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A frictionless contact problem with adhesion between two elastic bodies

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Abstract. We consider a mathematical model which describes the bilateral, frictionless, adhesive contact between two elastic bodies. The adhesion process on the common contact surface is modelled by a surface variable, the bonding field, the tangential shear due to the bonding field being included. We obtain an existence and uniqueness result by construction of an appropriate mapping which is shown to be a contraction on a Hilbert space.

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1. Introduction

In this paper we study a mathematical model which describes the adhesive contact between two elastic bodies, when the frictional tangential traction is negligible in comparison with the traction due to adhesion. As in [2, 3] we use the bonding field as an additional variable, defined on the common part of the boundary. We derive a variational formulation of the model then we prove its unique solvability, which provides the existence of a unique weak solution to the adhesive contact problem.

This work is a companion of the result in [8] where the frictionless contact between an elastic body and a rigid foundation was investigated when the contact is described by normal compliance with adhesion. The novelty here consist in the fact that we study the contact between two deformable bodies and we assume that on the common part of the boundary there is no separation between the bodies during the process, that is the contact is bilateral.

The paper is structured as follows. In Section 2 we present some notations and preliminary material. In Section 3 we describe the model and present its variational formulation. Finally, in Section 4 we state and prove our main existence and uniqueness result, Theorem 4.1. The proof is based on results of differential equations in Banach spaces and a fixed point argument.

2. Notations and preliminaries

We denote by $\mathbf{S}_3$ the space of second order symmetric tensors on $\mathbb{R}^3$; " $\cdot$ " and $|\cdot|$ represent the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbf{S}_3$, respectively. Thus, for every $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$, $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$, $|\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$, and for every $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbf{S}_3$, $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $|\boldsymbol{\tau}| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$. Here and below, the indices i and j run between 1 and 3 and the summation convention over repeated indices is adopted.

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Let Ω^1 and Ω^2 be two bounded domains in $\mathbb{R}^3$. Everywhere in this paper, we use a superscript k to indicate that a quantity is related to the domain Ω^k, $k = 1, 2$. For each domain Ω^k, we assume that its boundary Γ^k is Lipschitz continuous and is partitioned into three disjoint measurable parts Γ_1^k, Γ_2^k and Γ_3^k, with $meas\,\Gamma_1^k > 0$. The unit outward normal to Γ^k is denoted by $\boldsymbol{\nu}^k = (\nu_i^k)$.

We also use the notation

$$
\begin{aligned}
H^k &= \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega^k)\},\\
H_1^k &= \{\boldsymbol{u} = (u_i) \mid u_i \in H^1(\Omega^k)\},\\
\mathcal{H}^k &= \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega^k)\},\\
V^k &= \{\, \boldsymbol{v} \in H_1^k \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \Gamma_1^k\}.
\end{aligned}
$$

The spaces H^k, H_1^k and $\mathcal{H}^k$ are real Hilbert spaces with the canonical inner product denoted $(\cdot,\,\cdot)_{H^k}$, $(\cdot,\,\cdot)_{H_1^k}$ and $(\cdot,\,\cdot)_{\mathcal{H}}$, respectively. On the space V^k we use the inner product

$$
(\boldsymbol{u}^k,\, \boldsymbol{v}^k)_{V^k} = (\boldsymbol{\varepsilon}(\boldsymbol{u}^k), \boldsymbol{\varepsilon}(\boldsymbol{v}^k))_{\mathcal{H}^k} \quad \forall \boldsymbol{u}^k,\, \boldsymbol{v}^k \in V^k
$$

and the associated norm $\|\cdot\|_{V^k}$. Here, $\boldsymbol{\varepsilon}$ denotes the *deformation* operator for both function in V^1 and V^2, that is $\varepsilon(\boldsymbol{u})$ is the symmetric part of the gradient of $\boldsymbol{u}$: $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$. Since $meas\,\Gamma_1^k > 0$, it follows from Korn's inequality (see e.g. [5]) that $(V^k,\, (\cdot,\cdot)_{V^k})$ is a real Hilbert space.

Since the boundary Γ^k is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}^k$ on the boundary Γ^k is defined a.e. For every vector field $\boldsymbol{v}^k \in H_1^k$ we use the notation $\boldsymbol{v}^k$ for the trace of $\boldsymbol{v}^k$ on Γ^k and we denote by v_ν^k and $\boldsymbol{v}_\tau^k$ the *normal* and the *tangential* components of $\boldsymbol{v}^k$ on the boundary, given by

$$
v_\nu^k = \boldsymbol{v}^k \cdot \boldsymbol{\nu}^k, \quad \boldsymbol{v}_\tau^k = \boldsymbol{v}^k - v_\nu^k \boldsymbol{\nu}^k.
$$

