

Analysis of fractional Fokker-Planck equation with Caputo and Caputo-Fabrizio derivatives

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ABSTRACT. This research focus on the determination of the numerical solution for the mathematical model of Fokker-Planck equations utilizing a new method, in which Sumudu transformation and homotopy analysis method (SHAM) are used together. By SHAM analytical series solution of any mathematical model including fractional derivative can be obtained. By this method, we constructed the solution of fractional Fokker-Planck equations in Caputo and Caputo-Fabrizio senses. The results show that this method is advantageous and applicable to form the series resolution of the fractional mathematical models.

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1. Introduction

The importance of mathematical models including fractional derivatives increases effectively recently since these mathematical models reflects behaviour of real world phenomena better because of non-local behaviour of fractional derivatives. Using fractional derivatives in models, the memory and hereditary properties of phenomena are included which helps to analyze the complex behaviour of the any system much more better. Various fractional derivatives are defined and used to model real world phenomena since each of them has interesting properties which contributes to understand the behaviour of the system [6], [2], [9], [3], [21], [22]. For instance, fractional derivative in Caputo-Fabrizio sense has the property of characterizing heterogeneities and configurations [17]. Moreover this fractional derivative does not have any singularity. Hence its definition does not include a singular kernel which allows us to determine the effect of the memory without any difficulty. By making use of SHAM the fractional partial differential equation turn into a simpler form including a recursive relation which allows to obtain the solution as an analytical series. In this research, we utilize the SHAM to find analytical approximated solution for fractional Fokker-Planck equations. The utilized method consists of two methods [16]. Some writers have projected various systems for physical processes with two fractional operators. In [5], Dehghan practised the HAM to solve fractional models with Liouville-Caputo. In [23], is studied a fractional differential equation with a changeble coefficient. Jafari in [10] utilized the HAM to solve the higher order fractional models analized by Diethelmand Ford [7]. In [8], an analysis of an example with the Caputo-Fabrizio fractional derivative is studied, where analytical and advanced calculation are included.

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Morales-Delgado et al. [17] presented Laplace HAM to determine a new solution. In this study, SHAM is utilized to solve time fractional models including fractional derivative in Caputo and Caputo-Fabrizio senses to construct the series form of the solutions. The significant advantage of this method compare to the other semi analytic methods is that it does not entail any supplementary data other than boundary and initial conditions. This method allows us to transform the original problem into one for which the convergence and accuracy of the solution is very high especially for the Caputo fractional derivative [19]. The goal of this work is to establish approximate resolutions of the fractional model of Fokker-Planck equations (FPEs) as follows [20]:

$$D_t^\alpha f(x, t) = D_x^\beta \Phi(x, t, f) + D_x^{2\beta} \Omega(x, t, f), x \in \mathbb{R}, t > 0, 0 < \alpha, \beta \leq 1 \quad (1)$$

with the initial state $f(x, 0) = h(x)$. $\Phi(x, t, f)$ and $\Omega(x, t, f)$ denote drift and diffusion functions, α and β denote the orders of fractional derivatives, respectively. Notice that this equation is a classical FPE for $\alpha = 1, \beta = 1$. These equations are used in the pattern of divergent diffusion techniques. In [20], q -homotopy analysis transform method is utilized to obtain analytical solutions for Eqs. (1), stochastic expression and computer model of fractional FPE representing divergent diffusion is analysed in [15] and in [14], [13], approximated solutions are obtained by using Monte Carlo technique, exact solutions for fractional FPE has been determined by utilizing various methods, for example Laplace transform method [24], Homotopy perturbation method (HPM) [25], Homotopy perturbation transform method (HPTM) [11], Adomian decomposition method (ADM) [18], In the section 2 of this article, some basic definitions related to in case of every two fractional operators. In section 3 and 4, SHAM is applied to construct the solution of the fractional FPEs and some tables and graphical outcomes are contained to show the reliability and simplicity of the technique. Finally, in section 5, consequences are presented.

2. Preliminaries

Definition 2.1. A real function $f(t), t > 0$, belongs to the space $C_\mu, \mu \in \mathbb{R}$ if the condition $f(t) = t^p g(t)$ is satisfied for some real number $p (> \mu)$ where $g(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^m \in C_\mu, m \in \mathbb{N}$.

Definition 2.2. For a function $f(t) \in C_\mu, \mu \geq -1$ the following integral

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, t > 0, J^0 f(t) = f(t). \quad (2)$$

is called The Riemann Liouville Fractional integral operator of order $\alpha \geq 0$ [6]. The Riemann Liouville fractional integral of t^y is computed as follows:

$$J^\alpha t^y = \frac{\Gamma(y+1)}{\Gamma(\alpha+y+1)} t^{\alpha+y}. \quad (3)$$

Definition 2.3. The Caputo fractional derivative of $f(t)$ is defined in the following form [6]

$${}^C D_t^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-z)^{m-\alpha-1} \frac{d^m}{dt^m} f(z) dz, \quad m-1 < \alpha \leq m, m \in \mathbb{N}, t > 0. \quad (4)$$

Definition 2.4. The transformation

$$\bar{f}(w) = \mathbb{S}[f(t)] = \int_0^\infty f(wt) e^{-t} dt, w \in (-\tau_1, \tau_2), \tag{5}$$

is called Sumudu transformation defined on the following set [4]

