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Hammerstein equations in nonreflexive Banach spaces

Adrian Duma and Cristian Vladimirescu

Abstract. In this paper Hammerstein operators of type $T = I + AN$ are considered. One studies the continuity of the operator T^{-1} and one proves an existence result of the solution of the Hammerstein equation $Tu = w$ related to Banach lattices.

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1. Introduction

Let X be a real Banach space and X^* be its topological dual space. A nonlinear equation of Hammerstein type (in abstract form) is a functional equation of type

$$u + ANu = w, \tag{1}$$

where $w \in X$ is given, A is a linear application from X^* into X and N is in general a nonlinear application from X into X^*. The equation (1) has been constituted the object of an intensive study since it has been initially considered by A. Hammerstein in his paper from Acta Matematica, in 1930.

In reflexive Banach spaces there exists a surjectivity result for the operator $I + AN$ and a continuity result for the operator $(I + AN)^{-1}$, which have been proved by C.P. Gupta, in 1972. By completeness reason we shall mention it in what follows.

Theorem 1.1. *(see [3]) Let X be a reflexive Banach space, $A : D(A) \subset X^* \to X$ be a maximal monotone linear operator and $N : X \to X^*$ be a bounded operator of type (M), such that for every $v \in X$ the following condition is fulfilled:*

$$\liminf_{\|u\| \to \infty} (N(u + v), u) > 0.$$

Then, the mapping $I + AN$ is surjective.

Theorem 1.2. *(see [3]) Let X be a reflexive Banach space, $A : D(A) \subset X^* \to X$ be a maximal monotone linear operator and $N : X \to X^*$ be a bounded, hemicontinuous and strongly monotone operator.*

Then, there the operator $(I + AN)^{-1}$ does exist, is bounded and continuous.

In what follows let $A \in L(X^*, X)$ be arbitrary and $N : X \to X^*$ be an arbitrary continuous operator. We call the operator $T = I + AN$ to be Hammerstein operator. In Section 2 we shall prove that in every predual space of l_1 there exists an one-to-one and continuous Hammerstein operator which is not a homeomorphism. By using the properties of weakly sequentially complete Banach lattices, we prove in Section

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3 a surjectivity result for the operator T. In Section 4 we shall treat a generalized Hammerstein equation of type

$$u + A(u)\,Nu = w.$$

2. An existence result of a Hammerstein operator

Recall the following results, proved in [6].

Theorem 2.1. *(Bartle, Graves) Let X, Y be two Banach spaces and let $A \in L(X,Y)$ be a surjective operator. Then there exists a continuous operator $S : Y \to X$ such that $AS = I$ and the operator $H : X \to (Ker\ A) \times Y$ expressed by*

$$H(x) = (x - SAx, Ax)\,,\ \text{for every } x \in X$$

is a homeomorphism.

Theorem 2.2. *(Curtis) Let X be an infinite dimensional normed space. Then there exists a continuous and bijective operator $F : X \to X$ which is not a homeomorphism.*

Let X be a predual of l_1 (in particularly X is separable). We can state and prove the following result.

Theorem 2.3. *There exists a Hammerstein operator $F = I + AN : X \to X$ which is bijective and continuous and it is not homeomorphism.*

Proof. Let $F : X \to X$ given by Theorem 2.2. Consider a surjective operator $A \in L(X^*, X)$. Theorem 2.1 gives us a continuous operator $S : X \to X^*$, such that $AS = I$. Define therefore the continuous operator $N : X \to X^*$ through

$$N = S(F - I)\,.$$

Then, $I + AN = F$ and the theorem is proved. □

Remark 2.1. *Obviously, Theorem 2.3 holds in the case $X = c_0$. But there exists in addition predual spaces of l_1 which are not isomorphic with $C(K)-$ like spaces. Such an example has been given by Y. Benyamini and J. Lindenstrauss in 1973.*

3. A surjectivity result

In this Section denote by X a Banach lattice with weakly sequentially complete dual space and let $A \in L(X^*, X)$ be an arbitrary operator and $N : X \to X^*$ be a continuous operator.

One has the following existence result.

Theorem 3.1. *Suppose in addition that A is negative, i.e. $f \geq 0$ implies $Af \leq 0$ and N is isotone and odd. If there exists an operator $H : X^* \to X^*$, such that*

$$Hf \leq -(NAHf + |f|)_+\,, \tag{2}$$

for every $f \in X^*$, then the Hammerstein equation $u + ANu = w$ has at least one solution, for every $w \in R(A)$.