For a regular (say C^1) stress field $\boldsymbol{\sigma}^k$, the application of its trace on the boundary to $\boldsymbol{\nu}^k$ is the Cauchy stress vector $\boldsymbol{\sigma}^k\boldsymbol{\nu}^k$. We define, similarly, the *normal* and *tangential* components of the stress on the boundary by the formulas

$$
\sigma_\nu^k = (\boldsymbol{\sigma}^k\boldsymbol{\nu}^k) \cdot \boldsymbol{\nu}^k, \quad \boldsymbol{\sigma}_\tau^k = \boldsymbol{\sigma}^k\boldsymbol{\nu}^k - \sigma_\nu^k \boldsymbol{\nu}^k
$$

and we recall that the following Green's formula holds:

$$
(\boldsymbol{\sigma}^k, \boldsymbol{\varepsilon}(\boldsymbol{v}^k))_{\mathcal{H}^k} + (\mathrm{Div}\,\boldsymbol{\sigma}^k, \boldsymbol{v}^k)_{H^k} = \int_{\Gamma^k} \boldsymbol{\sigma}^k\boldsymbol{\nu}^k \cdot \boldsymbol{v}^k \, da \quad \forall \boldsymbol{v}^k \in H_1^k. \tag{1}
$$

Here and below we denote by *Div* the *divergence* operator for tensor valued functions defined on Ω^1 or Ω^2.

We recall that, by the Sobolev trace theorem, there exists c_0^k, depending only on Ω^k, Γ_1^k and Γ_3^k, such that

$$
\|\boldsymbol{v}^k\|_{L^2(\Gamma_3^k)^3} \leq c_0^k \|\boldsymbol{v}^k\|_{V^k} \qquad \forall \boldsymbol{v}^k \in V^k, \tag{2}
$$

and we denote by c_0 the constant given by

$$
c_0 = \, max\,\{c_0^1,\, c_0^2\}. \tag{3}
$$

Moreover, we need the following functional spaces:

$$
V = \{\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in V^1 \times V^2 \mid v_\nu^1 + v_\nu^2 = 0 \ \text{ on } \ \Gamma_3\},
$$

$$
\mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2.
$$

The spaces V and $\mathcal{H}$ are real Hilbert spaces endowed with the canonical inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_{\mathcal{H}}$, respectively, and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_{\mathcal{H}}$, respectively.

Finally, for every real Banach space X and $T > 0$ we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \le p \le +\infty$, $k = 1,2,\dots$, and we use the dot above to indicate the derivative with respect to the time variable.

3. The model and its variational formulation

The physical setting is as follows. We consider two elastic bodies that occupy the domains Ω^1 and Ω^2. The two bodies are in bilateral, frictionless, adhesive contact along the common part $\Gamma_3^1 = \Gamma_3^2$, which will be denoted in what follows Γ_3. Let $T > 0$ and let $[0,T]$ be the time interval of interest. We assume that the bodies are clamped on $\Gamma_1^k \times (0,T)$, body forces of density $\boldsymbol{f}_0^k$ act on $\Omega^k \times (0,T)$, and surface tractions of density $\boldsymbol{f}_2^k$ act on $\Gamma_2^k \times (0,T)$.

We denote by $\boldsymbol{u}^k$ the displacement vectors, by $\boldsymbol{\sigma}^k$ the stress tensors and by $\boldsymbol{\varepsilon}^k = \boldsymbol{\varepsilon}(\boldsymbol{u}^k)$ the linearized strain tensors. We model the materials with nonlinear elastic constitutive laws

$$\boldsymbol{\sigma}^k = \mathcal{E}^k \boldsymbol{\varepsilon}(\boldsymbol{u}^k)$$

where $\mathcal{E}^k$ are given nonlinear constitutive functions which will be described below. We denote $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ and, for simplicity, we shall also use the notation

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\boldsymbol{\varepsilon}(\boldsymbol{v}^1),\, \boldsymbol{\varepsilon}(\boldsymbol{v}^2)) \in \mathcal{H}^1 \times \mathcal{H}^2, \quad \forall \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in V,$$

$$\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\mathcal{E}^1\boldsymbol{\varepsilon}(\boldsymbol{v}^1),\, \mathcal{E}^2\boldsymbol{\varepsilon}(\boldsymbol{v}^2)), \quad \forall \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in V.$$

We describe now the conditions on the contact surface Γ_3. We assume that the contact is *bilateral*, i.e., there is no separation between the bodies during the process. Therefore,

$$u_\nu^1 + u_\nu^2 = 0 \qquad \text{on } \Gamma_3 \times (0,T).$$

Moreover, $\boldsymbol{\nu}^1 = -\boldsymbol{\nu}^2$ on Γ_3 and $\boldsymbol{\sigma}^1\boldsymbol{\nu}^1 = -\boldsymbol{\sigma}^2\boldsymbol{\nu}^2$ on $\Gamma_3 \times (0,T)$. Consequently,

$$\sigma_\nu^1 = \sigma_\nu^2 \quad \text{and} \quad \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \qquad \text{on } \Gamma_3 \times (0,T).$$

Following [2, 3], we introduce a surface state variable β, the *bonding* field, which is a measure of the fractional intensity of adhesion between the surface and the foundation. This variable is restricted to values $0 \le \beta \le 1$; when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds, and partial adhesion takes place.

