}}.$$

Remark 2.2. If in this Lemma we put $F(\tau, s) = \tau^s$, we have $[\mathbb{F}_+(u,\tau)]^{1-\frac{\alpha}{k}} = \left[\frac{u^{s+1}-\tau^{s+1}}{s+1}\right]^{1-\frac{\alpha}{k}}$, so the above becomes Theorem 2.1 of [1].

Lemma 2.3. *With the hypotheses of the Lemma 2.2, we obtain*

$$\int_0^1 |\chi_{\alpha,s}(\tau)| d\tau = \frac{1}{a_2 - a_1}\left[I_1 + I_2 + I_3 + I_4\right], \tag{19}$$

where

$$I_1 = \int_{\frac{a_1+a_2}{2}}^{a_2} [\mathbb{F}(w, a_1)]^{\frac{\alpha}{k}} dw - \int_{a_1}^{\frac{a_1+a_2}{2}} [\mathbb{F}(w, a_1)]^{\frac{\alpha}{k}} dw,$$

$$I_2 = \int_{\frac{a_1+a_2}{2}}^{a_2} [\mathbb{F}(a_2, a_1 + a_2 - w)]^{\frac{\alpha}{k}} dw - \int_{a_1}^{\frac{a_1+a_2}{2}} [\mathbb{F}(a_2, a_1 + a_2 - w)]^{\frac{\alpha}{k}} dw,$$

$$I_3 = \int_{a_1}^{\frac{a_1+a_2}{2}} [\mathbb{F}(a_2, w)]^{\frac{\alpha}{k}} dw - \int_{\frac{a_1+a_2}{2}}^{a_2} [\mathbb{F}(a_2, w)]^{\frac{\alpha}{k}} dw,$$

$$I_4 = \int_{a_1}^{\frac{a_1+a_2}{2}} [\mathbb{F}(a_2 + a_1 - w, a_1)]^{\frac{\alpha}{k}} dw - \int_{\frac{a_1+a_2}{2}}^{a_2} [\mathbb{F}(a_2 + a_1 - w, a_1)]^{\frac{\alpha}{k}} dw.$$

Proof. If we make $w = a_1\tau + (1-\tau)a_2$, we get

$$\int_0^1 |\chi_{\alpha,s}(\tau)| d\tau = \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} |\rho(w)| dw, \tag{20}$$

where

$$\rho(w) = [\mathbb{F}(w, a_1)]^{\frac{\alpha}{k}} - [\mathbb{F}(a_2 + a_1 - w, a_1)]^{\frac{\alpha}{k}} + [\mathbb{F}(a_2, a_2 + a_1 - w)]^{\frac{\alpha}{k}} - [\mathbb{F}(a_2, w)]^{\frac{\alpha}{k}}.$$

We observe that ρ is non-decreasing function on $[a_1, a_2]$. Thus, we have $\rho(a_1) < 0$ and $\rho\left(\frac{a_1+a_2}{2}\right) = 0$. Therefore , we deduce that

$$\begin{cases} \rho(w) \le 0 & \text{if } a_1 \le w \le \frac{a_1+a_2}{2}, \\ \rho(w) > 0 & \text{if } \frac{a_1+a_2}{2} < w \le a_2. \end{cases}$$

Thus, we have

$$\int_{a_1}^{a_2} |\rho(w)| dw = \int_{\frac{a_1+a_2}{2}}^{a_2} \rho(w) dw - \int_{a_1}^{\frac{a_1+a_2}{2}} \rho(w) dw. \tag{21}$$

From (20) and (21), we obtain (19). □

Remark 2.3. This result contains as a particular case Lemma 2 of [18].

Theorem 2.4. *If φ is a differentiable function on I^o such that $\varphi' \in L[a_1, a_2]$ and $|\varphi'|$ is quasi-convex on $[a_1, a_2]$, then*

$$\left|\frac{\varphi(a_1)+\varphi(a_2)}{2} - \frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left[\,{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) + {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right| \\ \leq \frac{I_1+I_2+I_3+I_4}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\max\{|\varphi'(a_1)|, |\varphi'(a_2)|\}, \tag{22}$$

where I_1, I_2, I_3 and I_4 were defined in the Lemma 2.3.

