Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 30(2), 2003, Pages 100–108
ISSN: 1223-6934

A representation of an uncertain body of evidence

Ion Iancu

Abstract. In this paper we present a kind of pairs (t-norm, t-conorm) dual with respect to a strong negation with n-threshold $a_1, ..., a_n \in (0, 1)$, $a_1 < a_2 < ... < a_n$. In this way we obtain an extension of operators with $1-$ and $2-$threshold from some of our papers. The new pair is obtained from given one.

2000 Mathematics Subject Classification. 68T35, 68T27, 68T30.
Key words and phrases. fuzzy sets, t-operators, possibility and necessity measures, belief functions, uncertainty, imprecision.

1. Introduction

Probability Theory, Shafer's belief theory and Zadeh's possibility theory are the main methods for study the uncertainty. Considering an event, we may to evaluate its probability, its feasibility, its possibility of occurence or how much it seems credible. All those evaluations are based on some sets of numbers, generally normalized in some sense, which have to combined in accordance with the characteristic axioms of those theories.

In 1972 Sugeno [19] introduced the concept of fuzzy measure in order to depart from the too rigid framework of probability theory.

Definition 1.1. *Given the universe X (supposed to be finite, for sake of simplicity) a fuzzy measure is a set function g from an algebra $\mathcal{A}$ (e.g., the set $\mathcal{P}(X)$ of subsets of X) defined on X to the interval $[0,1]$, such that*
i) $g(\emptyset) = 0$, *ii)* $g(X) = 1$
iii) $\forall A, B \in \mathcal{A}$, *if* $A \subset B$ *then* $g(A) \leq g(B)$.

The following inequalities hold

$$\forall A, B \in \mathcal{A}, \;\; g(A \cap B) \leq \min(g(A), g(B))$$
$$\forall A, B \in \mathcal{A}, \;\; g(A \cup B) \geq \max(g(A), g(B)).$$

The axioms from Definition 1.1 are very general and it is necessary to be particularised in order to obtain various classes of fuzzy measures. Thus, if

$$\forall A, B \in \mathcal{A}, \;\; g(A \cap B) = \min(g(A), g(B))$$

we obtain a *necessity measure* and for

$$\forall A, B \in \mathcal{A}, \;\; g(A \cup B) = \max(g(A), g(B))$$

we have a *possibility measure*.

In order to combine the uncertainties, the following axiom seems natural:

$$\forall A, B \in \mathcal{A}, \text{ if } A \cap B = \emptyset \text{ then } g(A \cup B) = g(A) * g(B) \tag{1}$$

Received: 31 October 2003.

where $*$ is some operator under which $[0,1]$ is closed. The algebraic structure of $\mathcal{A}$ induces the choosing $*$ among the triunghiular conorms.

If g is a fuzzy measure, the set function g' defined by

$$\forall A \in \mathcal{A},\, g'(A) = 1 - g(\bar{A})$$

is also a fuzzy measure; this last relation expresses the duality between g and g'. If g is a conorm-based set function satisfying (1), then the characteristic property of dual measure g' is

$$\forall A, B \in \mathcal{A}, \text{ if } A \cup B = X \text{ then } g\left(A \cap B\right) = g\left(A\right) \perp g\left(B\right) \tag{2}$$

where $\perp$ denotes the triangular norm dual of $*$ defined by

$$a \perp b = 1 - (1-a) * (1-b)\,.$$

Well-known t-norms are

$$T_M(x,y) = \min(x,y),\; T_P(x,y) = xy,\; T_L(x,y) = \max(0, x+y-1)$$

and well-known t-conorms are

$$S_M(x,y) = \max(x,y),\; S_P(x,y) = x+y-xy,\; S_L(x,y) = \min(1, x+y).$$

The relations (1) and (2) prove the importance of t-norms and t-conorms in representation of composed information from a body of evidence. For this reason, the construction of new t-norms and t-conorms seems to be an important tool not only for the theory but also for the applications. The construction of the new operators from given ones is a technique used in various paper [17], [18], [16], [4], [15], [5], [6], [7], [8], [9], [10], [12], [21], [11], [13]. From this kind we recall the operators of Pacholczyk type, named with threshold. They were introduced in order to improve the calculus of uncertainty in an expert system.