Remark 3.1. *If S is an isotone operator, then S keeps the interval $[a,b]$ if and only if $Sa \geq a$ and $Sb \leq b$.*

Proof. Let $w \in R(A)$. Then there exists $f \in X^*$ such that $Af = w$. Define the operator $S : X^* \to X^*$ by $S = f - NA$. Obviously, S is isotone and continuous. Assert that S keeps the interval $[Hf, -Hf]$. Since in X^* the monotone and bounded sequences are convergent, it will result the existence of a fixed point $g \in [Hf, -Hf]$ for S. Hence one gets

$$f - NAg = g. \tag{3}$$

By applying the operator A to the relation (3), one has

$$w - ANu = u,$$

with $u = Ag$. To complete the proof of the theorem it is sufficient, by using in addition the fact that N is odd, to prove the inequalities

$$f - NAHf \geq Hf \tag{4}$$

$$f + NAHf \leq -Hf. \tag{5}$$

By (2), it follows that

$$NAHf + |f| \leq (NAHf + |f|)_+ \leq -Hf.$$

Therefore the relations (4) and (5) are deduced immediately. The theorem is completely proved. □

4. On a generalized Hammerstein equation

Let X be a Banach space, X^* be its topological dual and the following operators:

1) $A : X \to LK(X^*, X)$, a compact operator for which

$$\|A(u)\| \leq M, \ \ (\forall) \ \ u \in X$$

and such that

$$(A(u)x, x) \geq 0, \ \ (\forall) \ \ x \in X^*$$

(here LK denotes the space of linear compact operators);

2) $N : X \to X^*$ a strong monotone and $L-$Lipschitz operator such that $ML < 1$.

Consider the equation

$$u + A(u)N(u) = w, \ \ w \in X. \tag{6}$$

We prove that, under the hypotheses 1) and 2), the equation (6) admits at least one solution $u_0 \in X$ with

$$\|u_0\| \leq \frac{\|w\| + M\|N(0)\|}{1 - ML} := C_w.$$

For every $u \in X$ and $t \in [0, 1]$ fixed consider the equation

$$v + tA(u)N(v) = w. \tag{7}$$

Step 1. The equation (7) **with** $t = 1$ **admits solutions**.

Firstly we prove the following á priori estimation: for every solution v_0 of the equation (7) one has

$$\|v_0\| \leq C_w.$$

Indeed, we have successively

$$v_0 = w - tA(u)N(v_0);$$

$$\begin{aligned} \|v_0\| &\leq \|w\| + t\|A(u)\| \cdot \|N(v_0)\| \leq \\ &\leq \|w\| + M(\|N(0)\| + \|N(v_0) - N(0)\|) \leq \\ &\leq \|w\| + ML\|v_0\| + M\|N(0)\| \end{aligned}$$

and therefore

$$\|v_0\| \leq C_w.$$

Since $C_w > \|w\|$, we have

$$\deg_{LS}(I, B(0, C_w), w) = 1$$

and, through the homotopic invariance of the Lerray-Schauder degree, it results

$$\deg_{LS}(I + tA(u)N, B(0, C_w), w) = 1 \neq 0,$$

for all $t \in [0,1]$. Therefore, the equation (7), with $t = 1$ admits solutions.

Step 2. The solution of the equation (7) with $t = 1$ is unique.

Indeed, let v_0' and v_0'' be two solutions of the equation (7). Hence

$$v_0' + A(u)N(v_0') = w, \tag{8}$$

$$v_0'' + A(u)N(v_0'') = w. \tag{9}$$

By relations (8) and (9), it follows that

$$v_0' - v_0'' + A(u)[N(v_0') - N(v_0'')] = 0$$

and

$$(v_0' - v_0'', N(v_0') - N(v_0'')) + (A(u)[N(v_0') - N(v_0'')], N(v_0') - N(v_0'')) = 0.$$

Since

$$(A(u)[N(v_0') - N(v_0'')], N(v_0') - N(v_0'')) \geq 0$$

it results

$$c\|v_0' - v_0''\|^2 \leq 0$$

and so

$$v_0' = v_0''.$$

Denote by $P(u)$ the unique solution of the equation (7) with $t = 1$. Obviously, one has

$$\|P(u)\| \leq C_w, (\forall)\ u \in X.$$

Step 3. The operator P is compact.

Indeed, one has

$$\begin{aligned} & \|P(u) - P(v)\| \\ = \ & \|A(u)NP(u) - A(u)NP(v) + A(u)NP(v) - A(v)NP(v)\| \leq \\ \leq \ & ML\|P(u) - P(v)\| + \|NP(v)\|\|A(u) - A(v)\|. \end{aligned}$$

It results

$$\begin{aligned} \|P(u) - P(v)\| & \leq \frac{1}{1 - ML}(L\|P(v)\| + \|N(0)\|)\|A(u) - A(v)\| \leq \\ & \leq \frac{L \cdot C_w + \|N(0)\|}{1 - ML}\|A(u) - A(v)\| := \\ & : = q\|A(u) - A(v)\|. \end{aligned}$$

Using the fact that A is compact, and from Schauder's theorem, it follows that P admits at least one fixed point, say u_0, with $\|u_0\| \leq C_w$. Therefore,

$$P(u_0) = u_0 \tag{10}$$

and

$$Pu_0 + A(u_0)NP(u_0) = w. \tag{11}$$

So, from (10) and (11), one has

$$u_0 + A(u_0)N(u_0) = w.$$

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(Adrian Duma) Centre for Nonlinear Analysis and its Applications,
University of Craiova,
Al. I. Cuza street, 13, Craiova RO-200585, Romania, Tel/Fax: 40-251412673
E-mail address: ady@royal.net

(Cristian Vladimirescu) Department of Mathematics, University of Craiova,
Al. I. Cuza street, 13, Craiova RO-200585, Romania, Tel/Fax: 40-251412673
E-mail address: cvladi@central.ucv.ro, vladimirescucris@hotmail.com