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Comments on the paper "A. Zada, B. Dayyan, Stability analysis for a class of implicit fractional differential equations with instantaneous impulses and Riemann–Liouville boundary conditions, *Ann. Univ. Craiova, Math. Comput. Sci. Ser.*, 47 (2020), 88–110"

Snezhana Hristova and Akbar Zada

Abstract. Caputo fractional differential equations with impulses are a very useful apparatus for adequate modeling of the dynamics of many real world problems. It requires developments of good and consistent theoretical proofs and the results for various problems. In this note we point out and correct the statement of the boundary value problem with Riemann–Liouville fractional integral for impulsive Caputo fractional differential equation studied in the paper " A. Zada, B. Dayyan, Stability analysis for a class of implicit fractional differential equations with instantaneous impulses and Riemann–Liouville boundary conditions, *Ann. Univ. Craiova, Math. Comput. Sci. Ser.*, **47** (2020), 88–110."

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1. Introduction

Recently, fractional differential equations with various types of fractional derivatives are intensively applied for theoretical study as well as more adequate modeling of dynamical processes. Contrary to the classical derivative, fractional derivative is nonlocal and it depends significantly on its lower limit. As it is mentioned in the remarkable paper [2], this leads to some obstacles for studying impulsive fractional differential equations (IFDE).

Recently, the paper[1] studied the implicit fractional differential equations with instantaneous impulses and Riemann–Liouville boundary conditions of the form

$$\begin{aligned} &{}^{C}D^{\beta}_{0,t}u(t) = y(t, u(t),\ {}^{C}D^{\beta}u(t)) \quad \text{for } t \neq t_m \in I,\\ &\Delta u(t_m) = J_m(u(t_m)), \quad m = 1, 2, \dots, q-1\\ &\eta_1 u(0) + \xi_1 I^{\beta}u(t)|_{t=0} = \nu_1,\\ &\eta_2 u(T) + \xi_2 I^{\beta}u(t)|_{t=T} = \nu_2, \end{aligned} \tag{1}$$

where $\eta_i, \nu_i, \xi_i,\ i = 1, 2$ are given constants, $I = [0, T]$, $t_0 = 0 < t_1 < t_2 < \cdots < t_q < T$ are given points, $I_m = (t_{m-1}, t_m],\ m = 1, 2, \dots, q,\ I_{q+1} = [t_q, T]$, $\Delta u(t_m) =$

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$u(t_m+0)-u(t_m-0)$, ${}^C D^{\beta}_{0,t}u(t)$ is a Caputo fractional derivative of order $\beta\in(0,1)$, defined by (see, for example [3])

$$ {}^C D^{\beta}_{0,t}u(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-gb}u'(s)ds \tag{2}$$

and $I^{\beta}_{0,t}u(t)$ is the Riemann–Liouville fractional integral of order $\beta>0$ defined by (see, for example, [3])

$$ I^{\beta}_{0,t}u(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds, t>0. \tag{3}$$

Initially we will comment the statement of the problem (1) to make and write more precisely and then we will discuss and correct the obtained in [1] results.

First, since in the paper [1] it is written "*implicit fractional differential equation*" it could be guessed that in the first equation of (1) the derivative ${}^C D^{\beta}u(t)$ means ${}^C D^{\beta}_{0,t}u(t)$, i.e. the lower limit of the Caputo fractional derivative is 0.

Additionally, it could be guessed that the Riemann–Liouville fractional integral $I^{\beta}u(t)|_{t=0}$, written in the third equation of (1) means the Riemann–Liouville fractional integral with a lower limit 0, i.e. $I^{\beta}_{0,t}u(t)$ and $I^{\beta}u(t)|_{t=0}=\lim_{t\to 0+}I^{\beta}_{0,t}u(t)=I^{\beta}_{0,t}u(t)|_{t=0+}$.

Similarly, if the lower limit of the Riemann–Liouville fractional integral $I^{\beta}u(t)|_{t=T}$ is 0, then $I^{\beta}u(t)|_{t=T}=\lim_{t\to T-0}I^{\beta}_{0,t}u(t)=I^{\beta}_{0,t}u(t)|_{t=T-0}$ because the function u is considered only on the interval $[0,T]$.