Given a pair (t-norm, t-conorm) denoted as (T,S) and the parameter $a \in (0,1)$, the first result of this type were presented under the form [1], [16]

$t-norm:$

$$T_a(x,y) = \begin{cases} \dfrac{a}{1-a} T\left(\dfrac{1-a}{a}x, \dfrac{1-a}{a}y\right) & if \quad x \le a \quad and \quad y \le a \\ \min(x,y) & if \quad x > a \quad or \quad y > a \end{cases}$$

corresponding to t-norm $T(x,y)$;

$t-conorm:$

$$S_a(x,y) = \begin{cases} S(x,y) & if \quad x \ge a \quad and \quad y \ge a \\ \max(x,y) & if \quad x < a \quad or \quad y < a \end{cases}$$

corresponding to t-conorm $S(x,y)$.

T_a and S_a are dual operators with respect to the negation with threshold

$$C_a(x) = \begin{cases} 1 - \dfrac{1-a}{a}x & if \quad x \le a \\ \dfrac{a}{1-a}(1-x) & if \quad x \ge a. \end{cases}$$

Such operators were used, with very satisfactory results, to construct the system SEQUI [1]- an expert system for processing the uncertain questions. In [5] we presented two kinds of operators with threshold: the first is a generalization of Pacholczyk's result [16] and the second is a new one.

In [6] we introduced the notion of operators with double threshold (in the Pacholczyk's meaning) and we presented two types of such operators; another types are given in [7] and [9]. In this paper we present a generalization of our results from [9].

2. Preliminaries

In the begining, we mention some definitions which will be used in the next section.

Definition 2.1. A t-norm T is an increasing, associative and commutative function from $[0,1]^2$ into $[0,1]$ that satisfies the boundary condition: $T(x,1)=x \quad \forall x\in[0,1]$. A continuous t-norm T is called Archimedean if $T(x,x)<x \quad \forall x\in(0,1)$.

Any t-norm T satisfies the relation $T(x,y)\leq T_M(x,y)=\min(x,y)$.

Definition 2.2. A t-conorm S is an increasing, associative and commutative function from $[0,1]^2$ into [0, 1] that satisfies the boundary condition: $S(x,0)=x \quad \forall x\in[0,1]$. A continuous t-conorm S is called Archimedean if $S(x,x)>x \quad \forall x\in(0,1)$.

Any t-conorm S satisfies the relation $S(x,y)\geq S_M(x,y)=\max(x,y)$.

Definition 2.3. A strong negation is an involutive decreasing function from [0, 1] into itself.

Theorem 2.1. [2] *If T is a t-norm and C is a strong negation then*

$$S(x,y)=C(T(C(x),C(y)))$$

is a t-conorm and reciprocally,

$$T(x,y)=C(S(C(x),C(y)));$$

namely, T and S are $C-$dual.

In order to obtain t-norms and negations one can used the following two theorems.

Theorem 2.2. [5] *Let $f:[0,1]\to I\subseteq[0,\infty)$ be a continuous strictly decreasing function and $\Delta:I\times I\to I$ with the following properties:*

(2.1.1) $\Delta(x,y)=\Delta(y,x)$,

(2.1.2) $\Delta(x,\Delta(y,z))=\Delta(\Delta(x,y),z)$,

(2.1.3) $\Delta(x,y)\leq\Delta(x,z)$ *if* $y\leq z$ *with equality iff* $y=z$,

(2.1.4) Δ *is continuous*

(2.1.5) $\Delta(x,f(1))=x$ *for all* $x,y,z\in I$ *. Then*

$$T(x,y)=f^{(-1)}(\Delta(f(x),f(y))) \qquad \forall x,y\in[0,1]$$

is a t-norm, where $f^{(-1)}$ is the pseudo-inverse of f , extended to the case $f(1)>0$:

$$f^{(-1)}(x)=\begin{cases}1 & \text{if } x\in[0,f(1)]\\ f^{-1}(x) & \text{if } x\in[f(1),f(0)]\\ 0 & \text{if } x\in[f(0),\infty)\end{cases}$$

Remark 2.1. For $\Delta=+$ and $f(1)=0$ we obtain the Ling′s result [14] for continuous Archimedean t-norms.

