A representation of an uncertain body of evidence

ION IANCU

Abstract. In this paper we present a kind of pairs (t-norm, t-conorm) dual with respect to a strong negation with n-threshold $a_1, ..., a_n \in (0, 1)$, $a_1 < a_2 < ... < a_n$. In this way we obtain an extension of operators with 1− and 2−threshold from some of our papers. The new pair is obtained from given one.

2000 Mathematics Subject Classification. 68T35, 68T27, 68T30.
Key words and phrases. fuzzy sets, t-operators, possibility and necessity measures, belief functions, uncertainty, imprecision.

1. Introduction

Probability Theory, Shafer’s belief theory and Zadeh’s possibility theory are the main methods for study the uncertainty. Considering an event, we may to evaluate its probability, its feasibility, its possibility of occurrence or how much it seems credible. All those evaluations are based on some sets of numbers, generally normalized in some sense, which have to combined in accordance with the characteristic axioms of those theories.

In 1972 Sugeno [19] introduced the concept of fuzzy measure in order to depart from the too rigid framework of probability theory.

Definition 1.1. Given the universe $X$ (supposed to be finite, for sake of simplicity) a fuzzy measure is a set function $g$ from an algebra $A$ (e.g., the set $\mathcal{P}(X)$ of subsets of $X$) defined on $X$ to the interval $[0, 1]$, such that

i) $g(\emptyset) = 0$,  
ii) $g(X) = 1$

iii) $\forall A, B \in A$, if $A \subset B$ then $g(A) \leq g(B)$.

The following inequalities hold

$$\forall A, B \in A, \ g(A \cap B) \leq \min (g(A), g(B))$$

$$\forall A, B \in A, \ g(A \cup B) \geq \max (g(A), g(B)).$$

The axioms from Definition 1.1 are very general and it is necessary to be particularised in order to obtain various classes of fuzzy measures. Thus, if

$$\forall A, B \in A, \ g(A \cap B) = \min (g(A), g(B))$$

we obtain a necessity measure and for

$$\forall A, B \in A, \ g(A \cup B) = \max (g(A), g(B))$$

we have a possibility measure.

In order to combine the uncertainties, the following axiom seems natural:

$$\forall A, B \in A, \text{ if } A \cap B = \emptyset \text{ then } g(A \cup B) = g(A) \ast g(B) \quad (1)$$

Received: 31 October 2003.
A REPRESENTATION OF AN UNCERTAIN BODY OF EVIDENCE

where $*$ is some operator under which $[0, 1]$ is closed. The algebraic structure of $A$ induces the choosing $*$ among the triangular conorms.

If $g$ is a fuzzy measure, the set function $g'$ defined by

$$\forall A \in A, g'(A) = 1 - g(\bar{A})$$

is also a fuzzy measure; this last relation expresses the duality between $g$ and $g'$. If $g$ is a conorm-based set function satisfying (1), then the characteristic property of dual measure $g'$ is

$$\forall A, B \in A, g(A \cup B) = g(A) \perp g(B) \quad (2)$$

where $\perp$ denotes the triangular norm dual of $*$ defined by

$$a \perp b = 1 - (1 - a) \ast (1 - b).$$

Well-known t-norms are

$$T_M(x, y) = \min(x, y), \quad T_P(x, y) = xy, \quad T_L(x, y) = \max(0, x + y - 1)$$

and well-known t-conorms are

$$S_M(x, y) = \max(x, y), \quad S_P(x, y) = x + y - xy, \quad S_L(x, y) = \min(1, x + y).$$

The relations (1) and (2) prove the importance of t-norms and t-conorms in representation of composed information from a body of evidence. For this reason, the construction of new t-norms and t-conorms seems to be an important tool not only for the theory but also for the applications. The construction of the new operators from given ones is a technique used in various paper [17], [18], [16], [4], [15], [5], [6], [7], [8], [9], [10], [12], [21], [11], [13]. From this kind we recall the operators of Pacholczyk type, named with threshold. They were introduced in order to improve the calculus of uncertainty in an expert system.