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}.$$

Definition 2.5. For $f(t) = t^\alpha$, the Sumudu transform is defined as [4]

$$\mathbb{S}[t^\alpha] = \int_0^\infty e^{-t} t^\alpha dt = \Gamma(\alpha + 1) w^\alpha, \text{ Re}(\alpha) > 0. \tag{6}$$

Definition 2.6. The Sumudu transformation $\mathbb{S}[f(t)]$ of the Riemann-Liouville fractional integral is defined as [4]

$$\mathbb{S}[J^\alpha f(t)] = w^\alpha F(w). \tag{7}$$

Definition 2.7. The Sumudu transformation $\mathbb{S}[f(t)]$ of the Caputo fractional derivative is defined as [4]

$$\mathbb{S} \left[{}_0^C D_t^\alpha f(x, t) \right] (w) = w^{-\alpha} \mathbb{S}[f(x, t)] - \sum_{k=0}^{n-1} \left[w^{-\alpha+k} \frac{\partial^k f(x, 0)}{\partial t^k} \right], n-1 < \alpha \leq n, n \in \mathbb{N}. \tag{8}$$

In the following, we provide some basic concepts and definitions in connection with the new Caputo-Fabrizio derivative. Caputo and Fabrizio defined derivative of the fractional order for a function f belongs to the Sobolev space $H^1(a, b) = \left\{ f(x), a < x < b, \int_a^b f^2(x) < \infty, \int_a^b (f')^2(x) < \infty \right\}$.

Definition 2.8. For a function $f \in H^1(a, b)$, the Caputo-Fabrizio derivative of fractional order $\alpha \in [0, 1]$ is defined as

$${}^C D_t^\alpha (f(t)) = \left(\frac{M(\alpha)}{1-\alpha} \right) \int_a^t f'(x) \exp \left[-\alpha \frac{t-x}{1-\alpha} \right] dx, \tag{9}$$

where $M(\alpha)$ is a normalization function under the conditions $M(0) = M(1) = 1$ [12]. But, if a certain function does not satisfy in the restriction $f \in H^1(a, b)$, then its fractional derivative is redefined as

$${}^C D_t^\alpha (f(t)) = \left(\frac{\alpha M(\alpha)}{1-\alpha} \right) \int_a^t (f(t) - f(x)) \exp \left[-\alpha \frac{t-x}{1-\alpha} \right] dx. \tag{10}$$

Clearly, as mentioned in [12], if one chooses $\sigma = \frac{1-\alpha}{\alpha} \in [0, \infty]$ and $\alpha = \frac{1}{1+\sigma} \in [0, 1]$, then the Caputo-Fabrizio definition becomes

$${}^C D_t^\alpha (f(t)) = \left(\frac{N(\sigma)}{\sigma} \right) \int_a^t f'(x) \exp \left[-\frac{t-x}{\sigma} \right] dx, N(0) = N(\infty) = 1, \tag{11}$$

where

$$\lim_{\sigma \rightarrow 0} \exp \left[-\frac{t-x}{\sigma} \right] = \delta(x-t). \tag{12}$$

For $n \geq 1$ and $\alpha \in [0, 1]$, the fractional derivative of order $(n + \alpha)$ is defined by

$$D_t^{(\alpha+n)} (f(t)) = D_t^{(\alpha)} \left(D_t^{(n)} (f(t)) \right). \tag{13}$$

Definition 2.9. The fractional arbitrary order integral of a function $f \in H^1(a, b)$ is defined as follows

$$I_t^\alpha(f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(s)ds, t \geq 0. \quad (14)$$

In view of the above definition, it is clear that the α^{th} Caputo-Fabrizio derivative is average between f and its one order integral. Therefore,

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1. \quad (15)$$

So, we arrived to prove that

$$M(\alpha) = \frac{2}{2-\alpha}, 0 \leq \alpha \leq 1.$$

By virtue of this formula, Losada and Nieto [12] remarked that Caputo-Fabrizio fractional derivative can be redefined as

$${}_0^{CF}D_t^\alpha(f(t)) = \left(\frac{1}{1-\alpha}\right)\int_a^t f'(x)\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx, \quad (16)$$

Applying the Laplace transform to (2.15), one has

$$\mathbb{L}\left[{}_0^{CF}D_t^\alpha(f(x,t))\right] = \frac{w\mathbb{L}[f(t)] - f(x,0)}{w + \alpha(1-w)}, \quad (17)$$

So

$$\mathbb{L}\left[{}_0^{CF}D_t^{\alpha+1}(f(x,t))\right] = \frac{w\mathbb{L}[f(t)] - wf(x,0) - \frac{\partial f(x,0)}{\partial t}}{w + \alpha(1-w)}, \quad (18)$$

In a general form

$$\mathbb{L}\left[{}_0^{CF}D_t^{\alpha+1}(f(x,t))\right] = \frac{w^{n+1}\mathbb{L}[f(x,t)] - w^n f(x,0) - w^{n-1}\frac{\partial f(x,0)}{\partial t} - \dots - \frac{\partial^n f(x,0)}{\partial t^n}(0)}{w + \alpha(1-w)}. \quad (19)$$

Sumudu transform is an integral transform which is defined by the following formula

$$F(w) = \mathbb{S}[f(t); w] = \frac{1}{w}\int_0^\infty e^{-\left(\frac{t}{w}\right)}f(t)dt.$$