Example 2.1. For $I=[0,\infty)$, $\Delta(x,y)=x+y+xy$ and $f(x)=1-x$ in Theorem 2.2 we obtain the t-norm $T(x,y)=\max(0,2x+2y-xy-2)$.

Theorem 2.3. [5]*Let $I\subseteq\mathbb{R}$ and $\Delta:I\times I\to I$ be an application satisfying the following conditions, for all $x,y,z\in I$:*

(2.2.1) - (2.2.4) *identical to (2.1.1)-(2.1.4)*

(2.2.5) *there is* $e\in I$ *such that* $\Delta(x,e)=x \quad \forall x\in I$

(2.2.6) $\forall x\in I$ *there is* $x'\in I$ *such that* $\Delta(x,x')=e$ *and* $\varphi:I\to I$, $\varphi(x)=x'$ *is a continuous strictly decreasing function*

(2.2.7) *let* $J=[0,\infty)\subseteq I$ *and* $t:[0,1]\to J$ *be a continuous strictly increasing function with* $t(0)=e$ *and* $t(1)$ *is a finite number.*

Then $C(x)=t^{-1}(\Delta(t(1),\varphi(t(x))))$ *is a strong negation for every* $x\in[0,1]$.

Remark 2.2. *For $\Delta = +$ and $t(x) = x$ we obtain the Trillas's result* [20].

Example 2.2. For $I = \mathbb{R}$, $\Delta(x, y) = x + y - 1$, $e = 1$, $J = [1, \infty)$, $t(x) = \dfrac{2x+1}{x+1}$ and $\varphi(x) = 2 - x$ we obtain, from Theorem 2.3, $C(x) = \dfrac{1-x}{1+3x}$.

Remark 2.3. Simultaneously using of functions Δ and f (respectively t) in previous lemmas allows obtaining of t-norms (respectively negations) on a easier way than in the case $\Delta = +$. For instance, if $\Delta(x, y) = x + y$ we don't work with functions of the type $f(x) = ax + b$ in order to obtain the t-norm from Example 2.1, being necessary more complicated forms.

3. Operators with $n-$threshold

We remain in the conditions of Theorem 2.3 and we use the operation $\oplus : I \times I \to I$ instead of Δ; therefore we write $x \oplus y = \Delta(x, y)$. We take $\otimes : I \times I \to I$ with the following properties:
(i) $x \otimes y < x \otimes z$ iff $y < z \quad \forall x, y, z \in I$ and $x > e$,
(ii) $(I, \oplus, \otimes)$ is a field.
We note $\ominus x$ and $\dfrac{1}{x}$ the inverse element of x corresponding to $\oplus$ and $\otimes$, respectively. For the simplification of writing we note $x \otimes \dfrac{1}{y} = \dfrac{x}{y}$ and $x \otimes x = x^2$.

Theorem 3.1. *If the operations $\oplus$ and $\otimes$ and the function t have the previous meaning and $n \in \mathbb{N}$, $n \geq 1$, $0 < a_1 < a_2 < ... < a_n < 1$,*

$$\delta(i) = \frac{t(a_{n-i}) \ominus t(a_{n-i+1})}{t(a_{i+1}) \ominus t(a_i)} \qquad \textit{and} \quad \theta(i) = \frac{t(a_{i+1}) \otimes t(a_{n-i+1}) \ominus t(a_i) \otimes t(a_{n-i})}{t(a_{i+1}) \ominus t(a_i)}$$

then

$$C_{a_1,\ldots,a_n}(x) = \begin{cases} t^{-1}\left(\dfrac{t\,(a_1) \otimes t\,(a_n) \otimes t\,(1)}{t\,(a_1) \otimes t\,(a_n) \oplus (t\,(1) \ominus t\,(a_n)) \otimes t\,(x)}\right) \\ \qquad\qquad\qquad\qquad \textit{if} \quad x \leq a_1 \\ t^{-1}(t(x) \otimes \delta(i) \oplus \theta(i)) \quad \textit{if} \quad a_i \leq x \leq a_{i+1} \quad \textit{and} \quad 1 \leq i < n \\ t^{-1}\left(\dfrac{t\,(a_1) \otimes t\,(a_n) \otimes (t\,(1) \ominus t\,(x))}{(t\,(1) \ominus t\,(a_n)) \otimes t\,(x)}\right) \\ \qquad\qquad\qquad\qquad \textit{if} \quad x \geq a_n \end{cases}$$

