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Characterizations of some fractional-order operators in complex domains and their extensive implications to certain analytic functions

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Abstract. The main target of this research note is firstly to introduce certain fundamental information in relation to various operators of fractional-order calculus in the complex plane, then create some comprehensive results associating with certain analytic functions as implications of those operators, and also present numerous conclusions and recommendations for the related researchers.

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1. Information on definitions, notations and motivation

As it is known from literature, specially, in mathematical sciences, the written materials present us a large number of scientific works associating with various operators constituted by fractional-order calculus (that is fractional-order integral and fractional-order derivative). By making simple literature review, one can easily be well up on detailing documents consisting of those operators and their properties and applications. For some of them, one can refer to some main works presented by the references in [3], [7], [18] and [19]-[22], and see also, as certain examples, [1], [2], [8]-[10]. Since this research will be related to some of the mentioned operators of fractional-order calculus (that is fractional-order derivative(s)) and a variety of their applications to certain functions (with complex variable), in special, there is in need of introducing certain basic information therewith. Accordingly, first of them, let

$$\zeta := \zeta(z) \,:\, \mathbb{U}_\rho(z) := \Big\{ z \in \mathbb{C} \,:\, |z| \le \rho < 1 \Big\} \to \mathbb{C}$$

be an analytic function, *which* is also normalized by the MacLaurin series expansion in the forms given by

$$\zeta(z) = z + \eta_{n+1} z^{n+1} + \eta_{n+2} z^{n+2} + \cdots \quad \left(\eta_{n+1} \neq 0\right), \tag{1.1}$$

where $\eta_{n+1} \in \mathbb{C}$ for all $n+1 \in \mathbb{N} = \{1, 2, 3, \cdots\}$. As a matter of course, the mentioned notations $\mathbb{N}$ and $\mathbb{C}$ are the sets of natural numbers and complex numbers, respectively.

Additionally, we then begin to introduce (*or* re-evoke) those fractional-order operators, *which* are well-known as the fractional-order derivative(s) operator and the

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Tremblay operator. Indeed, for a function $\varsigma := \varsigma(z)$ given by (1.1), the fractional derivative(s) of order $\widetilde{\mu}$ is then denoted by

$$\mathrm{D}_z^{\widetilde{\mu}}[\varsigma] \equiv \mathrm{D}_z^{\widetilde{\mu}}\big[\varsigma(z)\big] \quad \big(0 \leq \widetilde{\mu} < 1\big)$$

and also defined by

$$\mathrm{D}_z^{\widetilde{\mu}}[\varsigma] = \begin{cases} \frac{1}{\Gamma(1-\widetilde{\mu})} \frac{d}{dz} \left(\int_0^z \frac{\varsigma(t)}{(z-t)^{\widetilde{\mu}}} dt \right) \; when \; \widetilde{\mu} \in [0,1) \\ \frac{d^s}{dz^s} \left(\mathrm{D}_z^{\widetilde{\mu}-s}[\varsigma] \right) \qquad\qquad when \; s \in \mathbb{N} \; \& \; \widetilde{\mu} \in [s, s+1) \end{cases}, \tag{1.2}$$

where ς is an analytic function in any simply-connected region of the complex plane comprising the origin, and the multiplicity of $(z-t)^{\widetilde{\mu}}$ is removed by requiring $log(z-t)$ to be real *when* $z - t > 0$.

In addition, through the instrument of the fractional derivative(s) operator of order $\widetilde{\mu}$, for a function $\varsigma := \varsigma(z)$ like the form given in (1.1), the Tremblay operator is also defined by

$$\begin{aligned} \mathrm{T}_{\widetilde{\tau},\widetilde{\mu}}[\varsigma] &\equiv \mathrm{T}_{\widetilde{\tau},\widetilde{\mu}}[\varsigma(z)] \\ &= \frac{\Gamma(\widetilde{\mu})}{\Gamma(\widetilde{\tau})} z^{1-\widetilde{\mu}}\, \mathrm{D}_z^{\widetilde{\tau}-\widetilde{\mu}}\big[z^{\widetilde{\tau}-1}\varsigma(z)\big] \quad \big(z \in \mathbb{U}\big), \end{aligned} \tag{1.3}$$

where

$$\widetilde{\tau} \in (0,1] \quad , \quad \widetilde{\mu} \in (0,1] \quad \text{and} \quad \widetilde{\tau} - \widetilde{\mu} \in [0,1)\,. \tag{1.4}$$