Proof. As $|\varphi'|$ is quasi-convex on $[a_1, a_2]$, for all $\tau \in [0,1]$, we get

$$|\varphi'(\tau a_1 + (1-\tau)a_2)| \leq \max\{|\varphi'(a_1)|, |\varphi'(a_2)|\}.$$

Thus, by virtue of Lemmas 2.2 and 2.3, we have

$$\left|\frac{\varphi(a_1)+\varphi(a_2)}{2} - \frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left[\,{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) + {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right| \\ \leq \frac{(a_2-a_1)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\int_0^1 |\chi_{\alpha,s}(\tau)||\varphi'(\tau a_1+(1-\tau)a_2)|d\tau \\ \leq \frac{(a_2-a_1)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\max\{|\varphi'(a_1)|, |\varphi'(a_2)|\})|d\tau\right)\int_0^1 |\chi_{\alpha,s}(\tau)|d\tau \\ \leq \frac{I_1+I_2+I_3+I_4}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\max\{|\varphi'(a_1)|, |\varphi'(a_2)|\}. \tag{23}$$

□

Remark 2.4. If we choose $F \equiv 1$ and $\alpha = k = 1$ in the previous result, we obtain Theorem 3.3 of [6]. If we only take $F \equiv 1$ then the previously proved result contains as a particular case Lemma 3 of [13].

Theorem 2.5. *If φ is a differentiable function on I^o such that $\varphi' \in L[a_1, a_2]$ and $|\varphi'|^q$ is quasi-convex on $[a_1, a_2]$, then*

$$\left|\frac{\varphi(a_1)+\varphi(a_2)}{2} - \frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left[\,{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) + {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right| \\ \leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\int_0^1 |\chi_{\alpha,s}(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\max\{|\varphi'(a_1)|^q, |\varphi'(a_2)|^q\}\right)^{\frac{1}{q}}. \tag{24}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. For $\tau \in [0,1]$, since $|\varphi'|^q$ is quasi-convex on $[a_1, a_2]$, we obtain

$$|\varphi'(\tau a_1 + (1-\tau)a_2)|^q \leq \max\{|\varphi'(a_1)|^q, |\varphi'(a_2)|^q\}. \tag{25}$$

The Lemma 2.2, Holder's inequality and the inequality (25), implies that

$$\begin{aligned}
&\left|\frac{\varphi(a_1)+\varphi(a_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right| \\
&\leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\int_0^1 |\chi_{\alpha,s}(\tau)||\varphi'(\tau a_1+(1-\tau)a_2)|d\tau \\
&\leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\int_0^1 |\chi_{\alpha,s}(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |\varphi'(\tau a_1+(1-\tau)a_2)|^q d\tau\right)^{\frac{1}{q}} \\
&\leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\int_0^1 |\chi_{\alpha,s}(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \max\{|\varphi'(a_1)|^q,|\varphi'(a_2)|^q\}d\tau\right)^{\frac{1}{q}} \\
&\leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\int_0^1 |\chi_{\alpha,s}(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\max\{|\varphi'(a_1)|^q,|\varphi'(a_2)|^q\}\right)^{\frac{1}{q}}. \qquad (26)
\end{aligned}$$

□

Remark 2.5. Under appropriate choices of F, and therefore of $\mathbb{F}$, we have Theorem 1 of [14], Theorem 4 of [13] and Theorem 8 of [18].

Theorem 2.6. *If φ is a differentiable function on I^o such that $\varphi' \in L[a_1,a_2]$ and $|\varphi'|^q$ is quasi-convex on $[a_1,a_2]$, then*

$$\begin{aligned}
&\left|\frac{\varphi(a_1)+\varphi(a_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right| \\
&\leq \frac{I_1+I_2+I_3+I_4}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\max\{|\varphi'(a_1)|^q,|\varphi'(a_2)|^q\}\right)^{\frac{1}{q}}. \qquad (27)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and I_1,I_2,I_3 and I_4 were defined in the Lemma 2.3.