2. Some auxiliary remarks

The main results in [1] are based on the following auxiliary result:

Lemma 1. [3] The general solution of the Caputo fractional differential equation ${}^C D^{\beta}_{0,t}\varsigma(t)=0$ is $\varsigma(t)=a_0,\ t>0$ and the general solution of the Caputo fractional differential equation ${}^C D^{\beta}_{0,t}\varsigma(t)=g(t)$ is $\varsigma(t)=c_0+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g(s)ds,\ t>0$ where a_0,c_0 are arbitrary constants.

If the lower limit of Caputo fractional derivative is an arbitrary point a then Lemma 1 is changed to the following well known in the literature result:

Lemma 2. [3] The general solution of the Caputo fractional differential equation ${}^C D^{\beta}_{a,t}\varsigma(t)=g(t)$ is $\varsigma(t)=c_0+\frac{1}{\Gamma(\beta)}\int_a^t(t-s)^{\beta-1}g(s)ds,\ t>a$, where c_0 is an arbitrary constant.

Let us discuss the proof of Lemma 3.1 [1] which is done inductively with respect to the interval. On the interval $[0,t_1]$ everything is fine and formula (3.4)[1] is correct. But, for $t\in(t_1,t_2]$ since the equation is ${}^C D^{\beta}_{0,t}u(t)=y(t)$, i.e. the lower limit of Caputo fractional derivative is zero, then Lemma 2 has to be applied for $a=0$ and formula (3.5)[1] is not correct. The lower limit of the integral has to be 0. Instead of (3.5)[1] it has to be written $u(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)ds$. Then $u(t_1+0)\neq d_1$ and the rest of the proof of Lemma 3.1[1] is mistaken and the relation (3.4)[1] is not true.

Additionally, in the proof of Lemma 3.1[1] it is used incorrectly Lemma 2.5[1] to be written that $u(t)=I^{\beta}_{0,t}y(t)+d+ct$ (see Eq. (3.8)[1] and the line above it). Actually, if Lemma 2.5[1] is applied then it has to be written $u(t) = I^{\beta}_{0,t}y(t) + d$. This does not allow the authors to use the boundary condition which is presented by the last line of (1).

All results in the paper [1] are based on the mistaken relation (3.4)[1].

3. Correction of the statement of the problem (1).

To discuss the boundary value conditions in (1), we will use the following result:

Proposition 1. ([3]) Let $q \in (0, 1)$ and $b > 0$, $m : [0, b] \to \mathbb{R}$ be a Lebesgue measurable function.

(a) If there exists a.e. a limit $\lim_{t\to 0+}[t^{1-q}m(t)] = c \in \mathbb{R}$, then there also exists a limit

$$I^{1-q}_{0,t}m(t)|_{t=0} := \lim_{t\to 0+} \frac{1}{\Gamma(1-q)} \int_0^t \frac{m(s)}{(t-s)^q} ds = c\Gamma(q) = \Gamma(q) \lim_{t\to 0+}[t^{1-q}m(t)].$$

(b) If there exists a.e. a limit $\lim_{t\to 0+} I^{1-q}_{0,t}m(t) = c \in \mathbb{R}$, and if there exists the limit $\lim_{t\to 0+}[t^{1-q}m(t)]$, then

$$\lim_{t\to 0+}[t^{1-q}m(t)] = \frac{c}{\Gamma(q)} = \frac{1}{\Gamma(q)} \lim_{t\to 0+} {}_0I^{1-q}_t m(t).$$

According to Proposition 1 with $q = 1-\beta \in (0, 1)$ we get $I^{\beta}u(t)|_{t=0} = \lim_{t\to 0+} I^{\beta}_{0,t}u(t) = I^{\beta}_{0,t}u(t)|_{t=0+} = \Gamma(1-\beta)\lim_{t\to 0+}[t^{\beta}u(t)]$. Note that in Caputo fractional differential equations, similarly to ordinary differential equations, the initial value $u(0) < \infty$. Then $I^{\beta}u(t)|_{t=0} = 0$ and the term $I^{\beta}u(t)|_{t=0}$ is useless in the boundary conditions of (1).