Here and also in the definition in (1.2), we specially note that the apparent operator $\mathrm{D}_z^{\widetilde{\tau}-\widetilde{\mu}}[\,\cdot\,]$ represents a form of the Srivastava-Owa operator of fractional derivative of order $\widetilde{\tau} - \widetilde{\mu}$ $\;(\widetilde{\tau} - \widetilde{\mu} \in [0,1))$, *which* is presented by (1.2). For it and some of its applications, it can be looked over the references in [20] and [22].

As two basic applications of the fractional-order operators, under the admissible values of the related parameters restricted by the conditions given by (1.4), by applying the fractional-order operators presented in (1.2) and (1.3) to a simple analytic function $\Xi(z)$ being of the form like

$$\Xi := \Xi(z) = z^{\mathcal{M}},$$

one can easily determine that

$$\mathrm{D}_z^{\widetilde{\mu}}\big[\Xi\big] = \frac{\Gamma(\mathcal{M}+1)}{\Gamma(\mathcal{M}-\widetilde{\mu}+1)} z^{\mathcal{M}-\widetilde{\mu}} \tag{1.5}$$

and

$$\mathrm{T}_{\widetilde{\tau},\widetilde{\mu}}\big[\Xi\big] = \frac{\Gamma(\widetilde{\mu})\Gamma(\mathcal{M}+\widetilde{\tau})}{\Gamma(\widetilde{\tau})\Gamma(\mathcal{M}+\widetilde{\mu})} z^{\mathcal{M}} \quad \big(\mathcal{M} \in \mathbb{N}\big). \tag{1.6}$$

Since fractional-order calculations are tools that have very important roles in science and technology, the extensive resources of this research have also been enriched. Hence, for both certain instances and the extensive information thereunto appertaining the operators of fractional-order derivative(s), advertised by (1.2) and (1.3), and some of their implications, one may center on the earlier works cited in [5], [8], [12], [15], and, for example, also check certain earlier results presented by the papers in [1], [10], [11], [13], [14], [17] and [18].

In consideration of the comprehensive information given by both previous sections and the mentioned references of this investigation, by the help of the Tremblay Operator along with using certain (elementary) operators of fractional-order calculus (that is derivative(s)), and also under the conditions contained in

$$\lambda \in [0,1] \quad , \quad \beta \in (0,1] \quad , \quad \alpha \in (0,1] \quad \text{and} \quad \alpha-\beta \in [0,1)\,, \tag{1.7}$$

for an analytic function $\zeta := \zeta(z)$ like the forms given by (1.1), we then present the extensive operator of fractional-order derivative(s), *which* is denoted by

$$\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta] := \mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z)]\,,$$

and also defined by

$$\begin{aligned}\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta] &:= \lambda \mathrm{T}_{\alpha,\beta}[\zeta(z)] + (1-\lambda) z \frac{d}{dz}\Big(\mathrm{T}_{\alpha,\beta}[\zeta(z)]\Big) \\ &\equiv \lambda \mathrm{T}_{\alpha,\beta}[\zeta] + (1-\lambda) z \Big(\mathrm{T}_{\alpha,\beta}[\zeta]\Big)'\,,\end{aligned} \tag{1.8}$$

where $z \in \mathbb{U}_{\rho}(z)$ and, of course, for any functions being of the complex-series forms like (1.1.), $\mathrm{T}_{\alpha,\beta}[\cdot]$ is the well-known operator like the forms given by (1.3), and, most especially, the operator $\mathbb{T}^{\lambda}_{\alpha,\beta}[\cdot]$ has been recently defined in the reference in [14] (and then considered as its certain application in [11]) and it can be also seen the related results there, as certain examples.

As the last words of this section, it would be appropriate to provide some remarkable information for both the purpose and the details of this scientific research. Under certain suitable values of all parameters determined by the conditions presented in (1.7), when considering the main definitions in (1.2), (1.3) and (1.8) together with using their applications in (1.5) and (1.6), it can be easily seen that there are both several relationships between those operators and various extensive effects on certain analytic functions like (1.1). Further, some special results of those relate analytic and geometric properties of both operators and their implications. For those special properties, it can be focused on the related topics given by the works in [4], [6] and (also see) [12]-[15]. Specially, a few extra-special implications in relation with those will be given (*or* pointed out) in the second section.