Proof. Using the Lemmas 2.2 and 2.3, the Holder's inequality and the inequality (25), we deduce that

$$\begin{aligned}
&\left|\frac{\varphi(a_1)+\varphi(a_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right| \\
&\leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\int_0^1 |\chi_{\alpha,s}(\tau)|^{\frac{1}{p}}|\varphi'(\tau a_1+(1-\tau)a_2)||\chi_{\alpha,s}(\tau)|^{\frac{1}{q}}d\tau \\
&\leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\int_0^1 |\chi_{\alpha,s}(\tau)|d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |\varphi'(\tau a_1+(1-\tau)a_2)|^q|\chi_{\alpha,s}(\tau)|d\tau\right)^{\frac{1}{q}} \\
&\leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\int_0^1 |\chi_{\alpha,s}(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\max\{|\varphi'(a_1)|^q,|\varphi'(a_2)|^q\}\int_0^1 |\chi_{\alpha,s}(\tau)|d\tau\right)^{\frac{1}{q}} \\
&\leq \frac{a_2-a_1}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\int_0^1 |\chi_{\alpha,s}(\tau)|^p d\tau\right)\left(\max\{|\varphi'(a_1)|^q,|\varphi'(a_2)|^q\}\right)^{\frac{1}{q}} \\
&\leq \frac{I_1+I_2+I_3+I_4}{4[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(\max\{|\varphi'(a_1)|^q,|\varphi'(a_2)|^q\}\right)^{\frac{1}{q}}. \qquad (28)
\end{aligned}$$

□

Remark 2.6. As before, it is easy to see that this result contains as particular cases Theorem 2 of [2], Theorem 6 of [13] and Theorem 10 of [18].

3. Conclusions

In this work we obtained several inequalities of the Hermite-Hadamard type for quasi-convex function. The results achieved in this paper generalize [18], from (k,s)-Riemann-Liouville fractional integrals to a generalized fractional integral operator, which contain, for a proper choice of kernel F, several well-known integral operators reported in the literature.

By other hand, these generalized results of the Hermite-Hadamard inequality for the quasi-convex function through generalized fractional integrals, are a tool in obtaining various results of integral inequalities and some applications are also presented in the fields of approximation theory, optimization theory and analysis, among others, are a tool in obtaining various results of integral inequalities and some applications are also presented in the fields of approximation theory, optimization theory and analysis, among others.

References

[1] P. Agarwal, M. Jleli, and M. Tomar, Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals, *Journal of Inequalities and Applications* **55** (2017).

[2] M. Alomari, M. Darus, and S.S. Dragomir, New inequalities of Hermite–Hadamard type for functions whose second derivatives absolute values are quas-iconvex, *RGMIA Res. Rep. Coll.* **12** (2009), Supplement, Article 14.

[3] M.A. Ali, J.E. Nápoles, A. Kashuri, and Z. Zhang, Fractional non conformable Hermite-Hadamard inequalities for generalized ϕ-convex functions, *Fasciculi Mathematici* **64** (2020), 5–16.

[4] S. Bermudo, P. Kórus, and J.E. Nápoles, On q-Hermite-Hadamard inequalities for general convex functions, *Acta Math. Hungar.* **162** (2020), 364–374.

[5] R. Díaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulg. Mat.* **15** (2007), no. 2, 179–192.

[6] S.S. Dragomir and C.E.M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377–385.

[7] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[8] J.D. Galeano, J.E. Nápoles, and E. Pérez, Several integral inequalities for generalized Riemann-Liouville fractional operators, *Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.* **70** (2021), no. 1, 269–278.

[9] J.D. Galeano, J.E. Nápoles, and E. Pérez, Concerning to the generalized Hermite-Hadamard integral inequality, submitted.