We assume that the resistance to tangential motion is generated by the glue, in comparison to which the frictional traction can be neglected. Moreover, the tangential traction depends only on the bonding field and on the relative tangential displacement, that is

$$-\boldsymbol{\sigma}_\tau^1 = \boldsymbol{\sigma}_\tau^2 = \boldsymbol{p}_\tau(\beta, \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2) \quad \text{on } \Gamma_3 \times (0,T).$$

We assume that the evolution of the bonding field is governed by the differential equation,

$$\dot{\beta} = H_{ad}(\beta, R(|\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2|)) \qquad \text{on } \Gamma_3 \times (0,T).$$

Here, H_{ad} is a general function discussed below, which vanishes when its first argument vanishes. The function $R : \mathbb{R}_+ \to \mathbb{R}_+$ is a truncation and is defined as

$$R(s) = \begin{cases} s & \text{if } 0 \le s \le L \\ L & \text{if } s > L, \end{cases} \tag{4}$$

where $L > 0$ is a characteristic length of the bonds (see, e.g., [6]). We use it in H_{ad} since usually, when the glue is streched beyond the limit L it does not contribute more to the bond strength.

Let β_0 the initial bonding field. We assume that the process is quasistatic and therefore we neglect the inertial term in the equation of motion. Then, the classical formulation of the mechanical problem may be stated as follows.

Problem *P. For $k = 1, 2$, find a displacement field $\boldsymbol{u}^k = (u_i^k) : \Omega^k \times [0, T] \to \mathbb{R}^3$, a stress field $\boldsymbol{\sigma}^k = (\sigma_{ij}^k) : \Omega^k \times [0, T] \to \mathbf{S}_3$ and an bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ which satisfy*

$$
\begin{array}{rcll}
\boldsymbol{\sigma}^k & = & \mathcal{E}^k \boldsymbol{\varepsilon}(\boldsymbol{u}^k)) & \text{in } \Omega^k \times (0, T), \qquad (5)\\
\operatorname{Div} \boldsymbol{\sigma}^k + \boldsymbol{f}_0^k & = & \mathbf{0} & \text{in } \Omega^k \times (0, T), \qquad (6)\\
\boldsymbol{u}^k & = & \mathbf{0} & \text{on } \Gamma_1^k \times (0, T), \qquad (7)\\
\boldsymbol{\sigma}^k \boldsymbol{\nu}^k & = & \boldsymbol{f}_2^k & \text{on } \Gamma_2^k \times (0, T), \qquad (8)\\
\sigma_\nu^1 = \sigma_\nu^2, \qquad u_\nu^1 + u_\nu^2 & = & 0 & \text{on } \Gamma_3 \times (0, T), \qquad (9)\\
-\boldsymbol{\sigma}_\tau^1 = \boldsymbol{\sigma}_\tau^2 & = & \boldsymbol{p}_\tau(\beta, \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2) & \text{on } \Gamma_3 \times (0, T), \qquad (10)\\
\dot{\beta} & = & H_{ad}(\beta, R(|\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2|)) & \text{on } \Gamma_3 \times (0, T), \qquad (11)\\
\beta(0) & = & \beta_0 & \text{on } \Gamma_3. \qquad (12)
\end{array}
$$

To obtain a variational formulation of the problem P we assume that the elastic operators $\mathcal{E}^k$, the tangential contact function p_τ and the adhesive rate function H_{ad} satisfy:

$$
\left\{
\begin{array}{l}
\text{(a) } \mathcal{E}^k : \Omega^k \times \mathbf{S}_3 \to \mathbf{S}_3;\\[4pt]
\text{(b) There exists } L_{\mathcal{E}}^k > 0 \text{ such that}\\
\qquad |\mathcal{E}^k(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}^k(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \le L_{\mathcal{E}}^k \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|\\
\qquad \forall \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbf{S}_3,\ \text{a.e. } \boldsymbol{x} \in \Omega;\\[4pt]
\text{(c) There exists } m_{\mathcal{E}}^k > 0 \text{ such that}\\
\qquad (\mathcal{E}^k(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}^k(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{E}}^k \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2\\
\qquad \forall \, \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbf{S}_3,\ \text{a.e. } \boldsymbol{x} \in \Omega^k;\\[4pt]
\text{(d) The map } \boldsymbol{x} \mapsto \mathcal{E}^k(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega^k\\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbf{S}_3;\\[4pt]
\text{(e) The map } \boldsymbol{x} \mapsto \mathcal{E}^k(\boldsymbol{x}, \mathbf{0}) \in \mathcal{H}^k.
\end{array}
\right. \qquad (13)
$$