is strong negation having $t^{-1}\left(\dfrac{t\left(a_{\left[\frac{n}{2}\right]+1}\right) \oplus t\left(a_{\left[\frac{n+1}{2}\right]}\right)}{2}\right)$ *as fixed point, where $[x]$ is the greatest integer which is smaller than or equal to x.*

Proof. It is easy to verify the demands from definition of negation. For instance, for involution property one proves first that
(i): $x \leq a_1 \Leftrightarrow C_{a_1,\ldots,a_n}(x) \geq a_n$
(ii): $x \geq a_n \Leftrightarrow C_{a_1,\ldots,a_n}(x) \leq a_1$
(iii): $x \in [a_i, a_{i+1}] \Leftrightarrow C_{a_1,\ldots,a_n}(x) \in [a_{n-i}, a_{n-i+1}]$, $\forall i \in \{1, 2, ..., n-1\}$.
After, these relations are used to verify (by a simple calculus) the demands from Definition 2.3. The fixed point becomes $t^{-1}\left(\dfrac{t\,(a_k) \oplus t\,(a_{k+1})}{2}\right)$ for $n = 2k$ and a_{k+1} for $n = 2k + 1$. $\square$

Remark 3.1. The relation (ii) says that if the confidence in a proposition p is greater than or equal to the threshold a_n, then the confidence in *non* p is smaller than or equal to the threshold a_1.

Example 3.1. For $\oplus = +$, $\otimes = \times$, $n = 2$ and $t(x) = \dfrac{2x}{x+1}$ we obtain

$$C_{a_1,a_2}(x) = \begin{cases} \dfrac{a_1a_2(1+x)}{a_1a_2 + x(1+a_1-a_2)} & \textit{if} \quad x \le a_1 \\ \dfrac{(a_1a_2-1)x + a_1 + a_2 + 2a_1a_2}{x(a_1+a_2+2)+1-a_1a_2} & \textit{if} \quad a_1 \le x \le a_2 \\ \dfrac{(1-x)a_1a_2}{x(1+a_1-a_2)-a_1a_2} & \textit{if} \quad x \ge a_2 \end{cases}$$

Theorem 3.2. *Let S be a t-conorm and S' one from the t-conorms S_M or S. For $0 < a_1 < ... < a_n < 1$*

$$S_{a_1,\ldots,a_n;S'}(x,y) = \begin{cases} \max(x,y) & \textit{if} \quad x < a_1 \quad \textit{or} \quad y < a_1 \\ S(x,y) & \textit{if} \quad x \ge a_n \quad \textit{and} \quad y \ge a_n \\ S'(x,y) & \quad \textit{otherwise} \end{cases}$$

is a t-conorm.

Proof. For $S' \equiv S$ we obtain

$$S_{a_1,\ldots,a_n;S}(x,y) = \begin{cases} \max(x,y) & \textit{if} \quad x < a_1 \quad \textit{or} \quad y < a_1 \\ S(x,y) & \textit{if } x \ge a_1 \textit{ and } y \ge a_1 \end{cases}$$

that is the Pacholczyk's t-conorm S_{a_1}[16].

For $S' \equiv S_M$ we obtain

$$S_{a_1,\ldots,a_n;S_M}(x,y) = \begin{cases} \max(x,y) & \textit{if} \quad x < a_n \quad \textit{or} \quad y < a_n \\ S(x,y) & \textit{if } x \ge a_n \textit{ and } y \ge a_n \end{cases}$$

that is the Pacholczyk's t-conorm S_{a_n}[16]. □

Remark 3.2. We cannot choose S' as an arbitrary t-conorm because the associativity property is not always verified. For instance, for $n = 2$ and $a_1 < x < a_2 < y < z$ we have

$$S_{a_1,a_2;S'}(S_{a_1,a_2;S'}(x,y),z) = S_{a_1,a_2;S'}(x, S_{a_1,a_2;S'}(y,z))$$

which is equivalent with

$$S(S'(x,y),z) = S'(x, S(y,z)).$$

But for $S(x,y) = \max(x,y)$ and $S'(x,y) = xy$ the last equality becomes

$$\max(xy,z) = x \times \max(y,z) \Leftrightarrow z = xz$$

which is false because from $0 < x < a_2 < z < 1$ we obtain $x < 1$ and $z > 0$.