According to the above remarks, the problem (1) will be written precisely in the form

$$\begin{aligned} &{}^CD^{\beta}_{t_m,t}u(t) = y(t, u(t), {}^CD^{\beta}_{t_m,t}u(t)) \ \text{ for } t \in I_m,\ m = 0, 1, \dots, q+1, \\ &\Delta u(t_m) = J_m(u(t_m)), \quad m = 1, 2, \dots, q, \end{aligned} \tag{4}$$

with Riemann-Liouvlle integral boundary condition

$$\eta u(0) + \xi I^{\beta}_{0,t}u(t)|_{t=T-0} = \nu \tag{5}$$

where η, ξ, ν are real constants.

4. Correction of the integral presentation of the solution

Consider the following impulsive Caputo fractional differential equation

$$\begin{aligned} &{}^CD^{\beta}_{t_m,t}u(t) = y(t) \ \text{ for } t \in I_m,\ m = 0, 1, 2, \dots, q+1, \\ &\Delta u(t_m) = J_m(u(t_m)), \quad m = 1, 2, \dots, q, \end{aligned} \tag{6}$$

with a boundary condition (5).

Note problem (6), (5) is totally different then the studied (3.1) [1] because the Caputo fractional derivative changes its lower limit at each impulsive time t_i. It changes totally the behavior of the solution.

Lemma 3. Let $y : I \to \mathbb{R}$ be a continuous function on each interval I_m, $m = 0, 1, 2, \ldots, q$. Then the solution of (6), (5) satisfies

$$u(t) = \begin{cases} c_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds & t \in (0, t_1] \\ c_0 + \sum_{k=1}^{m} J_k(u(t_k)) + \frac{1}{\Gamma(\beta)} \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\beta-1} y(s) ds & \\ \quad + \frac{1}{\Gamma(\beta)} \int_{t_m}^t (t-s)^{\beta-1} y(s) ds & t \in (t_m, t_{m+1}] \\ c_0 + \sum_{m=1}^{q} J_m(u(t_m)) + \frac{1}{\Gamma(\beta)} \sum_{m=0}^{q-1} \int_{t_m}^{t_{m+1}} (t_{m+1}-s)^{\beta-1} y(s) ds & \\ \quad + \frac{1}{\Gamma(\beta)} \int_{t_q}^t (t-s)^{\beta-1} y(s) ds & t \in (t_p, T]. \end{cases}$$

where

$$\begin{aligned} c_0 = \frac{1}{\Gamma(\beta)\eta + \frac{\xi T^\beta}{\beta}} \Big\{ & \Gamma(\beta)\nu - \xi \sum_{m=1}^{q} \frac{(T - t_m)^\beta}{\beta} J_m(u(t_m)) \\ & - \xi \sum_{m=0}^{q-1} \frac{(T - t_{m+1})^\beta}{\Gamma(1+\beta)} \int_{t_m}^{t_{m+1}} (t_{m+1} - \sigma)^{\beta-1} y(\sigma) d\sigma \\ & - \xi \sum_{m=0}^{q-1} \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_{m+1}} (T-s)^{\beta-1} \int_{t_m}^{s} (s-\sigma)^{\beta-1} y(\sigma) d\sigma ds \\ & - \xi \frac{1}{\Gamma(\beta)} \int_{t_q}^{T} \int_{t_q}^{s} (T-s)^{\beta-1} (s-\sigma)^{\beta-1} y(\sigma) d\sigma ds \Big\}. \end{aligned} \tag{7}$$

P r o o f: We will use induction w.r.t. to the interval to prove Lemma 3.
For $t \in (0, t_1]$ we get

$$u(t) = c_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds. \tag{8}$$

For $t \in (t_1, t_2]$ by Lemma 2 with $a = t_1$we get

$$\begin{aligned} u(t) &= u(t_1+) + \frac{1}{\Gamma(\beta)} \int_{t_1}^t (t-s)^{\beta-1} y(s) ds \\ &= J_1(u(t_1)) + c_0 + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} y(s) ds + \frac{1}{\Gamma(\beta)} \int_{t_1}^t (t-s)^{\beta-1} y(s) ds. \end{aligned} \tag{9}$$