Let us now start to introduce the necessary information for stating and then proving of our main results *which* will be also created with the help of a different method.

2. Identification of related lemmas and main results

In this section, two auxiliary theorems and our main results proved by them will be given. For the details of the related auxiliary theorems *which* are Lemmas 2.1 and 2.2, the references cited in [23] and [17] can be then reviewed, respectively. Moreover, it can be also focused on the earlier researches in [11]-[15] as certain their applications. Let us now present those theorems.

Lemma 2.1. *Let* $x+iy \in \mathbb{C}-\{0\}$ *and* $u+iv \in \mathbb{C}$. *Then,*

$$(x+iy)^{u+iv} = \left(x^2+y^2\right)^{\frac{u+iv}{2}} e^{i(u+iv)Arg(x+iy)},$$

or, equivalently,

$$(x+iy)^{u+iv} = \left(x^2+y^2\right)^{\frac{u}{2}} e^{-vArg(x+iy)} e^{i\left[uArg(x+iy)+\frac{v}{2}\log(x^2+y^2)\right]}. \tag{1.9}$$

Lemma 2.2. *Let $\varphi(z)$ be an analytic function in $\mathbb{U}_\rho(z)$ with $\varphi(0) = 1$. If there exists a point $z_0 \in \mathbb{U}_\rho(z)$ such that*

$$\Re e\big(\varphi(z)\big) > 0 \quad \big(|z| < |z_0| < \rho < 1\big) \tag{1.10}$$

and

$$\varphi(z_0) \neq 0 \quad \textit{and} \quad \Re e\big(\varphi(z_0)\big) = 0\,, \tag{1.11}$$

then

$$\frac{z\varphi'(z_0)}{\varphi(z_0)} = i\lambda\,, \tag{1.12}$$

where $\lambda \in \mathbb{I} := (-\infty, -1] \cup [1, \infty)$.

In order to establish the main results and some of their implications and also since some extensive collections (concerning our main results) will consist of various rational-type functions with complex variable, for convenience, here and also in parallel with this section, we will take into account the equivalent forms given by

$$\frac{z\Big(\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]\Big)'}{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]} \equiv \frac{z\frac{d}{dz}\Big(\frac{\beta}{\alpha}\,\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z)]\Big)}{\frac{\beta}{\alpha}\,\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z)]} \equiv \frac{z\frac{d}{dz}\Big(\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z)]\Big)}{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z)]} \tag{1.13}$$

and

$$\frac{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]}{z} \equiv \frac{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z)]}{z} \tag{1.14}$$

for all $z \in \mathbb{U}_\rho(z)$ and for some values of the parameters identified by the conditions in (1.7). Indeed, *when* having regard to all analytic functions like $\zeta := \zeta(z)$ being of the forms in (1.1), it can be easily seen that the rational type-complex functions, presented by both (1.13) and (1.14), have a removable-singular point at $z := 0$. Because of this important reason, there is no problem for stating (and also proving) our main results associating with those complex-type functions (designated by the definitions in (1.13) and (1.14)) and also their (more) special forms.

By taking into account the extensive information between (1.8)-(1.12), we now begin by setting and then proving our main results associating with various comprehensive relationships between those rational-type functions as constituted in the definitions given in (1.13) and (1.14), *which* are the following theorems (just below).

Theorem 2.3. *Let*

$$\kappa \in \mathbb{N}\ ,\ \Omega \in \mathbb{C} - \{0\}\ ,\ \nabla \in \mathbb{I}\ ,\ z \in \mathbb{U}_\rho(z) \quad \textit{and} \quad \frac{\pi}{4} \leq \big|\tilde{\Theta}\big| < \frac{\pi}{2}\,. \tag{1.15}$$

Then, under the conditions determined with (1.7) and also the definitions stated in (1.2), (1.3), (1.8), (1.13) and (1.14), for all functions like $\zeta := \zeta(z)$ having the series form in (1.1), the following proposition is true:

$$arg\left\{\left(\frac{z\Big(\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]\Big)'}{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]}\right)^{\Omega}\right\} \neq 2\kappa\pi + \tilde{\Theta}\,\Re e\big(\Omega\big) + \Im m\big(\Omega\big)log\sqrt{1+\nabla^2} \tag{1.16}$$

$$\Longrightarrow \quad \Re e\left(\frac{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]}{z}\right) > 0\,. \tag{1.17}$$