[10] J.D. Galeano, J.E. Nápoles, and E. Pérez, On a general formulation of the fractional operator Riemann-Liouville and related inequalities, submitted.

[11] J. Hadamard, Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann, *J. Math. Pures App.* **9** (1893), 171–216.

[12] Ch. Hermite, Sur deux limites d'une intégrale définie, *Mathesis* **3** (1883), 82.

[13] R. Hussain, A. Ali, A. Latif, and G. Gulshan, Some k-fractional associats of Hermite-Hadamard's inequality for quasi-convex functions and applications to special means, *Fractional Differential Calculus*, **7** (2017), no. 2, 301–309.

[14] D.A. Ion, Some estimates on the Hermite–Hadamard inequality through quasi–convex functions, *Annals of the University of Craiova - Mathematics and Computer Science Series* **34** (2007), 82–87.

[15] P. Kórus, L.M. Lugo, and J.E. Nápoles Valdés, Integral inequalities in a generalized context, *Studia Scientiarum Mathematicarum Hungarica* **57** (2020), no. 3, 312–320.
[16] D.S. Mitrinovi'c and I.B. Lackovi'c, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.
[17] P.O. Mohammed, Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function, *Math Meth Appl Sci.* **2019** (2019), 1–11.
[18] E.R. Nwaeze, Inequalities of the Hermite-Hadamard type for quasi-convex functions via the (k,s)-Riemann-Liouville fractional integrals, *Fractional Differential Calculus*, **8** (2018), no. 2, 327–336.
[19] J.E. Nápoles, Hermite-Hadamard inequality in generalized context, *VI Colloquium on Applied Mathematics and II International Meeting of Applied Mathematics*, UNIMILITAR, BOGOTÁ, COLOMBIA, NOVEMBER 11-13, 2020.
[20] J.E. Nápoles, New generalized fractional integral inequalities of Hermite-Hadamard type for harmonically convex functions, *XVI International Meeting of Mathematics*, Barranquilla, Colombia NOVEMBER 17-20, 2020.
[21] J.E. Nápoles Valdes, F. Rabossi, and A.D. Samaniego, Convex functions: Ariadne's thread or Charlotte's spiderweb? , *Advanced Mathematical Models & Applications* **5** (2020), no. 2, 176–191.
[22] J.E. Nápoles Valdes, J.M. Rodríguez, and J.M. Sigarreta, New Hermite-Hadamard Type Inequalities Involving Non-Conformable Integral Operators, *Symmetry* **2019** (2019), **11**, 1108.
[23] F. Qi, and B.N. Guo, Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.* **111** (2017), no. 2, 425—434.
[24] F. Qi, S. Habib, S. Mubeen, and M.N. Naeem, Generalized k-fractional conformable integrals and related inequalities, *AIMS Mathematics* **4**(2019), no. 3, 343–358.
[25] E.D. Rainville, *Special Functions*, Macmillan Co., New York, 1960.
[26] M.Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.* **57** (2013), no. 9, 2403–2407.
[27] Z.H. Yang and J.F. Tian, Monotonicity and inequalities for the gamma function, *J. Inequal. Appl.* **2017** (2017), Article number 317.
[28] Z.H. Yang and J. F. Tian, Monotonicity and sharp inequalities related to gamma function, *J. Math. Inequal.* **12** (2018), no. 1, 1–22.

(J. G. Galeano D.) Departamento de Matemáticas y Estadística, Universidad de Córdoba, Montería, Colombia
E-mail address: jgabrielgaleano@correo.unicordoba.edu.co

(J. E. Nápoles V.) UNNE, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina
jnapoles@exa.unne.edu.ar
and
UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina
E-mail address: jnapoles@frre.utn.edu.ar

(E. Pérez R.) Facultad de Ciencias e Ingeniería, Universidad del Sinú Elías Bechara Zainúm, Montería, Colombia
E-mail address: edgardomath@gmail.com