$$
\left\{
\begin{array}{l}
\text{(a) } \boldsymbol{p}_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3;\\[4pt]
\text{(b) There exists } \ L_\tau > 0 \text{ such that}\\
\qquad |\boldsymbol{p}_\tau(\boldsymbol{x}, \beta_1, \boldsymbol{r}_1) - \boldsymbol{p}_\tau(\boldsymbol{x}, \beta_2, \boldsymbol{r}_2)| \le L_\tau \left(|\beta_1 - \beta_2| + |\boldsymbol{r}_1 - \boldsymbol{r}_2|\right)\\
\qquad \forall \, \beta_1,\ \beta_2 \in \mathbb{R},\ \boldsymbol{r}_1,\ \boldsymbol{r}_2 \in \mathbb{R}^3,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3;\\[4pt]
\text{(c) The map } \boldsymbol{x} \mapsto \boldsymbol{p}_\tau(\boldsymbol{x}, \beta, \boldsymbol{r}) \text{ is Lebesgue measurable}\\
\qquad \text{on } \Gamma_3 \ \forall \, \beta \in \mathbb{R},\ \boldsymbol{r} \in \mathbb{R}^3;\\[4pt]
\text{(d) The map } x \mapsto \boldsymbol{p}_\tau(\boldsymbol{x}, 0, 0) \in L^\infty(\Gamma_3)^3;\\[4pt]
\text{(e) } \boldsymbol{p}_\tau(\boldsymbol{x}, \beta, \boldsymbol{r}) \cdot \boldsymbol{\nu}(x) = 0 \ \ \forall \boldsymbol{r} \in \mathbb{R}^3 \text{ such that } \boldsymbol{r} \cdot \boldsymbol{\nu}(\boldsymbol{x}) = 0,\\
\qquad \text{a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}
\right. \qquad (14)
$$

$$\left\{\begin{array}{l}\text{(a) } H_{ad}:\Gamma_3\times \mathbb{R}\times\mathbb{R}_+\to\mathbb{R};\\[2mm] \text{(b) There exists } \ L_{H_{ad}}>0 \text{ such that}\\ \qquad |H_{ad}(\boldsymbol{x},b_1,r_1)-H_{ad}(\boldsymbol{x},b_2,r_2)|\\ \qquad \le L_{H_{ad}}\,(|b_1-b_2|+|r_1-r_2|)\\ \qquad \forall\, b_1,\, b_2\in\mathbb{R},\ \forall r_1,\, r_2\in[0,L],\ \text{a.e. } \boldsymbol{x}\in\Gamma_3;\\[2mm] \text{(c) The map } \boldsymbol{x}\mapsto H_{ad}(\boldsymbol{x},b,r) \text{ is Lebesgue measurable}\\ \qquad \text{on } \Gamma_3,\ \forall\, b\in\mathbb{R},\ r\in[0,L];\\[2mm] \text{(d) The map } (b,r)\mapsto H_{ad}(\boldsymbol{x},b,r) \text{ is continuous on}\\ \qquad \mathbb{R}\times[0,L],\ \text{a.e. } \boldsymbol{x}\in\Gamma_3;\\[2mm] \text{(e) } H_{ad}(\boldsymbol{x},0,r)=0 \quad \forall\, r\in[0,L],\ \text{a.e. } \boldsymbol{x}\in\Gamma_3;\\[2mm] \text{(f) } H_{ad}(\boldsymbol{x},b,r)\ge 0 \quad \forall\, b\le 0,\ r\in[0,L],\ \text{a.e. } \boldsymbol{x}\in\Gamma_3 \quad \text{and}\\ \qquad H_{ad}(\boldsymbol{x},b,r)\le 0 \quad \forall\, b\ge 1,\ r\in[0,L],\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{15}$$

Examples of functions $\mathcal{E}^k$, p_τ and H_{ad} which satisfy conditions (13), (14) and (15) can be found in [1, 4, 8]. We conclude that all the results below are valid for the corresponding contact problems.