Proof. In the light of the conditions in relation with the parameters restricted in (1.7) and by using the operators in (1.2), (1.3) and (1.8) for an analytic function $\zeta := \zeta(z)$ having the series forms in (1.1), the following-equivalent results:

$$\begin{aligned} \mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta] &\equiv \lambda \mathrm{T}_{\alpha,\beta}[\zeta] + (1-\lambda) z \Big(\mathrm{T}_{\alpha,\beta}[\zeta]\Big)' \\ &= \frac{\alpha}{\beta}\left(z + \sum_{m=n+1}^{\infty} \aleph_m(\lambda;\alpha,\beta) z^m\right) \\ &= \frac{\alpha}{\beta} z \left(1 + \sum_{m=n+1}^{\infty} \aleph_m(\lambda;\alpha,\beta) z^{m-1}\right) \end{aligned} \tag{1.18}$$

can be easily determined, *where*

$$\aleph_m(\lambda;\alpha,\beta) := \big[\lambda + m(1-\lambda)\big] \frac{\Gamma(m+\alpha)\Gamma(1+\beta)}{\Gamma(m+\beta)\Gamma(1+\alpha)} \eta_m \,.$$

With the help of (1.18), if define a function $\varphi(z)$ in the implicit form given by

$$\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta] = \frac{\alpha}{\beta} z\varphi(z)\,, \tag{1.19}$$

then it is easily seen that $\varphi(z)$ is an analytic function in the domain $\mathbb{U}_\rho(z)$ and it also satisfies the condition $\varphi(0) = 1$ of Lemma 2.2. Therefore, since the implicit function $\varphi(z)$ defined by (1.19) is a suitable function for making use of Lemma 2.2, of course, it can be considered for the proof of the theorem above. For this, it then follows from (1.19) that

$$\frac{z\Big(\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]\Big)'}{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]} = 1 + \frac{z\varphi'(z)}{\varphi(z)} \quad \big(z \in \mathbb{U}_\rho(z)\big)\,, \tag{1.20}$$

and also supposing that there exists a point $z_0 \in \mathbb{U}_\rho(z)$ satisfying the condition of Lemma 2.2, namely, the following condition given by

$$\Re e\Big(\varphi(z_0)\Big) = 0\,. \tag{1.21}$$

Accordingly, from (1.16) (and also (1.20)), $\varphi(z_0) \neq 0$. Then, by applying of (the assertions of) Lemma 2.2 together with (the related assertion of) Lemma 2.1 to the rational-type function composed as in (1.20), one can easily arrive at the results contained in the following-equivalent forms given by

$$\begin{aligned} \left(\frac{z\Big(\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z)]\Big)'}{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z)]}\Bigg|_{z:=z_0}\right)^{\Omega} &= \left(1 + \frac{z\varphi'(z)}{\varphi(z)}\Bigg|_{z:=z_0}\right)^{\Omega} \\ &= \big(1+i\xi\big)^{\Omega} \\ &= \big|1+i\xi\big|^{\Re e(\Omega)} e^{-\Theta\, \Im m(\Omega) + i\Delta} \\ &= \Big(\sqrt{1+\xi^2}\Big)^{\Re e(\Omega)} e^{-\Theta\, \Im m(\Omega)} e^{i\Delta}\,, \end{aligned} \tag{1.22}$$

where

$$\Theta := Arg(1+i\xi) \quad \big(\xi \in \mathbb{I}\big)\,, \tag{1.23}$$

$$\Delta := \Theta\, \Re e(\Omega) + \Im m(\Omega) log\sqrt{1+\xi^2} \tag{1.24}$$

and

$$\Theta \in \begin{cases} \left[\frac{\pi}{4}, \frac{\pi}{2}\right) & when \quad \xi \geq 1 \\ \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right] & when \quad \xi \leq -1 \end{cases} . \tag{1.25}$$