Theorem 3.3. *Let (T,S) be a pair (t-norm, t-conorm) dual with respect to $C(x) = t^{-1}(t(1) \ominus t(x))$, $\quad S' \in \{S_M, S\}$ and T' is the dual t-norm of S' with respect to the* *same negation (i. e. $T' \in \{T_M, T\}$). For $0 < a_1 < a_2 < ... < a_n < 1$, we define*

$$T_{a_1,\dots a_n;T'}(x,y) = \begin{cases} t^{-1}\left(k \otimes \dfrac{t\left(T\left(\alpha\left(x\right),\alpha\left(y\right)\right)\right)}{t\left(1\right) \ominus t\left(T\left(\alpha\left(x\right),\alpha\left(y\right)\right)\right)}\right) \\ \qquad \textit{if } x \le a_1 \textit{ and } y \le a_1 \\ t^{-1}\left(k \otimes \dfrac{t\left(T'\left(\alpha\left(x\right),\beta\left(y,i\right)\right)\right)}{t\left(1\right) \ominus t\left(T'\left(\alpha\left(x\right),\beta\left(y,i\right)\right)\right)}\right) \\ \qquad \textit{if } x \le a_1 \textit{ and } y \in (a_i, a_{i+1}], 1 \le i \le n-1 \\ t^{-1}\left(k \otimes \dfrac{t\left(T'\left(\beta\left(x,i\right),\alpha\left(y\right)\right)\right)}{t\left(1\right) \ominus t\left(T'\left(\beta\left(x,i\right),\alpha\left(y\right)\right)\right)}\right) \\ \qquad \textit{if } y \le a_1 \textit{ and } x \in (a_i, a_{i+1}], 1 \le i \le n-1 \\ t^{-1}\left((t(1) \ominus t(T'(\beta(x,i),\beta(y,j)))) \otimes \delta(k) \oplus \theta(k)\right) \\ \qquad \textit{if } x \in (a_i, a_{i+1}], y \in (a_j, a_{j+1}],\ l = \max(i,j) \textit{ and} \\ \qquad \textit{there is an integer } k \in [n-l, n-1] \textit{ such that} \\ \qquad T'(\beta(x,i),\beta(y,j)) \in [C(a_{k+1}), C(a_k)) \\ t^{-1}\left(k \otimes \dfrac{t\left(T'\left(\beta\left(x,i\right),\beta\left(y,j\right)\right)\right)}{t\left(1\right) \ominus t\left(T'\left(\beta\left(x,i\right),\beta\left(y,j\right)\right)\right)}\right) \\ \qquad \textit{if } x \in (a_i, a_{i+1}], y \in (a_j, a_{j+1}], T'(\beta(x,i),\beta(y,j)) < C(a_n) \\ \min(x,y) \qquad \textit{if } x > a_n \textit{ or } y > a_n. \end{cases}$$

where $k = \dfrac{t\left(a_1\right) \otimes t\left(a_n\right)}{t\left(1\right) \ominus t\left(a_n\right)}$, $\alpha\left(z\right) = t^{-1}\left(\dfrac{\left(t\left(1\right) \ominus t\left(a_n\right)\right) \otimes t\left(z\right) \otimes t\left(1\right)}{t\left(a_1\right) \otimes t\left(a_n\right) \oplus \left(t\left(1\right) \ominus t\left(a_n\right)\right) \otimes t\left(z\right)}\right)$ *and* $\beta\left(z,i\right) = t^{-1}\left(t\left(1\right) \ominus t\left(z\right) \otimes \delta\left(i\right) \ominus \theta\left(i\right)\right)$, $\delta\left(i\right)$ *and* $\theta\left(i\right)$ *having the significance from Theorem 3.1.*

Then $T_{a_1,\dots a_n;T'}$ is a t-norm $C_{a_1,\dots,a_n}$ - dual with t-conorm $S_{a_1,\dots a_n;S'}$.