Atangana [1] proved that for the Caputo-Fabrizio derivative of fractional order for $f(t)$, the Sumudu transform is obtained as

$$\mathbb{S}\left[{}_0^{CF}D_t^\alpha(f(x,t))\right] = M(\alpha)\frac{\mathbb{S}[f(x,t)] - f(0)}{1 - \alpha + \alpha w}, \quad (20)$$

and in a general form

$$\mathbb{S}\left[{}_0^{CF}D_t^\alpha(f(x,t))\right] = \frac{M(\alpha)}{1 - \alpha + \alpha w}\left[\frac{\mathbb{S}[f(x,t)]}{w^n} - \sum_{k=0}^n \frac{1}{w^{n-k}}\frac{\partial^k f(x,0)}{\partial t^k}\right]. \quad (21)$$

3. Approximate solution of Caputo time-fractional differential equation via SHAM

$${}^C D_t^\alpha f(x, t) + \vartheta(x) \frac{\partial f(x, t)}{\partial x} + \gamma(x) \frac{\partial^2 f(x, t)}{\partial x^2} + \varphi(x) f(x, t) = \sigma(x, t), \tag{22}$$

where $(x, t) \in [0, 1] \times [0, \check{T}]$, $n - 1 < \alpha \leq n$

$$\frac{\partial^i f(x, 0)}{\partial t^i} = f_i(x), i = 0, 1, \dots, n - 1, \tag{23}$$

and

$$f(0, t) = \epsilon_0(t), f(1, t) = \epsilon_1(t), t \geq 0. \tag{24}$$

Ignoring all boundary and initial conditions make the computation simpler. Now, the methodology involves using the Sumudu transformation on both sides of the Eq. (22) to obtain

$$\begin{aligned} \mathbb{S}[f(x, t)] - \sum_{k=0}^{n-1} \left[w^k \frac{\partial^k f(x, 0)}{\partial t^k} \right] + w^\alpha \left[\vartheta(x) \frac{\partial}{\partial x} + \gamma(x) \frac{\partial^2}{\partial x^2} + \varphi(x) \right] \mathbb{S}[f(x, t)] \\ - w^\alpha \mathbb{S}[\sigma(x, t)] = 0 \end{aligned} \tag{25}$$

The nonlinear operator becomes

$$\begin{aligned} N[\phi(x, t; p)] = \mathbb{S}[\phi(x, t; p)] - \sum_{k=0}^{n-1} \left[w^k \frac{\partial^k \phi(x, 0; p)}{\partial t^k} \right] \\ + w^\alpha \left[\vartheta(x) \frac{\partial}{\partial x} + \gamma(x) \frac{\partial^2}{\partial x^2} + \varphi(x) \right] \mathbb{S}[\phi(x, t; p)] - w^\alpha \mathbb{S}[\sigma(x, t)] \\ = 0 \end{aligned} \tag{26}$$

where $\phi(x, t; p)$ is a real function of x, t and embedding parameter $p \in [0, 1]$. By homotopy, we have

$$(1 - p) \mathbb{S}[\phi(x, t; p) - f_0(x, t)] = phH(x, t)N[\phi(x, t; p)]; \tag{27}$$

where $\phi(x, t; p)$ is an unknown function, $H(x, t) \neq 0, h \neq 0$ is a auxiliary parameter and an auxiliary function $f_0(x, t)$ is an initial guess of $f(x, t)$. Auxiliary parameter can be chosen arbitrarily in SHAM. Clearly, If $p = 0, \phi(x, t; 0) = f_0(x, t)$ and if $p = 1, \phi(x, t; 1) = f(x, t)$. Thus, the solution converges to the solution $f(x, t)$ from first prediction $f_0(x, t)$ as p varies from 0 to 1. Now, writing $\phi(x, t; p)$ in the form of Taylor’s series with respect to p leads to

$$\phi(x, t; p) = f_0(x, t) + \sum_{m=1}^{\infty} p^m f_m(x, t) \tag{28}$$

where

$$f_m(x, t) = \frac{1}{\Gamma(m + 1)} \left. \frac{\partial^m \phi(x, t; p)}{\partial p^m} \right|_{p=0} \tag{29}$$

The parameter h controls the convergence of numerical solution (28). The series (28) converges at $p = 1$ if we make the correct choices of necessary parameter and

predictions. From here,

$$f(x, t) = f_0(x, t) + \sum_{m=1}^{\infty} f_m(x, t) \quad (30)$$

which leads to one of the solutions of Eq. (26) is obtained. It is seen from the above expression that exact solution $f(x, t)$ and the first prediction $f_0(x, t)$ have a relationship in terms of $f_m(x, t)$ ($m = 1, 2, 3, \dots$). Differentiating Eq. (27) m times with respect to p , plugging $p = 0$, and multiplying by $\frac{1}{\Gamma(m+1)}$ leads to:

$$\mathbb{S}[f_m(x, t) - \chi_m f_{m-1}(x, t)] = hH(x, t) R_m(\vec{f}_{m-1}, x, t). \quad (31)$$

where

$$\vec{f} = \{f_0(x, t), f_1(x, t), f_2(x, t), \dots, f_m(x, t)\}. \quad (32)$$

Applying inverse Sumudu transform to both sides of Eq. (31), then the expression below is obtained:

$$f_m(x, t) = \chi_m f_{m-1}(x, t) + \mathbb{S}^{-1} \left[hH(x, t) R_m(\vec{f}_{m-1}, x, t) \right] \quad (33)$$

where

$$R_m(\vec{f}_{m-1}, x, t) = \frac{1}{\Gamma(m)} \left. \frac{\partial^{m-1} \phi(x, t; p)}{\partial p^{m-1}} \right|_{p=0} \quad (34)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (35)$$