Given a pair (t-norm, t-conorm) denoted as $(T, S)$ and the parameter $a \in (0, 1)$, the first result of this type were presented under the form [1], [16]

**t – norm:**

$$T_a(x, y) = \begin{cases} \frac{a}{1 - a} T\left(\frac{1 - a}{a} x, \frac{1 - a}{a} y\right) & \text{if } x \leq a \text{ and } y \leq a \\ \min(x, y) & \text{if } x > a \text{ or } y > a \end{cases}$$

corresponding to t-norm $T(x, y)$;

**t – conorm:**

$$S_a(x, y) = \begin{cases} S(x, y) & \text{if } x \geq a \text{ and } y \geq a \\ \max(x, y) & \text{if } x < a \text{ or } y < a \end{cases}$$

corresponding to t-conorm $S(x, y)$.

$T_a$ and $S_a$ are dual operators with respect to the negation with threshold

$$C_a(x) = \begin{cases} 1 - \frac{1 - a}{a} x & \text{if } x \leq a \\ \frac{a}{1 - a} (1 - x) & \text{if } x \geq a. \end{cases}$$

Such operators were used, with very satisfactory results, to construct the system SEQUI [1]- an expert system for processing the uncertain questions. In [5] we presented two kinds of operators with threshold: the first is a generalization of Pacholczyk’s result [16] and the second is a new one.

In [6] we introduced the notion of operators with double threshold (in the Pacholczyk’s meaning) and we presented two types of such operators; another types are given in [7] and [9]. In this paper we present a generalization of our results from [9].
2. Preliminaries

In the beginning, we mention some definitions which will be used in the next section.

**Definition 2.1.** A t-norm $T$ is an increasing, associative and commutative function from $[0, 1]^2$ into $[0, 1]$ that satisfies the boundary condition: $T(x, 1) = x \quad \forall x \in [0, 1]$. A continuous t-norm $T$ is called Archimedean if $T(x, x) < x \quad \forall x \in (0, 1)$.

Any t-norm $T$ satisfies the relation $T(x, y) \leq T_M(x, y) = \min(x, y)$.

**Definition 2.2.** A t-conorm $S$ is an increasing, associative and commutative function from $[0, 1]^2$ into $[0, 1]$ that satisfies the boundary condition: $S(x, 0) = x \quad \forall x \in [0, 1]$. A continuous t-conorm $S$ is called Archimedean if $S(x, x) > x \quad \forall x \in (0, 1)$.

Any t-conorm $S$ satisfies the relation $S(x, y) \geq S_M(x, y) = \max(x, y)$.

**Definition 2.3.** A strong negation is an involutive decreasing function from $[0, 1]$ into itself.

**Theorem 2.1.** [2] If $T$ is a t-norm and $C$ is a strong negation then

$$S(x, y) = C(T(C(x), C(y)))$$

is a t-conorm and reciprocally,

$$T(x, y) = C(S(C(x), C(y)))$$

namely, $T$ and $S$ are $C$-dual.

In order to obtain t-norms and negations one can used the following two theorems.

**Theorem 2.2.** [5] Let $f : [0, 1] \to I \subseteq [0, \infty)$ be a continuous strictly decreasing function and $\Delta : I \times I \to I$ with the following properties:

1. $\Delta(x, y) = \Delta(y, x)$,
2. $\Delta(x, \Delta(y, z)) = \Delta(\Delta(x, y), z)$,
3. $\Delta(x, y) \leq \Delta(x, z)$ if $y \leq z$ with equality iff $y = z$,
4. $\Delta$ is continuous

Then

$$T(x, y) = f^{-1}(\Delta(f(x), f(y))) \quad \forall x, y \in [0, 1]$$

is a t-norm, where $f^{-1}$ is the pseudo-inverse of $f$, extended to the case $f(1) > 0$:

$$f^{-1}(x) = \begin{cases} 1 & \text{if } x \in [0, f(1)] \\ f^{-1}(x) & \text{if } x \in [f(1), f(0)] \\ 0 & \text{if } x \in [f(0), \infty) \end{cases}$$

**Remark 2.1.** For $\Delta = -$ and $f(1) = 0$ we obtain the Ling's result [14] for continuous Archimedean t-norms.

**Example 2.1.** For $I = [0, \infty)$, $\Delta(x, y) = x + y + xy$ and $f(x) = 1 - x$ in Theorem 2.2 we obtain the t-norm $T(x, y) = \max(0, 2x + 2y - xy - 2)$.