We also suppose that the body forces and surface tractions satisfy

$$\boldsymbol{f}_0^k\in L^\infty(0,T;H^k),\qquad \boldsymbol{f}_2^k\in L^\infty(0,T;L^2(\Gamma_2^k)^3) \tag{16}$$

and, finally, the initial data satisfies

$$\beta\in L^\infty(\Gamma_3),\quad 0\le\beta_0\le 1 \text{ a.e. in } \Gamma_3. \tag{17}$$

Using (1) and (6), we deduce that for $k=1,2$ we have

$$(\boldsymbol{\sigma}^k(t),\,\boldsymbol{\varepsilon}(\boldsymbol{v}^k))_{\mathcal{H}^k}=(\boldsymbol{f}_0^k(t),\,\boldsymbol{v}^k)_{H^k}+\int_{\Gamma_2^k}\boldsymbol{f}_2^k(t)\cdot\boldsymbol{v}^k\,da+ \tag{18}$$

$$+\int_{\Gamma_3}(\sigma_\nu^k(t)\,v_\nu^k+\boldsymbol{\sigma}_\tau^k(t)\cdot\boldsymbol{v}_\tau^k)da \quad \forall \boldsymbol{v}^k\in V^k,\ \text{a.e. } t\in(0,T).$$

We define the map $\boldsymbol{f}:[0,T]\to V$ by the equality

$$(\boldsymbol{f}(t),\boldsymbol{v})_V=\sum_{k=1}^{2}\Big((\boldsymbol{f}_0^k(t),\boldsymbol{v}^k)_{H^k}+\int_{\Gamma_2^k}\boldsymbol{f}_2^k(t)\cdot\boldsymbol{v}^k\,da\Big) \tag{19}$$

$\forall\boldsymbol{v}=(\boldsymbol{v}^1,\,\boldsymbol{v}^2)\in V$, a.e. $t\in(0,T)$. We note that, using (16) we obtain the following regularity

$$\boldsymbol{f}\in L^\infty(0,T;V). \tag{20}$$

From (18) and (19) we deduce

$$(\boldsymbol{\sigma}(t),\,\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}=(\boldsymbol{f}(t),\boldsymbol{v})_V+\sum_{k=1}^{2}\int_{\Gamma_3}\sigma_\nu^k(t)\,v_\nu^k\,da+ \tag{21}$$

$$+\sum_{k=1}^{2}\int_{\Gamma_3}\boldsymbol{\sigma}_\tau^k(t)\cdot\boldsymbol{v}_\tau^k da \qquad \forall\boldsymbol{v}=(\boldsymbol{v}^1,\,\boldsymbol{v}^2)\in V,\ \text{a.e. } t\in(0,T).$$

Keeping in mind (9) and (10) we deduce

$$\sum_{k=1}^{2}\int_{\Gamma_3}(\sigma_\nu^k\,v_\nu^k+\boldsymbol{\sigma}_\tau^k\cdot\boldsymbol{v}_\tau^k)da=-\int_{\Gamma_3}\boldsymbol{p}_\tau(\beta,\boldsymbol{u}_\tau^1-\boldsymbol{u}_\tau^2)\cdot(\boldsymbol{v}_\tau^1-\boldsymbol{v}_\tau^2)da. \tag{22}$$

Let define the functional $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \boldsymbol{p}_\tau(\beta, \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2) \cdot (\boldsymbol{v}_\tau^1 - \boldsymbol{v}_\tau^2) da \tag{23}$$

$\forall \beta \in L^\infty(\Gamma_3)$, $\forall \boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2), \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in V$. Taking into account (21)-(23) we can write

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V, \ \text{a.e. } t \in (0, T).$$

Then, the variational formulation of the Problem P may be stated as follows.

Problem P_V. *Find a displacement field $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2) : [0, T] \to V$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) : [0, T] \to \mathcal{H}$, and a bonding field $\beta : [0, T] \to L^\infty(\Gamma_3)$ such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{24}$$

$$\dot{\beta}(t) = H_{ad}(\beta(t), R(|\boldsymbol{u}_\tau^1(t) - \boldsymbol{u}_\tau^2(t)|)), \ 0 \le \beta(t) \le 1, \tag{25}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V, \tag{26}$$

a.e. $t \in (0, T)$, and

$$\beta(0) = \beta_0. \tag{27}$$

The unique solvability of problem P_V will be proved in the next section.

4. An existence and uniqueness result

Our main existence and uniqueness result is the following.

Theorem 4.1. *Assume that* (13)–(17) *hold and, assume moreover that*

$$\frac{c_0^2 \, L_\tau}{m_{\mathcal{E}}} < \frac{1}{2\sqrt{2e}}\,. \tag{28}$$

Then there exists a unique solution $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta\}$ of problem P_V. Moreover, the solution satisfies

$$\boldsymbol{u} \in L^\infty(0, T; V), \tag{29}$$

$$\boldsymbol{\sigma} \in L^\infty(0, T; \mathcal{H}), \tag{30}$$

$$\beta \in W^{1,\infty}(0, T; L^\infty(\Gamma_3)), \tag{31}$$

$$0 \le \beta(t) \le 1 \quad \forall t \in [0, T], \ \text{a.e. on } \Gamma_3. \tag{32}$$

Theorem 4.1 states the well posedness of the variational problem P_V. By this theorem we conclude that, under the assumptions (13)–(17) and (28), then the mechanical problem P has a unique *weak solution* with regularity (29)–(31).