In our case

$$\begin{aligned} R_m(\vec{f}_{m-1}, x, t) &= {}_0^C D_t^\alpha f_{m-1}(x, t) + \vartheta(x) \frac{\partial f_{m-1}(x, t)}{\partial x} + \gamma(x) \frac{\partial^2 f_{m-1}(x, t)}{\partial x^2} \\ &+ \varphi(x) f_{m-1}(x, t) - (1 - \chi_m) \sigma(x, t) \end{aligned} \quad (36)$$

$f_m(x, t)$ for $m \geq 1$, at M^{th} order can be obtained easily from (33) which leads to accurate approximation of the Eq. (22)

$$f(x, t) = \sum_{m=0}^M f_m(x, t) \quad (37)$$

as $M \rightarrow \infty$.

Theorem 3.1. *If the approximation (37) converges as $M \rightarrow \infty$. The exact solution (22) is obtained.*

Proof. Assume that the approximation (37) is a convergent series then

$$\sum_{m=0}^{\infty} f_m(x, t) = f_0(x, t) + \sum_{m=1}^{\infty} f_m(x, t) = K(x, t).$$

Now we have $\lim_{M \rightarrow \infty} f_m(x, t) = 0$. Taking definition of Eq. (31) into account leads to

$$\begin{aligned} \lim_{M \rightarrow \infty} \left[hH(x, t) \sum_{m=1}^M R_m(\vec{f}_{m-1}, x, t) \right] &= \lim_{M \rightarrow \infty} \left(\sum_{m=1}^M \mathbb{S}[f_m(x, t) - \chi_m f_{m-1}(x, t)] \right) \\ &= \mathbb{S} \left[\lim_{M \rightarrow \infty} \sum_{m=1}^M [f_m(x, t) - \chi_m f_{m-1}(x, t)] \right] \\ &= \mathbb{S} \left[\lim_{M \rightarrow \infty} f_m(x, t) \right] = 0. \end{aligned}$$

Since $h \neq 0, H(x, t) \neq 0$, therefore, $\sum_{m=1}^{\infty} R_m(\vec{f}_{m-1}, x, t) = 0$. From (36)

$$\begin{aligned} \sum_{m=1}^{\infty} R_m(\vec{f}_{m-1}, x, t) &= \sum_{m=1}^{\infty} {}_0^C D_t^\alpha f_{m-1}(x, t) \\ &+ \sum_{m=1}^{\infty} \left[\vartheta(x) \frac{\partial f_{m-1}(x, t)}{\partial x} + \gamma(x) \frac{\partial^2 f_{m-1}(x, t)}{\partial x^2} + \varphi(x) f_{m-1}(x, t) \right] - \sum_{m=1}^{\infty} (1 - \chi_m) \sigma(x, t). \\ \sum_{m=1}^{\infty} R_m(\vec{f}_{m-1}, x, t) &= {}_0^C D_t^\alpha \sum_{m=0}^{\infty} f_m(x, t) + \vartheta(x) \sum_{m=0}^{\infty} \frac{\partial f_m(x, t)}{\partial x} \\ &+ \gamma(x) \sum_{m=0}^{\infty} \frac{\partial^2 f_m(x, t)}{\partial x^2} + \varphi(x) \sum_{m=0}^{\infty} f_m(x, t) - \sigma(x, t). \end{aligned}$$

$${}_0^C D_t^\alpha K(x, t) + \vartheta(x) \frac{\partial K(x, t)}{\partial x} + \gamma(x) \frac{\partial^2 K(x, t)}{\partial x^2} + \varphi(x) K(x, t) - \sigma(x, t) = 0. \tag{38}$$

Above equation (38) shows that, $K(x, t)$ satisfies the original problem (22). □

Example 3.1.

$${}_0^C D_t^\alpha f(x, t) = -\frac{\partial(xf(x, t))}{\partial x} + \frac{\partial^2 \left(\frac{x^2 f(x, t)}{2} \right)}{\partial x^2}, x, t > 0, 0 < \alpha \leq 1 \tag{39}$$

By the initial condition

$$f(x, 0) = x, \tag{40}$$

$$f(x, t) = xe^t \tag{41}$$

is the solution for $\alpha = 1$. Utilizing the Sumudu transformation to Eq. (39) and using the definition (8) leads to

$$\mathbb{S}[f(x, t)] + w^\alpha \mathbb{S} \left[\frac{\partial(xf(x, t))}{\partial x} - \frac{\partial^2 \left(\frac{x^2 f(x, t)}{2} \right)}{\partial x^2} \right] = 0, t > 0 \tag{42}$$

The operator which is nonlinear becomes

$$N[\phi(x, t; p)] = \mathbb{S}[\phi(x, t; p)] + w^\alpha \mathbb{S} \left[\frac{\partial(x\phi(x, t; p))}{\partial x} - \frac{\partial^2 \left(\frac{x^2 \phi(x, t; p)}{2} \right)}{\partial x^2} \right] = 0, t > 0, 0 \leq p \leq 1 \tag{43}$$