Proof. Because $S_{a_1,\dots a_n;S'}$ is a t-conorm, in order to prove the theorem it is sufficient to verify, in accordance with Theorem 2.1, the equality

$$T_{a_1,\dots a_n;T'}(x,y) = C_{a_1,\dots,a_n}(S_{a_1,\dots a_n;S'}(C_{a_1,\dots,a_n}(x), C_{a_1,\dots,a_n}(y))).$$

The same theorem yields

$$S(x,y) = C(T(C(x),C(y))) = t^{-1}(t(1) \ominus t(T(C(x),C(y)))). \tag{3}$$

In order to simplify the writing we denote

$$\eta(z,i) = t^{-1}\left(t(x) \otimes \delta\left(i\right) \oplus \theta\left(i\right)\right)$$

and we analyze the following cases:

i1): For $x, y \le a_1$ we have

$$C_{a_1,\dots,a_n}(x) \ge a_n, \quad C_{a_1,\dots,a_n}(y) \ge a_n \text{ and}$$
$$S_{a_1,\dots a_n;S'}(C_{a_1,\dots,a_n}(x), C_{a_1,\dots,a_n}(y)) = S(C_{a_1,\dots,a_n}(x), C_{a_1,\dots,a_n}(y)) \ge \max(C_{a_1,\dots,a_n}(x), C_{a_1,\dots,a_n}(y)) \ge a_n.$$

Using the last relations and the identity (3) we have

$$C_{a_1,\dots,a_n}(S_{a_1,\dots a_n;S'}(C_{a_1,\dots,a_n}(x), C_{a_1,\dots,a_n}(y))) =$$

$$t^{-1}\left(\frac{t(a_1) \otimes t(a_n)}{t(1) \ominus t(a_n)} \otimes \frac{t(1) \ominus t\left(S\left(C_{a_1,\dots,a_n}\left(x\right), C_{a_1,\dots,a_n}\left(y\right)\right)\right)}{t\left(S\left(C_{a_1,\dots,a_n}\left(x\right), C_{a_1,\dots,a_n}\left(y\right)\right)\right)}\right) =$$

$$= t^{-1}\left(k \otimes \frac{t\left(T\left(C\left(C_{a_1,\dots,a_n}\left(x\right)\right), C\left(C_{a_1,\dots,a_n}\left(y\right)\right)\right)\right)}{t\left(1\right) \ominus t\left(T\left(C\left(C_{a_1,\dots,a_n}\left(x\right)\right), C\left(C_{a_1,\dots,a_n}\left(y\right)\right)\right)\right)}\right) =$$

$$= t^{-1}\left(k \otimes \frac{t(T(\alpha(x),\alpha(y)))}{t\,(1) \ominus t(T(\alpha(x),\alpha(y)))}\right).$$

i2): For $x \leq a_1$ and $y \in (a_i, a_{i+1}]$, $1 \leq i \leq n-1$, we have
$C_{a_1,\ldots,a_n}(x) \geq a_n$, $C_{a_1,\ldots,a_n}(y) \in [a_{n-i}, a_{n-i+1})$ and

$$S_{a_1,\ldots a_n;S'}(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)) = S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)) \geq$$

$$\geq \max(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)) \geq a_n.$$

Using the same reasoning as in the previous case, we have

$$C_{a_1,\ldots,a_n}(S_{a_1,\ldots a_n;S'}(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y))) =$$

$$= C_{a_1,\ldots,a_n}(S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y))) =$$

$$= t^{-1}\left(k \otimes \frac{t\,(1) \ominus t(S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)))}{t(S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)))}\right) =$$

$$= t^{-1}\left(k \otimes \frac{t(T'(C\,(C_{a_1,\ldots,a_n}(x))\,, C\,(C_{a_1,\ldots,a_n}(y))))}{t\,(1) \ominus t(T'(C\,(C_{a_1,\ldots,a_n}(x))\,, C\,(C_{a_1,\ldots,a_n}(y))))}\right) =$$