Now, for desired proof, since the operation taking a look at the results in (1.22) is the argument of a complex expression, in the light of the information between (1.23)-(1.25), by taking the argument of the both sides of (1.22), it can easily determined that

$$\begin{aligned} arg\left\{\left(\frac{z\left(\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z_0)]\right)'}{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta(z_0)]}\right)^{\Omega}\right\} \\ &= arg\left\{\left(1+\frac{z\varphi'(z_0)}{\varphi(z_0)}\right)^{\Omega}\right\} \\ &= arg\left\{(1+i\xi)^{\Omega}\right\} \\ &= arg\left\{\left(\sqrt{1+\xi^2}\right)^{\Re e(\Omega)} e^{-\Theta\,\Im m(\Omega)} e^{i\Delta}\right\} \\ &= arg\left\{\left(\sqrt{1+\xi^2}\right)^{\Re e(\Omega)} e^{-\Theta\,\Im m(\Omega)}\right\} + arg\left\{e^{i\Delta}\right\} \\ &= 2k\pi + \Delta \quad \left(k \in \mathbb{N}\right), \end{aligned}$$

which also is a contradiction with the hypothesis of Theorem 2.3, namely, the mentioned inequality given by (1.16) *when* setting

$$k := \kappa \quad , \quad \xi := \nabla \quad \text{and} \quad \tilde{\Theta} := \Theta ,$$

where Δ is given by (1.24). This is to say us that there is *no* a point $z_0 \in \mathbb{U}_{\rho}(z)$ satisfying the condition in (1.21) (*or,* in (1.11) of Lemma 2.2). Therefore, it gives us the inequality, *which* also is one of the hypotheses of Lemma 2.2, given by

$$\Re e\left(\varphi(z)\right) > 0$$

for all $z \in \mathbb{U}_{\rho}(z)$. At this stage, the expression in (1.19) immediately yields that the provision of Theorem 2.3, *which* is also given by (1.17). Thus and so the desired proof is finished. □

Another one of our main-comprehensive results is also given as Theorem 2.4, *which* is below and has several special results of the relationships between the rational-type functions given by (1.13) and (1.14). Its proof is so similar to the proof of Theorem 2.3. For it, it will be enough to use the definition of the analytic function $\varphi(z)$ constituted in (1.19) and then follow the same steps taken into account in the proof of Theorem 2.3. We think anyone can easily take care of this problem. For this reason, its detail is omitted in this research.

Theorem 2.4. *Let the parameters κ, Ω, ∇ and $\tilde{\Theta}$ satisfy the conditions given by (1.15). Then, under the conditions determined in (1.7) and also the definitions constituted by (1.2), (1.3), (1.8), (1.13) and (1.14), for all functions like $\zeta := \zeta(z)$ being of the complex-series form in (1.1), the following proposition is satisfied:*

$$\left| \left(\frac{z\left(\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]\right)'}{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]} \right)^{\Omega} \right| \neq e^{-\tilde{\Theta}\,\Im m(\Omega)} \left(1+\nabla^2\right)^{\frac{\Re e(\Omega)}{2}}$$

$$\Longrightarrow \quad \Re e\left(\frac{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]}{z} \right) > 0 \quad \big(z \in \mathbb{U}_{\rho}(z)\big)\,.$$

3. Concluding remarks and some implications

As has been presented in the first section and the second section, we have firstly introduced various information in relation with some operators of fractional-order derivatives and some special definitions specified by those operators. Afterwards, we have also constituted and then demonstrated two extensive-main results by the help of using of those derivative operators (in (1.2), (1.3) and (1.8)) along with the special functions (in (1.13) and (1.14)). Clearly, as we have emphasized in both sections of this paper, this scientific note contains important relations and implications in many ways for the literature. Specially, the indicated relations are various relationships between the operators of those fractional-order derivatives(s) introduced there. For those relationships, it may be helpful to consider the earlier-main works (*or* papers) cited in [5] and [11]-[15] in the references. The other-indicated results also are possible special results *which* will be obtained by considering of the main results. In order to reveal these special results, it will also be enough to choose the suitable values of the parameters used in the theorems. For you, let us now present (*or* reveal) only four of those implications of them, *which* include both related relations-results, and leave others to the relevant researchers. In addition, we leave both the sampling of all the main results and their specific results to the concerned researchers.