TABLE 1. Comparison between approximate solution f_{SHAM} and exact solution f_{exact} for problem (39)-(40) in case of the Caputo at $t = 0.01$. (First 10 term)

x	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
0.25	0.028398296778751	0.006280575846505	$5.551115123125783 \times 10^{-17}$
0.5	0.056796593557503	0.012561151693010	$1.110223024625157 \times 10^{-16}$
0.75	0.085194890336254	0.018841727539515	0
1	0.113593187115006	0.025122303386020	$2.220446049250313 \times 10^{-16}$

and thus

$$R_m \left(\vec{f}_{m-1}, x, t \right) = \mathbb{S} [f_{m-1}(x, t)] + w^\alpha \mathbb{S} \left[\frac{\partial (x f_{m-1}(x, t))}{\partial x} - \frac{\partial^2 \left(\frac{x^2 f_{m-1}(x, t)}{2} \right)}{\partial x^2} \right] = 0, t > 0, \quad (44)$$

Utilizing the inverse Sumudu transformation to Eq. (31), we get

$$f_m(x, t) = \chi_m f_{m-1}(x, t) + \mathbb{S}^{-1} \left[h H(x, t) R_m \left(\vec{f}_{m-1}, x, t \right) \right] \quad (45)$$

Taking $H(x, t) = 1$ above and solving for $m = 1, 2, \dots$ leads to

$$\begin{aligned} f_1(x, t) &= \mathbb{S}^{-1} \left[h \left[w^\alpha \mathbb{S} \left[\frac{\partial (x f_0(x, t))}{\partial x} - \frac{\partial^2 \left(\frac{x^2 f_0(x, t)}{2} \right)}{\partial x^2} \right] \right] \right] = -xh \frac{t^\alpha}{\Gamma(\alpha + 1)}. \\ f_2(x, t) &= -xh \frac{t^\alpha}{\Gamma(\alpha + 1)} - xh^2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + xh^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}. \\ f_3(x, t) &= -xh \frac{t^\alpha}{\Gamma(\alpha + 1)} - 2xh^2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + 2xh^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - xh^3 \frac{t^\alpha}{\Gamma(\alpha + 1)} \\ &\quad + 2xh^3 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - xh^3 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \end{aligned}$$

Computing the first 10 terms allow us to construct a formula for the solution of Eq. (39) as

$$f(x, t) = f_0(x, t) + \sum_{m=1}^{\infty} f_m(x, t). \quad (46)$$

Taking $h = -1$ leads to the following approximate solution:

$$f(x, t) = x + x \frac{t^\alpha}{\Gamma(\alpha + 1)} + x \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + x \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots = x \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \quad (47)$$

For $\alpha = 1$,

$$f(x, t) = xe^t.$$

This solution is the same as exact solution for Eq. (39).

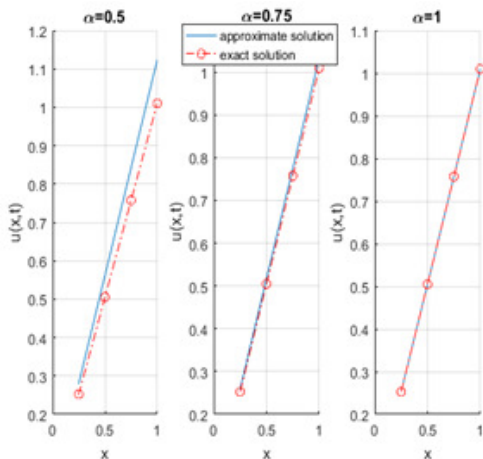


FIGURE 1. Figures of approximate solution and exact solution for $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 1$ at $t = 0.01$

4. Approximate solution of Caputo-Fabrizio time-fractional differential equation via SHAM

$${}_0^{CF}D_t^\alpha f(x, t) + \vartheta(x) \frac{\partial f(x, t)}{\partial x} + \gamma(x) \frac{\partial^2 f(x, t)}{\partial x^2} + \varphi(x) f(x, t) = \sigma(x, t), \quad (48)$$

where $(x, t) \in [0, 1] \times [0, \check{T}]$, $m - 1 < \alpha + n \leq m$, the initial positions are

$$\frac{\partial^i f(x, 0)}{\partial t^i} = f_i(x), i = 0, 1, \dots, n - 1, \quad (49)$$

and

$$f(0, t) = \epsilon_0(t), f(1, t) = \epsilon_1(t), t \geq 0, \quad (50)$$

Ignoring all boundary and initial conditions make the computation simpler. Now, the methodology involves using the Sumudu transformation on each side of the Eq. (48) to obtain

$$\begin{aligned} \mathbb{S}[f(x, t)] - \sum_{k=0}^{n-1} \left[\frac{1}{w^{-k}} \frac{\partial^k f(x, 0)}{\partial t^k} \right] + \frac{(1 - \alpha + \alpha w) w^n}{M(\alpha)} \left[\vartheta(x) \frac{\partial}{\partial x} + \gamma(x) \frac{\partial^2}{\partial x^2} + \varphi(x) \right] \\ \times \mathbb{S}[f(x, t)] - \frac{(1 - \alpha + \alpha w) w^n}{M(\alpha)} \mathbb{S}[\sigma(x, t)] = 0. \end{aligned} \quad (51)$$