**Theorem 2.3.** [5] Let $I \subseteq \mathbb{R}$ and $\Delta : I \times I \to I$ be an application satisfying the following conditions, for all $x, y, z \in I$:

1. $\Delta(x, y) = \Delta(y, x)$ identical to (2.1.1)-(2.1.4)
2. $\forall x \in I$ there is $e \in I$ such that $\Delta(x, e) = x$ \quad $\forall x \in I$
3. $\forall x \in I$ there is $x' \in I$ such that $\Delta(x, x') = e$ and $\varphi : I \to I$, $\varphi(x) = x'$ is a continuous strictly decreasing function
4. $\forall x \in [0, 1] \to J$ is a continuous strictly increasing function with $t(0) = e$ and $t(1)$ is a finite number.

Then $C(x) = t^{-1}(\Delta(t(1), \varphi(t(x))))$ is a strong negation for every $x \in [0, 1]$. 


Remark 2.2. For $\Delta = +$ and $t(x) = x$ we obtain the Trillas’s result [20].

Example 2.2. For $I = \mathbb{R}$, $\Delta(x, y) = x + y - 1$, $e = 1$, $J = [1, \infty)$, $t(x) = \frac{2x + 1}{x + 1}$ and $\varphi(x) = 2 - x$ we obtain, from Theorem 2.3, $C(x) = \frac{1 - x}{1 + 3x}$.

Remark 2.3. Simultaneously using of functions $\Delta$ and $f$ (respectively $t$) in previous lemmas allows obtaining of $t$-norms (respectively negations) on a easier way than in the case $\Delta = +$. For instance, if $\Delta(x, y) = x + y$ we don’t work with functions of the type $f(x) = ax + b$ in order to obtain the $t$-norm from Example 2.1, being necessary more complicated forms.

3. Operators with $n$-threshold

We remain in the conditions of Theorem 2.3 and we use the operation $\odot: I \times I \to I$ instead of $\Delta$; therefore we write $x \odot y = \Delta(x, y)$. We take $\odot : I \times I \to I$ with the following properties:

(i) $x \odot y < x \odot z$ if $y < z \ \forall x, y, z \in I$ and $x > e$,

(ii) $(I, \odot, \otimes)$ is a field.

We note $\ominus x$ and $\frac{1}{x}$ the inverse element of $x$ corresponding to $\oplus$ and $\otimes$, respectively. For the simplification of writing we note $x \otimes \frac{1}{y} = \frac{x}{y}$ and $x \otimes x = x^2$.

Theorem 3.1. If the operations $\oplus$ and $\odot$ and the function $t$ have the previous meaning and $n \in \mathbb{N}$, $n \geq 1$, $0 < a_1 < a_2 < \ldots < a_n < 1$ ,

$$\delta(i) = \frac{t(a_{n-i}) \odot t(a_{n-i+1})}{t(a_{i+1}) \odot t(a_i)} \quad \text{and} \quad \theta(i) = \frac{t(a_{i+1}) \odot t(a_{n-i+1}) \odot t(a_i) \odot t(a_{n-i})}{t(a_{i+1}) \odot t(a_i)}$$

then

$$C_{a_1,\ldots,a_n}(x) = \begin{cases} 
  t^{-1} \left( \frac{t(a_1) \odot t(a_n) \odot t(1)}{t(a_1) \odot t(a_n) \odot t(1) \odot t(a_n) \odot t(x)} \right) & \text{if} \quad x \leq a_1 \\
  t^{-1} \left( t(x) \odot \delta(i) \odot \theta(i) \right) & \text{if} \quad a_i \leq x \leq a_{i+1} \quad \text{and} \quad 1 \leq i < n \\
  t^{-1} \left( \frac{t(a_1) \odot t(a_n) \odot t(1) \odot t(x)}{t(1) \odot t(a_n) \odot t(x)} \right) & \text{if} \quad x \geq a_n 
\end{cases}$$

is strong negation having $t^{-1} \left( \frac{t \left( a_{\left\lfloor \frac{n}{2} \right\rfloor + 1} \right) \oplus t \left( a_{\left\lfloor \frac{n}{2} + 1 \right\rfloor} \right)}{2} \right)$ as fixed point, where $[x]$ is the greatest integer which is smaller than or equal to $x$.