$$= t^{-1}\left(k \otimes \frac{t(T'(\alpha(x),\beta(y,i)))}{t\,(1) \ominus t(T'(\alpha(x),\beta(y,i)))}\right).$$

i3): The case $x \in (a_i, a_{i+1}]$ and $y \leq a_1$ is similar to i2).
i4): For $x \in (a_i, a_{i+1}]$ and $y \in (a_j, a_{j+1}]$, $1 \leq i, j \leq n-1$ we have

$$C_{a_1,\ldots,a_n}(x) \in [a_{n-i}, a_{n-i+1}),\;\; C_{a_1,\ldots,a_n}(y) \in [a_{n-j}, a_{n-j+1})$$

and

$$S_{a_1,\ldots a_n;S'}(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)) =$$

$$= S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)) \geq \min(a_{n-i}, a_{n-j}) \geq a_{n-l}.$$

where $l = \max(i, j)$. Further on we have two possibilities
i4a): If there is an integer $k \in [n-l, n-1]$ such that
$C(a_k) > T'(\beta(x,i), \beta(y,j)) \geq C(a_{k+1})$ then , using the relation (3), we have

$$S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)) =$$

$$= S'(t^{-1}(t(x) \otimes \delta(i) \oplus \theta(i)), t^{-1}(t(y) \otimes \delta(j) \oplus \theta(j))) =$$

$$= t^{-1}(t(1) \ominus t(T'(\beta(x,i), \beta(y,j))) \in (a_k, a_{k+1}].$$

Because $a_1 \leq a_k$ and $a_{k+1} \leq a_n$ we have

$$C_{a_1,\ldots,a_n}(S_{a_1,\ldots a_n;S'}(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y))) =$$

$$= C_{a_1,\ldots,a_n}(S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y))) =$$

$$= t^{-1}(t(S'(\eta(x,i), \eta(y,j))) \otimes \delta(k) \oplus \theta(k)) =$$

$$= t^{-1}((t(1) \ominus t(T'(\beta(x,i), \beta(y,j)))) \otimes \delta(k) \oplus \theta(k)).$$

i4b): If $T'(\beta(x,i),\beta(y,j)) < C(a_n)$ then, using again the relation (3), we have

$$S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)) =$$

$$= S'(t^{-1}(t(x) \otimes \delta(i) \oplus \theta(i)), t^{-1}(t(y) \otimes \delta(j) \oplus \theta(j))) > a_n$$

and therefore

$$C_{a_1,\ldots,a_n}(S_{a_1,\ldots a_n;S'}(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y))) =$$

$$= C_{a_1,\ldots,a_n}(S'(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y))) =$$

$$= t^{-1}\left(k \otimes \frac{t(1) \ominus t(S'(\eta(x,i),\eta(y,j)))}{t(S'(\eta(x,i),\eta(y,j)))}\right) =$$

$$= t^{-1}\left(k \otimes \frac{t(T'(\beta(x,i),\beta(y,j)))}{t\,(1) \ominus t(T'(\beta(x,i),\beta(y,j)))}\right).$$

i5): If $x > a_n$ or $y > a_n$ then $C_{a_1,\ldots,a_n}(x) < a_1$ or $C_{a_1,\ldots,a_n}(y) < a_1$ and therefore

$$C_{a_1,\ldots,a_n}\left(S_{a_1,\ldots,a_n;S'}\left(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)\right)\right) =$$

$$= C_{a_1,\ldots,a_n}\left(\max\left(C_{a_1,\ldots,a_n}(x), C_{a_1,\ldots,a_n}(y)\right)\right) = \min(x,y).$$

□

Example 3.2. For $n = 1$ the last theorem gives the results from [8] and for $n = 2$ we obtain the results from [9].

Example 3.3. Taking $n = 2, a = a_1, b = a_2$ (evidently $0 < a < b < 1$), $\oplus = +, \otimes = \times, t(x) = x$ and $T' = T_M$ in the Theorem 3.3 we obtain a new extension of t-norms with 1-threshold, namely *parametrized t-norms*

$$T_{a/b}(x,y) = \begin{cases} kT\left(\alpha(x), \alpha(y)\right) & if \quad x \le a \ and \ y \le a \\ \min(x,y) & otherwise \end{cases}$$

where $k = \dfrac{ab}{1-b}$ *and* $\alpha(z) = \dfrac{(1-b)\,z}{(1-b)\,z + ab}$, which are t-norms with 1-threshold a and parameter b.