The operator which is not linear becomes

$$\begin{aligned} N[\phi(x, t; p)] &= \mathbb{S}[\phi(x, t; p)] - \sum_{k=0}^{n-1} \left[\frac{1}{w^{-k}} \frac{\partial^k \phi(x, 0; p)}{\partial t^k} \right] \\ &+ \frac{(1 - \alpha + \alpha w) w^n}{M(\alpha)} \left[\vartheta(x) \frac{\partial}{\partial x} + \gamma(x) \frac{\partial^2}{\partial x^2} + \varphi(x) \right] \mathbb{S}[\phi(x, t; p)] \\ &- \frac{(1 - \alpha + \alpha w) w^n}{M(\alpha)} \mathbb{S}[\sigma(x, t)] = 0 \end{aligned} \quad (52)$$

where $\phi(x, t; p)$ is a real valued function and $p \in [0, 1]$ is an embedding parameter. By homotopy, we have

$$(1 - p) \mathbb{S}[\phi(x, t; p) - f_0(x, t)] = pHH(x, t)N[\phi(x, t; p)]; \quad (53)$$

where $\phi(x, t; p)$ is an unknown function, $H(x, t) \neq 0$, $h \neq 0$ is a auxiliary parameter and an auxiliary function $f_0(x, t)$ is an first prediction of $f(x, t)$. Auxiliary parameter can be choosen arbitrarily in SHAM. Clearly, If $p = 0$, $\phi(x, t; 0) = f_0(x, t)$ and if $p = 1$, $\phi(x, t; 1) = f(x, t)$. Thus, the solution converges to the solution $f(x, t)$ from first prediction $f_0(x, t)$ as p varies from 0 to 1. Now, writing $\phi(x, t; p)$ in the form of Taylor's series with respect to p leads to

$$\phi(x, t; p) = f_0(x, t) + \sum_{m=1}^{\infty} p^m f_m(x, t) \quad (54)$$

where

$$f_m(x, t) = \frac{1}{\Gamma(m+1)} \left. \frac{\partial^m \phi(x, t; p)}{\partial p^m} \right|_{p=0} \quad (55)$$

The parameter h controls the convergence of numerical solution (54). The series (54) converges at $p = 1$ if we make the right choices of necessary parameter and guesses.. From here,

$$f(x, t) = f_0(x, t) + \sum_{m=1}^{\infty} f_m(x, t) \quad (56)$$

which leads to one of the solutions of Eq. (52) is obtained. It is seen from the above expression that exact solution $f(x, t)$ and the initial guess $f_0(x, t)$ have a relationship in terms of $f_m(x, t)$, ($m = 1, 2, 3, \dots$). Differentiating Eq. (53) m times with respect to p , plugging $p = 0$, and multiplying by $\frac{1}{\Gamma(m+1)}$ leads to:

$$\mathbb{S}[f_m(x, t) - \chi_m f_{m-1}(x, t)] = hH(x, t) R_m(\vec{f}_{m-1}, x, t). \quad (57)$$

where

$$\mathbb{S}[f_m(x, t) - \chi_m f_{m-1}(x, t)] = hH(x, t) R_m(\vec{f}_{m-1}, x, t).$$

If both sides of Eq. (57) is operated the inverse Sumudu transform, then the expression below is obtained:

$$f_m(x, t) = \chi_m f_{m-1}(x, t) + \mathbb{S}^{-1} \left[hH(x, t) R_m(\vec{f}_{m-1}, x, t) \right] \quad (58)$$

where

$$R_m(\vec{f}_{m-1}, x, t) = \frac{1}{\Gamma(m)} \left. \frac{\partial^{m-1} \phi(x, t; p)}{\partial p^{m-1}} \right|_{p=0} \quad (59)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{60}$$

In our case

$$\begin{aligned} R_m \left(\vec{f}_{m-1}, x, t \right) &= {}_0^{CF}D_t^\alpha f_{m-1}(x, t) + \vartheta(x) \frac{\partial f_{m-1}(x, t)}{\partial x} + \gamma(x) \frac{\partial^2 f_{m-1}(x, t)}{\partial x^2} \\ &+ \varphi(x) f_{m-1}(x, t) - (1 - \chi_m)\sigma(x, t) \end{aligned} \tag{61}$$

$f_m(x, t)$ for $m \geq 1$, at M^{th} order can be obtained easily from (58) which leads to accurate approximation of the Eq. (48)

$$f(x, t) = \sum_{m=0}^M f_m(x, t) \tag{62}$$

as $M \rightarrow \infty$.

Example 4.1.

$${}_0^{CF}D_t^\alpha f(x, t) = -\frac{\partial(xf(x, t))}{\partial x} + \frac{\partial^2\left(\frac{x^2 f(x, t)}{2}\right)}{\partial x^2}, \quad x, t > 0, \quad 0 < \alpha \leq 1 \tag{63}$$

By the initial condition

$$f(x, 0) = x, \tag{64}$$

$f(x, t) = xe^t$ is the solution for $\alpha = 1$. Utilizing the Sumudu transformation to Eq. (63) and using the definition (8) leads to

$$\mathbb{S}[f(x, t)] - f(x, 0) + \frac{(1 - \alpha + \alpha w)w}{M(\alpha)} \mathbb{S} \left[\frac{\partial(xf(x, t))}{\partial x} - \frac{\partial^2\left(\frac{x^2 f(x, t)}{2}\right)}{\partial x^2} \right] = 0, \quad t > 0. \tag{65}$$