The proof of Theorem 4.1 is carried out in several steps that we present in what follows. Everywhere below we assume that (13)–(17) hold. We use Riesz's representation theorem to define the operator $E : V \to V$ by

$$(E\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \qquad \forall \boldsymbol{u}, \, \boldsymbol{v} \in V. \tag{33}$$

It follows from (13) that E is a strongly monotone Lipschitz continuous operator. More exactly, it satisfies

$$\|E\boldsymbol{u} - E\boldsymbol{v}\|_V \le L_{\mathcal{E}} \|\boldsymbol{u} - \boldsymbol{v}\|_V$$

and

$$(E\boldsymbol{u} - E\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \ge m_{\mathcal{E}} \|\boldsymbol{u} - \boldsymbol{v}\|_V^2$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$, where $L_{\mathcal{E}} = L^1_{\mathcal{E}} + L^2_{\mathcal{E}}$ and $m_{\mathcal{E}} = min\,\{m^1_{\mathcal{E}},\, m^2_{\mathcal{E}}\}$. Therefore, it follows that E is invertible and its inverse, denoted $E^{-1} : V \to V$, satisfies

$$\|E^{-1}(\boldsymbol{w}_1) - E^{-1}(\boldsymbol{w}_2)\|_V \le \frac{1}{m_{\mathcal{E}}} \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_V \qquad \forall \boldsymbol{w}_1,\, \boldsymbol{w}_2 \in V. \tag{34}$$

Let $\boldsymbol{\eta}$ be an arbitrary element of the space $L^\infty(0,T;V)$ and denote

$$\boldsymbol{u}^\eta(t) = E^{-1}(\boldsymbol{f}(t) - \boldsymbol{\eta}(t)) \quad \text{ a.e. } t \in (0,T). \tag{35}$$

It follows from (20) that

$$\boldsymbol{u}^\eta \in L^\infty(0,T;V) \tag{36}$$

and, moreover, (33) and (35) imply that a.e. $t \in (0,T)$

$$(\mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}^\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\eta}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V. \tag{37}$$

Let consider now the following evolutionary problem.

Problem P^η_V. *Find a bonding field $\beta_\eta : [0,T] \to L^\infty(\Gamma_3)$ such that*

$$\dot{\beta}_\eta(t) = H_{ad}(\beta_\eta(t), R(|\boldsymbol{u}^{1\eta}_\tau(t) - \boldsymbol{u}^{2\eta}_\tau(t)|)) \quad a.e.\ t \in (0,T), \tag{38}$$

$$\beta_\eta(0) = \beta_0. \tag{39}$$

We have the following result.

Lemma 4.1. *There exists a unique solution β_η of problem P^η_V and it satisfies* (31). *Moreover,*

$$0 \le \beta_\eta(t) \le 1 \quad \forall t \in [0,T],\ a.e.\ on\ \Gamma_3. \tag{40}$$

Proof. For the sake of simplicity we suppress the dependence of various functions on $\boldsymbol{x} \in \Gamma_3$. Notice that the equalities and inequalities below are valid a.e. $\boldsymbol{x} \in \Gamma_3$. We consider the map $F : (0,T) \times L^\infty(\Gamma_3) \to L^\infty(\Gamma_3)$ defined by

$$F(t,\beta) = H_{ad}(\beta, R(|\boldsymbol{u}^{1\eta}_\tau(t) - \boldsymbol{u}^{2\eta}_\tau(t)|)) \quad \text{ a.e. } t \in (0,T),\ \forall \beta \in L^\infty(\Gamma_3).$$

It is easy to check that F is Lipschitz continuous with respect to the second variable, uniformly in time; moreover, for all $\beta \in L^\infty(\Gamma_3)$, $t \mapsto F(t,\beta)$ belongs to $L^\infty(0,T;L^\infty(\Gamma_3))$. Thus, the existence of a unique function β_η which satisfies (38)–(39), follows from a version of the Cauchy-Lipschitz theorem.

Finally, the proof of (40) is a consequence of the assumptions (15) and (17), see [8] for details. □

We now study the dependence of the solution of problem P^η_V with respect to $\boldsymbol{\eta}$.

Lemma 4.2. *Let $\boldsymbol{\eta}_i \in L^\infty(0,T;V)$ and let β_i denote the solution of problem $P^{\eta_i}_V$, $i = 1,2$. Then:*

$$\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|^2_{L^2(\Gamma_3)} \le \frac{2\,L^2_{H_{ad}}\,c_0^2}{m^2_{\mathcal{E}}}\,T\,e^{2\,T\,L_{H_{ad}}} \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|^2_V\,ds \qquad \forall t \in [0,T]. \tag{41}$$