Similarly, for $t \in I_q$ by Lemma 2 with $a = t_q$ we get

$$\begin{aligned} u(t) &= u(t_q+) + \int_{t_q}^t (t-s)^{\beta-1} y(s) ds \\ &= c_0 + \sum_{m=1}^{q} J_m(u(t_m)) + \frac{1}{\Gamma(\beta)} \sum_{m=0}^{q-1} \int_{t_m}^{t_{m+1}} (t_{m+1} - s)^{\beta-1} y(s) ds \\ &\quad + \frac{1}{\Gamma(\beta)} \int_{t_q}^t (t-s)^{\beta-1} y(s) ds. \end{aligned} \tag{10}$$

Therefore,

$$
\begin{aligned}
\Gamma(\beta) I_{0,t}^{\beta} u(t)|_{t=T-0} &= \lim_{t\to T-0} \int_0^t (t-s)^{\beta-1} u(s)ds = \int_0^T (T-s)^{\beta-1} u(s)ds \\
&= \sum_{m=0}^{q-1} \int_{t_m}^{t_{m+1}} (T-s)^{\beta-1} u(s)ds + \int_{t_q}^{T} (T-s)^{\beta-1} u(s)ds \\
&= \sum_{m=0}^{q-1} \Big\{ \frac{(T-t_m)^{\beta} - (T-t_{m+1})^{\beta}}{\beta} \Big[c_0 + \sum_{i=1}^{m} J_i(u(t_i)) \\
&\quad + \frac{1}{\Gamma(\beta)} \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (t_{i+1}-\sigma)^{\beta-1} y(\sigma) d\sigma \Big] \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_{m+1}} (T-s)^{\beta-1} \int_{t_m}^{s} (s-\sigma)^{\beta-1} y(\sigma) d\sigma ds \Big\} \\
&\quad + \frac{(T-t_q)^{\beta}}{\beta} \Big[c_0 + \sum_{m=1}^{q} J_m(u(t_m)) + \frac{1}{\Gamma(\beta)} \sum_{m=0}^{q-1} \int_{t_m}^{t_{m+1}} (t_{m+1}-s)^{\beta-1} y(\sigma) d\sigma \Big] \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_q}^{T} \int_{t_q}^{s} (T-s)^{\beta-1} (s-\sigma)^{\beta-1} y(\sigma) d\sigma ds \\
&= c_0 \frac{T^{\beta}}{\beta} + \sum_{m=1}^{q} \frac{(T-t_m)^{\beta}}{\beta} J_m(u(t_m)) \\
&\quad + \sum_{m=0}^{q-1} \frac{(T-t_{m+1})^{\beta}}{\Gamma(1+\beta)} \int_{t_m}^{t_{m+1}} (t_{m+1}-\sigma)^{\beta-1} y(\sigma) d\sigma \\
&\quad + \sum_{m=0}^{q-1} \frac{1}{\Gamma(\beta)} \int_{t_m}^{t_{m+1}} (T-s)^{\beta-1} \int_{t_m}^{s} (s-\sigma)^{\beta-1} y(\sigma) d\sigma ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{t_q}^{T} \int_{t_q}^{s} (T-s)^{\beta-1} (s-\sigma)^{\beta-1} y(\sigma) d\sigma ds.
\end{aligned}
\tag{11}
$$

From Eq. (11) and the boundary condition (5) we get Eq. (7). □

5. Conclusions

The formula in Lemma 3 for the solution of (6), (5) could be applied to prove the existence, uniqueness as well as stability properties of the boundary value problem (4), (5) following the ideas of the proofs in [1].

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(Snezhana Hristova) Faculty of Mathematics and Informatics, University of Plovdiv
E-mail address: snehri@gmail.com

(Akbar Zada) Department of Mathematics, University of Peshawar, Pakistan
E-mail address: zadababo@yahoo.com