The operator which is nonlinear becomes

$$\begin{aligned} N[\phi(x, t; p)] &= \mathbb{S}[\phi(x, t; p)] + \frac{(w - \alpha w + \alpha w^2)}{M(\alpha)} \mathbb{S} \left[\frac{\partial(x\phi(x, t; p))}{\partial x} - \frac{\partial^2\left(\frac{x^2 \phi(x, t; p)}{2}\right)}{\partial x^2} \right] = 0, \\ t > 0, 0 \leq p \leq 1 \end{aligned} \tag{66}$$

and thus

$$\begin{aligned} R_m \left(\vec{f}_{m-1}, x, t \right) &= \mathbb{S}[f_{m-1}(x, t)] + \frac{(w - \alpha w + \alpha w^2)}{M(\alpha)} \mathbb{S} \left[\frac{\partial(xf_{m-1}(x, t))}{\partial x} - \frac{\partial^2\left(\frac{x^2 f_{m-1}(x, t)}{2}\right)}{\partial x^2} \right] \\ &= 0, t > 0, \end{aligned} \tag{67}$$

Utilizing the inverse Sumudu transformation to Eq. (57), we get

$$\mathbb{S}[f_m(x, t) - \chi_m f_{m-1}(x, t)] = hH(x, t)R_m \left(\vec{f}_{m-1}, x, t \right).$$

Applying the inverse Sumudu transform, we have

$$f_m(x, t) = \chi_m f_{m-1}(x, t) + \mathbb{S}^{-1} \left[hH(x, t)R_m \left(\vec{f}_{m-1}, x, t \right) \right] \tag{68}$$

TABLE 2. Comparison between approximate solution f_{SHAM} and exact solution f_{exact} for problem (63)-(64) in case of the Caputo-Fabrizio at $x = 0.05$. (First 10 term)

$\alpha = 1$			
t	f_{SHAM}	f_{exact}	$ f_{SHAM} - f_{exact} $
0.01	0.050002500020833	0.050502508354208	$5.000083333749905 \times 10^{-4}$
0.05	0.050062513021918	0.052563554818801	$2.501041796883 \times 10^{-3}$
0.1	0.050250208402790	0.055258545903782	$5.008337500992 \times 10^{-4}$
0.15	0.050563555478834	0.058091712136414	$7.528156657581 \times 10^{-3}$
0.2	0.051003337780954	0.061070137908009	$1.0066800127055 \times 10^{-2}$

Taking $H(x, t) = 1$ above and solving for $m = 1, 2, \dots$ leads to

$$\begin{aligned}
 f_1(x, t) &= \mathbb{S}^{-1} \left[h \left[\frac{(w - \alpha w + \alpha w^2)}{M(\alpha)} \mathbb{S}[-x] \right] \right] \\
 &= \frac{1}{M(\alpha)} \left[-hx \frac{t}{\Gamma(1+1)} + hx\alpha \frac{t}{\Gamma(1+1)} - hx\alpha \frac{t^2}{\Gamma(2+1)} \right]. \\
 f_2(x, t) &= \frac{xh}{M(\alpha)} \left[-\frac{t}{\Gamma(2)} + \alpha \frac{t}{\Gamma(2)} - \alpha \frac{t^2}{\Gamma(3)} \right] + \frac{xh^2}{M(\alpha)} \left[-\frac{t}{\Gamma(2)} + \alpha \frac{t}{\Gamma(2)} - \alpha \frac{t^2}{\Gamma(3)} \right] \\
 &+ \frac{h^2x}{M^2(\alpha)} \left[\left(\frac{t^2}{\Gamma(3)} - \alpha \frac{t^2}{\Gamma(3)} + \alpha \frac{t^3}{\Gamma(4)} \right) + \left(-\alpha \frac{t^2}{\Gamma(3)} + \alpha^2 \frac{t^2}{\Gamma(3)} - \alpha^2 \frac{t^3}{\Gamma(4)} \right) + \left(\alpha \frac{t^3}{\Gamma(4)} - \alpha^2 \frac{t^3}{\Gamma(4)} + \alpha^2 \frac{t^4}{\Gamma(5)} \right) \right]. \\
 f_3(x, t) &= \frac{xh}{M(\alpha)} \left[-\frac{t}{\Gamma(2)} + \alpha \frac{t}{\Gamma(2)} - \alpha \frac{t^2}{\Gamma(3)} \right] + \frac{xh^2}{M(\alpha)} \left[-\frac{t}{\Gamma(2)} + \alpha \frac{t}{\Gamma(2)} - \alpha \frac{t^2}{\Gamma(3)} \right] \\
 &+ \frac{h^2x}{M^2(\alpha)} \left[\left(\frac{t^2}{\Gamma(3)} - \alpha \frac{t^2}{\Gamma(3)} + \alpha \frac{t^3}{\Gamma(4)} \right) + \left(-\alpha \frac{t^2}{\Gamma(3)} + \alpha^2 \frac{t^2}{\Gamma(3)} - \alpha^2 \frac{t^3}{\Gamma(4)} \right) + \left(\alpha \frac{t^3}{\Gamma(4)} - \alpha^2 \frac{t^3}{\Gamma(4)} + \alpha^2 \frac{t^4}{\Gamma(5)} \right) \right] \\
 &+ \frac{xh^2}{M(\alpha)} \left[-\frac{t}{\Gamma(2)} + \alpha \frac{t}{\Gamma(2)} - \alpha \frac{t^2}{\Gamma(3)} \right] + \frac{xh^3}{M(\alpha)} \left[-\frac{t}{\Gamma(2)} + \alpha \frac{t}{\Gamma(2)} - \alpha \frac{t^2}{\Gamma(3)} \right] \\
 &+ \frac{h^3x}{M^2(\alpha)} \left[\left(\frac{t^2}{\Gamma(3)} - \alpha \frac{t^2}{\Gamma(3)} + \alpha \frac{t^3}{\Gamma(4)} \right) + \left(-\alpha \frac{t^2}{\Gamma(3)} + \alpha^2 \frac{t^2}{\Gamma(3)} - \alpha^2 \frac{t^3}{\Gamma(4)} \right) + \left(\alpha \frac{t^3}{\Gamma(4)} - \alpha^2 \frac{t^3}{\Gamma(4)} + \alpha^2 \frac{t^4}{\Gamma(5)} \right) \right] \\
 &+ \frac{xh^2}{M^2(\alpha)} \left[(1 - 2\alpha + \alpha^2) \frac{t^2}{\Gamma(3)} + (\alpha - \alpha^2) \frac{t^3}{\Gamma(4)} \right] + \frac{xh^3}{M^2(\alpha)} \left[(1 - 2\alpha + \alpha^2) \frac{t^2}{\Gamma(3)} + (\alpha - \alpha^2) \frac{t^3}{\Gamma(4)} \right] \\
 &+ \frac{xh^3}{M^3(\alpha)} \left[(-1 + 3\alpha - 3\alpha^2 + \alpha^3) \frac{t^3}{\Gamma(4)} + (-3\alpha + 6\alpha^2 - 3\alpha^3) \frac{t^4}{\Gamma(5)} + (-3\alpha^2 + 3\alpha^3) \frac{t^5}{\Gamma(6)} - \alpha^3 \frac{t^6}{\Gamma(7)} \right].
 \end{aligned}$$