Proof. Let $t \in [0,T]$. The equalities and inequalities below are valid a.e. on Γ_3. Using (38) and (39) we can write

$$\beta_i(t) = \beta_0 + \int_0^t H_{ad}(\beta_i(s), R(|\boldsymbol{u}_\tau^{1i}(s) - \boldsymbol{u}_\tau^{2i}(s)|)\, ds, \quad i = 1,2$$

where $\boldsymbol{u}^{\eta_i} = \boldsymbol{u}^i = (\boldsymbol{u}^{1i}, \boldsymbol{u}^{2i})$. Using now (15) and (4), we obtain

$$|\beta_1(t) - \beta_2(t)| \le L_{H_{ad}} \Big(\int_0^t |\beta_1(s) - \beta_2(s)|\, ds + \int_0^t |(\boldsymbol{u}_\tau^{11}(s) - \boldsymbol{u}_\tau^{21}(s)) - (\boldsymbol{u}_\tau^{12}(s) - \boldsymbol{u}_\tau^{22}(s))|\, ds \Big).$$

Next, we apply Gronwall's inequality to deduce

$$|\beta_1(t) - \beta_2(t)| \le L_{H_{ad}}\, e^{T\, L_{H_{ad}}} \int_0^t |(\boldsymbol{u}_\tau^{11}(s) - \boldsymbol{u}_\tau^{21}(s)) - (\boldsymbol{u}_\tau^{12}(s) - \boldsymbol{u}_\tau^{22}(s))| ds,$$

which implies

$$|\beta_1(t) - \beta_2(t)|^2 \le 2\, L_{H_{ad}}^2\, T\, e^{2\,T\,L_{H_{ad}}} \int_0^t (|\boldsymbol{u}^{11}(s) - \boldsymbol{u}^{12}(s)|^2 + |\boldsymbol{u}^{21}(s) - \boldsymbol{u}^{22}(s)|^2)\, ds.$$

Integrating the last inequality over Γ_3 and keeping in mind (2) we find

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 \le 2\, L_{H_{ad}}^2\, T\, e^{2\,T\,L_{H_{ad}}} \int_0^t \Big((c_0^1)^2 \|\boldsymbol{u}^{11}(s) - \boldsymbol{u}^{12}(s)\|_{V^1}^2 + +(c_0^2)^2 \|\boldsymbol{u}^{21}(s) - \boldsymbol{u}^{22}(s)\|_{V^2}^2 \Big)\, ds.$$

Taking into account (3) we deduce

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 \le 2\,(c_0)^2\, L_{H_{ad}}^2\, T\, e^{2\,T\,L_{H_{ad}}} \int_0^t \|\boldsymbol{u}^1(s) - \boldsymbol{u}^2(s)\|_V^2\, ds,$$

and using (35), we obtain (41). □

We consider now the operator $\Lambda : L^\infty(0,T;V) \to L^\infty(0,T;V)$ given by

$$(\Lambda\boldsymbol{\eta}(t), \boldsymbol{v})_V = j(\beta_\eta(t),\, \boldsymbol{u}^\eta(t),\, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V, \ \text{ a.e. } t \in (0,T), \tag{42}$$

where β_η denotes the solution of problem P_V^η, $\boldsymbol{u}^\eta$ is given by (35) and j is the functional (23).

We have the following result.

Lemma 4.3. *Under the smallness assumption* (28)*, there exists a unique element* $\boldsymbol{\eta}^* \in L^\infty(0,T;V)$ *such that* $\Lambda\, \boldsymbol{\eta}^* = \boldsymbol{\eta}^*$.

Proof. Let $\boldsymbol{\eta}_i \in L^\infty(0,T;V)$, $\boldsymbol{u}^{\eta_i} = \boldsymbol{u}^i = (\boldsymbol{u}^{1i},\, \boldsymbol{u}^{2i})$ and let β_i denote the solution of problem $P_V^{\eta_i}$, $i = 1,2$. The equalities and inequalities below are valid for all $\boldsymbol{v} \in V$, a.e. $t \in (0,T)$. Using (42), (23) and the properties of the function $\boldsymbol{p}_\tau$, after some computation we obtain

$$|(\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t), \boldsymbol{v})_V| \le L_\tau \Big(\|\boldsymbol{u}^{11}(t) - \boldsymbol{u}^{12}(t)\|_{L^2(\Gamma_3)^3} + \|\boldsymbol{u}^{21}(t) - \boldsymbol{u}^{22}(t)\|_{L^2(\Gamma_3)^3} + \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \Big) (\|\boldsymbol{v}^1\|_{L^2(\Gamma_3)^3} + \|\boldsymbol{v}^2\|_{L^2(\Gamma_3)^3}).$$