Proof. It is easy to verify the demands from definition of negation. For instance, for involution property one proves first that

(i): $x \leq a_1 \Leftrightarrow C_{a_1,\ldots,a_n}(x) \geq a_n$

(ii): $x \geq a_n \Leftrightarrow C_{a_1,\ldots,a_n}(x) \leq a_1$

(iii): $x \in [a_i, a_{i+1}] \Leftrightarrow C_{a_1,\ldots,a_n}(x) \in [a_{n-i}, a_{n-i+1} \right] \forall i \in \{1,2,\ldots,n-1\}$.

After, these relations are used to verify (by a simple calculus) the demands from Definition 2.3. The fixed point becomes $t^{-1} \left( \frac{t(a_k) \odot t(a_{k+1})}{2} \right)$ for $n = 2k$ and $a_{k+1}$ for $n = 2k + 1$. □
Remark 3.1. The relation (ii) says that if the confidence in a proposition \( p \) is greater than or equal to the threshold \( a_n \), then the confidence in \( \neg p \) is smaller than or equal to the threshold \( a_1 \).

Example 3.1. For \( \oplus = + \), \( \otimes = \times \), \( n = 2 \) and \( t(x) = \frac{2x}{x+1} \) we obtain

\[
C_{a_1,a_2}(x) = \begin{cases} 
\frac{a_1 a_2 (1 + x)}{a_1 a_2 + x (1 + a_1 - a_2)} & \text{if } x \leq a_1 \\
\frac{(a_1 a_2 - 1) x + a_1 + a_2 + 2 a_1 a_2}{x (a_1 + a_2 + 2) + 1 - a_1 a_2} & \text{if } a_1 \leq x \leq a_2 \\
\frac{x (1 - x) a_1 a_2}{x (1 + a_1 - a_2) - a_1 a_2} & \text{if } x \geq a_2 
\end{cases}
\]

Theorem 3.2. Let \( S \) be a t-conorm and \( S' \) one from the t-conorms \( S_M \) or \( S \). For \( 0 < a_1 < \ldots < a_n < 1 \)

\[
S_{a_1,\ldots,a_n;S'}(x,y) = \begin{cases} 
\max(x,y) & \text{if } x < a_1 \text{ or } y < a_1 \\
S(x,y) & \text{if } x \geq a_n \text{ and } y \geq a_n \\
S'(x,y) & \text{otherwise}
\end{cases}
\]

is a t-conorm.

Proof. For \( S' \equiv S \) we obtain

\[
S_{a_1,\ldots,a_n;S}(x,y) = \begin{cases} 
\max(x,y) & \text{if } x < a_1 \text{ or } y < a_1 \\
S(x,y) & \text{if } x \geq a_n \text{ and } y \geq a_n
\end{cases}
\]

that is the Pacholczyk's t-conorm \( S_{a_1}[16] \).

For \( S' \equiv S_M \) we obtain

\[
S_{a_1,\ldots,a_n;S_M}(x,y) = \begin{cases} 
\max(x,y) & \text{if } x < a_n \text{ or } y < a_n \\
S(x,y) & \text{if } x \geq a_n \text{ and } y \geq a_n
\end{cases}
\]

that is the Pacholczyk's t-conorm \( S_{a_n}[16] \). \( \square \)

Remark 3.2. We cannot choose \( S' \) as an arbitrary t-conorm because the associativity property is not always verified. For instance, for \( n = 2 \) and \( a_1 < x < a_2 < y < z \) we have

\[
S_{a_1,a_2;S'}(S_{a_1,a_2;S'}(x,y),z) = S_{a_1,a_2;S'}(x,S_{a_1,a_2;S'}(y,z))
\]

which is equivalent with

\[
S(S'(x,y),z) = S'(x,S(y,z)).
\]

But for \( S(x,y) = \max(x,y) \) and \( S'(x,y) = xy \) the last equality becomes

\[
\max(xy,z) = x \times \max(y,z) \Leftrightarrow z = xz
\]

which is false because from \( 0 < x < a_2 < z < 1 \) we obtain \( x < 1 \) and \( z > 0 \).