Remark 3.3. It is easy to observe the difference between these t-norms with threshold and those obtained as ordinal sum.

4. Conclusions

This paper present a method to construct t-norms with n-threshold from standard t-norms. In the beginning, a kind of negations with n-threshold, $C_{a_1,\ldots,a_n}$ is introduced and after it is used to obtain a family of t-norms and t-conorms $C_{a_1,\ldots,a_n}-$dual, starting from a pair (t-norm, t-conorm) dual with respect to an arbitrary negation $C(x)$.

References

[1] H. Akdag, D. Pacholczyk, SEQUI: Un système expert pour le traitement de questionnaires incertaines, *Journées orléanaises sur la gestion d'incertitude dans le systèmes decisionnels*, September (1987).
[2] C. Alsina, E. Trillas, L. Valverde, On non distributive local connectives for fuzzy theory, *BUSEFAL*, **3**, 18-29 (1980).
[3] J. F. Baldwin, Evidential Support Logic Programming, *Fuzzy Sets and Systems*, **24**, 1-26 (1987).
[4] J.C. Fodor, A remark on constructing t-norms, *Fuzzy Sets and Systems*, **41**, 195-199 (1991).
[5] I. Iancu, T-norms with threshold, *Fuzzy Sets and Systems*, **85**, 83-92 (1997).
[6] I. Iancu, Introduction of a double threshold in uncertainty management, In: *I. Plander, Ed., Proc. of the 7th Int. Conf. Artificial Intelligence and Information-Control Systems of Robots, World Scientific*, Singapore, 201-210 (1997).
[7] I. Iancu, A method for constructing t-norms, *Korean J. of Computational & Applied Mathematics*, **2**, 407-414 (1998).
[8] I. Iancu, Fuzzy connectives with applications in uncertainty management, *Proc. of the 3-rd annual meeting of the Romanian Society of Math. Science*, May 26-29, 40-47 (1999).
[9] I. Iancu, On a family of t-operators, *Analele Univ. din Craiova*, **XXVI**, 84-92 (1999).
[10] S. Jenei, Fibred triangular norms, *Fuzzy Sets and Systems*, **10**, 3-13 (1999).
[11] S. Jenei, New family of triangular norms via contrapositive symmetrization of reziduated implications, *Fuzzy Sets and Systems*, **110**, 154-174 (2000).
[12] E.P. Klement, R. Mesiar, E. Pap, Quasi- and pseudo-inverses of monotone functions, and the construction of t-norms, *Fuzzy Sets and Systems*, **104**, 3-13 (1999).
[13] E.P. Klement, R.Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.
[14] C. H. Ling, Representation of associative functions, *Publ. Math. Debrecen*, **12**, 189-212 (1965).
[15] R. Mesiar, On some construction of new triangular norma, *Mathware & Soft Computing*, **2**, 39-45, (1995).
[16] D. Pacholczyk, Introduction d'un seuil dans le calcul d'incertitude en logique floue, *BUSEFAL* **32**, 11-18 (1987).
[17] B. Schweizer, A. Sklar, Associative functions and statistical triangle inequalities, *Publ. Math. Debrecen*, **8**, 169-186 (1961).
[18] B. Schweizer, A. Sklar, Associative functions and abstract semigroups, *Publ. Math. Debrecen*, **10**, 69-81 (1963).
[19] M. Sugeno, *Theory of Fuzzy Integrals and its Applications*, Ph. d. Thesis, Tokyo, Institute of Technology (1974).
[20] E. Trillas, Sobre funciones de negation en la teoria de conjuntos difusos, *Stochastica*, **1**, 47-59 (1979).
[21] P. Vicenik, A note to construction of t-norms based on pseudo - inverses of monotone functions, *Fuzzy Sets and Systems*, **104**, 15-18 (1999).

(Ion Iancu) Department of Informatics, University of Craiova,
Al. I. Cuza street, 13, Craiova RO-200585, Romania, Tel/Fax: 40-251412673
E-mail address: i_iancu@yahoo.com