Moreover, keeping in mind (2) and (3) we can write

$$|(\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t),\boldsymbol{v})_V|\leq L_\tau\,c_0\Big(\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}+c_0(\|\boldsymbol{u}^{11}(t)-\boldsymbol{u}^{12}(t)\|_{V^1}+$$
$$+\|\boldsymbol{u}^{21}(t)-\boldsymbol{u}^{22}(t)\|_{V^2})\Big)\,(\|\boldsymbol{v}^1\|_{V^1}+\|\boldsymbol{v}^2\|_{V^2}),$$

and from this inequality we find

$$\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_V\leq\sqrt{2}c_0\,L_\tau\Big(\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}+ \tag{43}$$
$$+\sqrt{2}\,c_0\,\|\boldsymbol{u}^1(t)-\boldsymbol{u}^2(t)\|_V\Big).$$

Using now (43), (41), (34) and (35) we deduce that

$$\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_V^2\leq k\,\|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_V^2+M\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2ds \tag{44}$$

where

$$k=\frac{8\,c_0^4\,L_\tau^2}{m_{\mathcal{E}}^2},\qquad M=\frac{8\,L_{H_{ad}}^2\,c_0^4L_\tau^2\,T\,e^{2\,T\,L_{H_{ad}}}}{m_{\mathcal{E}}^2}. \tag{45}$$

Taking into account (44) we can use now a fixed point argument already used in [7]. To this end we denote

$$I_0(t)=\|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_V^2, \tag{46}$$

$$I_1(t)=\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2\,ds, \tag{47}$$

$$I_j(t)=\int_0^t\int_0^{s_{j-1}}...\int_0^{s_1}\|\boldsymbol{\eta}_1(r)-\boldsymbol{\eta}_2(r)\|_V^2\,dr\,ds_1...ds_{j-1},\ \forall j\in N,\ j\geq 2. \tag{48}$$

Notice that

$$I_j(t)\leq\frac{t^j}{j\,!}\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_{L^\infty(0,T;V)}^2,\qquad\forall\,j\in\mathbb{N}. \tag{49}$$

Reiterating the inequality (44) and using (46)–(49), we deduce that

$$\|\Lambda^p\boldsymbol{\eta}_1(t)-\Lambda^p\boldsymbol{\eta}_2(t)\|_V^2\leq\Big(\sum_{j=0}^p C_p^j\,k^{p-j}\,\frac{M^j\,T^j}{j\,!}\Big)\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_{L^\infty(0,T;V)}^2. \tag{50}$$

It is easy to check that

$$\sum_{j=0}^p C_p^j\,k^{p-j}\,\frac{M^j\,T^j}{j\,!}\leq\frac{(k\,p+M\,T)^p}{p\,!}$$

and therefore, (50) implies

$$\|\Lambda^p\boldsymbol{\eta}_1-\Lambda^p\boldsymbol{\eta}_2\|_{L^\infty(0,T;V)}^2\leq\frac{(k\,p+M\,T)^p}{p\,!}\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_{L^\infty(0,T;V)}^2.$$

Assume now that (28) hold. It follows that $0<k<\frac{1}{e}$ and, therefore, the series $\displaystyle\sum_{p=1}^\infty\frac{(k\,p+M\,T)^p}{p\,!}$ is convergent. Consequently,

$$\lim_{p\to\infty}\frac{(k\,p+M\,T)^p}{p\,!}=0.$$

We conclude that for a sufficiently large p, the mapping Λ^p is a contraction in the Banach space $L^\infty(0,T;V)$. Therefore, there exists a unique $\boldsymbol{\eta}^*\in L^\infty(0,T,V)$ such that $\Lambda^p\,\boldsymbol{\eta}^*=\boldsymbol{\eta}^*$ and, moreover, $\boldsymbol{\eta}^*$ is the unique fixed point of Λ. □

We have now all the ingredients to prove Theorem 4.1.

Proof of Theorem 4.1
Existence. Let $\boldsymbol{\eta}^* \in L^\infty(0,T;V)$ be the fixed point of the operator Λ and let $\boldsymbol{u}$, β be defined by (35), (38)–(39) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$, i.e. $\boldsymbol{u} = \boldsymbol{u}^{\eta^*}$, $\beta = \beta_{\eta^*}$. We denote by $\boldsymbol{\sigma}$ the function given by (24). Clearly, (24), (25) and (27) hold. Since $\Lambda\,\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$, we deduce that

$$(\Lambda\,\boldsymbol{\eta}^*(t), \boldsymbol{v})_V = (\boldsymbol{\eta}^*(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V, \ \text{ a.e. } t \in (0,T)$$

and, keeping in mind (42) and (37) we deduce that (26) hold too. The regularity (29) follows from (20) while the regularity (31) and property (32) are consequences of Lemma 4.1. Moreover, since $\boldsymbol{u} \in L^\infty(0,T;V)$, it follows from (24) that $\boldsymbol{\sigma} \in L^\infty(0,T;\mathcal{H})$. We conclude that the triplet $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta\}$ is a solution of problem P_V and it satisfies (29)-(32).

Uniqueness. The uniqueness part follows the uniqueness of the fixed point of the operator Λ (see [8] for details). □

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