By choosing the value of the parameter Ω as $\Omega := 1$ in Theorem 2.3, we get the first implication of our main results, *which* is the following assertion given as Proposition 3.1 (just below).

Proposition 3.1. *Let $\kappa \in \mathbb{N}$, $0 \leq \lambda \leq 1$, $z \in \mathbb{U}_{\rho}(z)$ and $\pi/4 \leq \left|\tilde{\Theta}\right| < \pi/2$. Then, under the conditions in (1.7) and also the definitions in (1.2), (1.3), (1.8), (1.13) and (1.14), for a analytic function $\zeta := \zeta(z)$ like the form in (1.1), the following proposition holds:*

$$arg\left(\frac{z\left(\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]\right)'}{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]} \right) \neq 2\kappa\pi + \tilde{\Theta} \quad \Rightarrow \quad \Re e\left(\frac{\mathbb{T}^{\lambda}_{\alpha,\beta}[\zeta]}{z} \right) > 0\,.$$

By selecting the values of the parameters α and β as $\alpha := 1$ and $\beta := 1$, in Proposition 3.1, we then get the equivalent relationships between the special forms of the operators (of fractional-order calculus) and the function $\zeta(z)$ like the form in

(1.1), *which* are

$$\begin{aligned}\mathbb{T}^{\lambda}_{1,1}[\zeta] &\equiv \mathbb{T}^{\lambda}_{1,1}[\zeta(z)]\\ &\equiv \lambda\zeta(z) + (1-\lambda)z\zeta'(z)\\ &\equiv \lambda \mathrm{T}_{1,1}[\zeta(z)] + (1-\lambda)z\Big(\mathrm{T}_{1,1}[\zeta(z)]\Big)',\end{aligned}$$

are received (see, for extra information (*or* certain examples), [11] and [14]). At this time, we also get that the second implication (of our main results), *which* is the following Proposition 3.2 (below).

Proposition 3.2. *Let $\kappa \in \mathbb{N}$, $z \in \mathbb{U}_\rho(z)$ and $\pi/4 \leq |\tilde{\Theta}| < \pi/2$. For an analytic function $\zeta := \zeta(z)$ like the form in (1.1), if the inequality:*

$$arg\left(\frac{z[\lambda\zeta(z) + (1-\lambda)z\zeta'(z)]'}{\lambda\zeta(z) + (1-\lambda)z\zeta'(z)}\right) \neq 2\kappa\pi + \tilde{\Theta}$$

or, equivalently,

$$arg\left(\frac{z\zeta'(z) + (1-\lambda)z^2\zeta''(z)}{\lambda\zeta(z) + (1-\lambda)z\zeta'(z)}\right) \neq 2\kappa\pi + \tilde{\Theta}$$

is provided, then the inequality:

$$\Re e\left(\lambda\frac{\zeta(z)}{z} + (1-\lambda)\zeta'(z)\right) > 0$$

is provided.

By taking the values of the parameter λ as $\lambda := 1$ and $\lambda := 0$ in Proposition 3.2, respectively, we then get the special implications (relating to (Analytic and) Geometric Function Theory. For their details, one may center on the main books in [4] and [6])), *which* are the following assertions which are Propositions 3.3 and 3.4.

Proposition 3.3. *Let $\kappa \in \mathbb{N}$, $z \in \mathbb{U}_\rho(z)$ and $\pi/4 \leq |\tilde{\Theta}| < \pi/2$. For an analytic function $\zeta := \zeta(z)$ like the form in (1.1), the following proposition is also true:*

$$arg\left(\frac{z\zeta'(z)}{\zeta(z)}\right) \neq 2\kappa\pi + \tilde{\Theta} \quad \Rightarrow \quad \Re e\left(\frac{\zeta(z)}{z}\right) > 0\,.$$

Proposition 3.4. *Let $\kappa \in \mathbb{N}$, $z \in \mathbb{U}_\rho(z)$ and $\pi/4 \leq |\tilde{\Theta}| < \pi/2$. For an analytic function $\zeta := \zeta(z)$ like the form in (1.1), the following proposition is then satisfied:*

$$arg\left(1 + \frac{z\zeta''(z)}{\zeta'(z)}\right) \neq 2\kappa\pi + \tilde{\Theta} \quad \Rightarrow \quad \Re e\Big(\zeta'(z)\Big) > 0\,.$$

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