Theorem 3.3. Let \((T, S)\) be a pair \((t-norm, t-conorm)\) dual with respect to \( C(x) = t^{-1}(t(1) \ominus t(x)) \), \( S' \in \{S_M, S\} \) and \( T' \) is the dual t-norm of \( S' \) with respect to the
same negation (i.e., $T' \in \{T_M, T\}$). For $0 < a_1 < a_2 < \ldots < a_n < 1$, we define

$$T_{a_1, \ldots, a_n}(x, y) = \begin{cases} 
    t^{-1} \left( k \otimes \frac{t(T'(\alpha(x), \beta(y)))}{t(1) \otimes t(T'(\alpha(x), \alpha(y)))} \right) & \text{if } x \leq a_1 \text{ and } y \leq a_1 \\
    t^{-1} \left( k \otimes \frac{t(T'(\alpha(x), \alpha(y)))}{t(1) \otimes t(T'(\alpha(x), \beta(y)))} \right) & \text{if } x \leq a_1 \text{ and } y \in (a_1, a_{i+1}], 1 \leq i \leq n-1 \\
    t^{-1} \left( k \otimes \frac{t(T'(\beta(x, i), \alpha(y)))}{t(1) \otimes t(T'(\beta(x, i), \alpha(y)))} \right) & \text{if } y \leq a_1 \text{ and } x \in (a_1, a_{i+1}], 1 \leq i \leq n-1 \\
    t^{-1} \left( \left( t(1) \otimes t(T'(\beta(x, i), \beta(y, j)), \delta(k) \oplus \theta(k)) \right) \otimes \delta(\beta(i)) \oplus \theta(\delta(i)) \right) & \text{if } x \in (a_1, a_{i+1}], y \in (a_j, a_{j+1}], l = \max(i, j) \text{ and } \\
    & \text{there is an integer } k \in [n-l, n-1] \text{ such that } \\
    & T'(\beta(x, i), \beta(y, j)) \in [C(a_{k+1}), C(a_k)) \\
    \min(x, y) & \text{if } x > a_n \text{ or } y > a_n.
\end{cases}$$

where

$$k = \frac{t(a_1) \otimes t(a_n)}{t(1) \otimes t(a_n)}, \quad \alpha(z) = t^{-1} \left( \frac{t(1) \otimes t(a_1)) \otimes t(z) \otimes t(1)}{t(a_1) \otimes t(a_n) \otimes t(1) \otimes t(a_n) \otimes t(z)} \right) \quad \text{and}$$

$$\beta(z, i) = t^{-1} \left( t(1) \otimes t(z) \otimes \delta(i) \oplus \theta(i) \right), \quad \delta(i) \text{ and } \theta(i) \text{ having the significance from Theorem 3.1.}$$

Then $T_{a_1, \ldots, a_n}$ is a t-norm $C_{a_1, \ldots, a_n}$- dual with t-conorm $S_{a_1, \ldots, a_n}$. $S'$.  

**Proof.** Because $S_{a_1, \ldots, a_n; S'}$ is a t-conorm, in order to prove the theorem it is sufficient to verify, in accordance with Theorem 2.1, the equality

$$T_{a_1, \ldots, a_n}(x, y) = C_{a_1, \ldots, a_n}(S_{a_1, \ldots, a_n; S'}(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y))).$$

The same theorem yields

$$S(x, y) = C(T(C(x), C(y))) = t^{-1}(t(1) \otimes t(T(C(x), C(y)))) \quad (3)$$

In order to simplify the writing we denote

$$\eta(z, i) = t^{-1}(t(x) \otimes \delta(i) \oplus \theta(i))$$

and we analyze the following cases:

**i):** For $x, y \leq a_1$ we have

$$C_{a_1, \ldots, a_n}(x) \geq a_n, \quad C_{a_1, \ldots, a_n}(y) \geq a_n \text{ and}$$

$$S_{a_1, \ldots, a_n; S'}(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y)) = S(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y)) \geq \max(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y)) = a_n.$$

Using the last relations and the identity (3) we have

$$C_{a_1, \ldots, a_n}(S_{a_1, \ldots, a_n; S'}(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y))) =$$

$$t^{-1} \left( \left( t(a_1) \otimes t(a_n) \otimes t(1) \otimes t(S(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y))) \right) \right) =$$

$$t^{-1} \left( k \otimes t(T(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y)))) \right) =$$
Because

For i4): If there is an integer

The case

Using the same reasoning as in the previous case, we have

where


Using the same reasoning as in the previous case, we have

and

where

where \( l = \max(i, j) \). Further on we have two possibilities

i4a): If there is an integer \( k \in [n - l, n - 1] \) such that

Then, using the relation (3), we have

Because \( a_1 \leq a_k \) and \( a_{k+1} \leq a_n \) we have


i4b): If $T'(\beta(x,i), \beta(y,j)) < C(a_n)$ then, using again the relation (3), we have

$$S'(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y)) =$$

$$= S'(t^{-1}(t(x) \otimes \delta(i) \oplus \theta(i)), t^{-1}(t(y) \otimes \delta(j) \oplus \theta(j))) > a_n$$

and therefore

$$C_{a_1, \ldots, a_n}(S_{a_1, \ldots, a_n}(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y))) =$$

$$= t^{-1} \left( k \otimes \frac{t(1) \otimes t(S'(\eta(x), \eta(y)))}{t(S'(\eta(x), \eta(y)))} \right) =$$

$$= t^{-1} \left( k \otimes \frac{t(T'(\beta(x,i), \beta(y,j)))}{t(1) \otimes t(T'(\beta(x,i), \beta(y,j)))} \right).$$

i5): If $x > a_n$ or $y > a_n$ then $C_{a_1, \ldots, a_n}(x) < a_1$ or $C_{a_1, \ldots, a_n}(y) < a_1$ and therefore

$$C_{a_1, \ldots, a_n}(S_{a_1, \ldots, a_n}(C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y))) =$$

$$= C_{a_1, \ldots, a_n} \left( \max (C_{a_1, \ldots, a_n}(x), C_{a_1, \ldots, a_n}(y)) \right) = \min (x, y).$$

□

Example 3.2. For $n = 1$ the last theorem gives the results from [8] and for $n = 2$ we obtain the results from [9].

Example 3.3. Taking $n = 2, a = a_1, b = a_2$ (evidently $0 < a < b < 1$), $\oplus = +, \otimes = \times, t(x) = x$ and $T' = T_M$ in the Theorem 3.3 we obtain a new extension of t-norms with 1-threshold, namely parametrized t-norms

$$T_{a/b}(x, y) = \begin{cases} kT(\alpha(x), \alpha(y)) & \text{if } x \leq a \text{ and } y \leq a \\ \min (x, y) & \text{otherwise} \end{cases}$$

where $k = \frac{ab}{1-b}$ and $\alpha(z) = \frac{(1-b)z}{(1-b)z + ab}$, which are t-norms with 1-threshold $a$ and parameter $b$.

Remark 3.3. It is easy to observe the difference between these t-norms with threshold and those obtained as ordinal sum.

4. Conclusions

This paper present a method to construct t-norms with n-threshold from standard t-norms. In the beginning, a kind of negations with $n$-threshold, $C_{a_1, \ldots, a_n}$ is introduced and after it is used to obtain a family of t-norms and t-conorms $C_{a_1, \ldots, a_n}$-dual, starting from a pair (t-norm, t-conorm) dual with respect to an arbitrary negation $C(x)$. 
References


(Ion Iancu) DEPARTMENT OF INFORMATICS, UNIVERSITY OF CRAIOVA, AL. I. CUZA STREET, 13, CRAIOVA RO-200585, ROMANIA, Tel./Fax: 40-251412673

E-mail address: i_iancu@yahoo.com