Computing the first 10 terms allow us to construct a formula for the solution of Eq. (63) as

$$f(x, t) = f_0(x, t) + \sum_{m=1}^{\infty} f_m(x, t). \tag{69}$$

Taking $h = -1$ leads to the following approximate solution:

$$\begin{aligned}
 f(x, t) &= x + \frac{x}{M(\alpha)} \left[\frac{t}{\Gamma(2)} - \alpha \frac{t}{\Gamma(2)} + \alpha \frac{t^2}{\Gamma(3)} \right] \\
 &+ \frac{x}{M^2(\alpha)} \left[\left(\frac{t^2}{\Gamma(3)} - \alpha \frac{t^2}{\Gamma(3)} + \alpha \frac{t^3}{\Gamma(4)} \right) + \left(-\alpha \frac{t^2}{\Gamma(3)} + \alpha^2 \frac{t^2}{\Gamma(3)} - \alpha^2 \frac{t^3}{\Gamma(4)} \right) + \left(\alpha \frac{t^3}{\Gamma(4)} - \alpha^2 \frac{t^3}{\Gamma(4)} + \alpha^2 \frac{t^4}{\Gamma(5)} \right) \right] \\
 &- \frac{x}{M^3(\alpha)} \left[(-1 + 3\alpha - 3\alpha^2 + \alpha^3) \frac{t^3}{\Gamma(4)} + (-3\alpha + 6\alpha^2 - 3\alpha^3) \frac{t^4}{\Gamma(5)} + (-3\alpha^2 + 3\alpha^3) \frac{t^5}{\Gamma(6)} - \alpha^3 \frac{t^6}{\Gamma(7)} \right].
 \end{aligned}$$

For $\alpha \rightarrow 1$ and $n \rightarrow \infty$

$$f(x, t) = x + x \frac{t^2}{\Gamma(3)} + x \frac{t^4}{\Gamma(5)} + x \frac{t^6}{\Gamma(7)} + \dots = x \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = x \cosh(t). \tag{70}$$

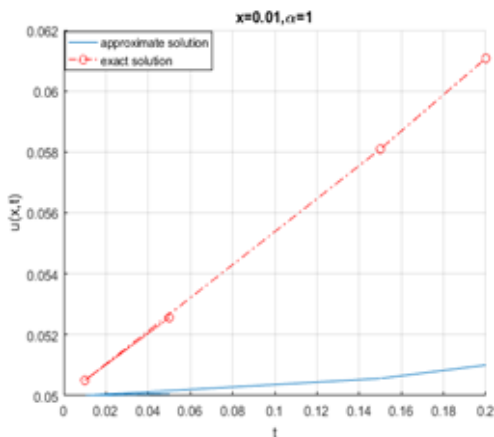


FIGURE 2. Graphs of approximate solution and exact solution

5. Conclusion

In this study the SHAM has been utilized in order to construct numerical solution of Caputo and Caputo-Fabrizio time-fractional Fokker-Planck equations. We have compared the approximate solutions received in the sight of SHAM with those outcomes received from the exact analytical solutions. This operation indicates an accurate understanding between the SHAM and exact outcomes. It is clear that the SHAM gives accurate and convergent series solutions applying only a few iterations in every two fractional derivative. Since the Sumudu transform permits one to get over the deficiency chiefly produced by unsatisfied boundary or initial conditions, the SHAM is a stronger method that requires inferior calculation time and this method is much more useful than the HPM. It is clear from the tables and solution graphics of the examples that the approximate solution of Caputo fractional Fokker-Planck equation get closer to the exact solution. However the approximate solution of Caputo-Fabrizio Fokker-Planck equation with time fractional derivative does not converge to the exact solution. As a result Fokker-Planck equation with Caputo time fractional derivative models the time evolution of the probability density function much more better than Fokker-Planck equation with Caputo-Fabrizio time fractional derivative.

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