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# New results on topological effect algebras

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Abstract. In this paper, by considering the notion of effect algebra and by using of a new ideal in an effect algebra $E$, we construct a topology $\tau$ on $E$, and we show that $(E, \tau)$ is a topological effect algebra. Then we obtain some conditions under which that $(E, \tau)$ is a Hausdorff space. Also, we obtain some results about connected components of this topological space, and we construct a quotient topological effect algebra.

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## 1. Introduction

In 1994, Foulis and Bennett [12] introduced the concept of effect algebras with a partially addition " + " in order to axiomatize some quantum measurements. They are additive counterparts to $D$-posets introduced by Kôpka and Chovanec (1994), where the subtraction of comparable elements is a primary notion. They met interest of mathematicians physicists while they give a common base for algebraic as well as fuzzy set properties of the system $\varepsilon(H)$ of all effects of a Hilbert space $H$, i.e., of all Hermitian operators $A$ on $H$ such that $O \leq A \leq I$, where $O$ and $I$ are the null and the identity operators on $H$. In many cases, effect algebras are intervals in unital po-groups, e.g., $\varepsilon(H)$ is the interval in the po-group $\beta(H)$ of all Hermitian operators on $H$; this group is of great importance for physics.

Effect algebras generalize many examples of quantum structures, like Boolean algebras, orthomodular lattices or posets, orthoalgebras, $MV$-algebras and etc. Since the field of effect algebras is bigger than the most of algebraic structures, stating and opening of any subject in this field can be useful.
In the study of effect algebras (or more general, quantum structures) as carriers of states and probability measures, an important tool is the study of topologies on them. In fact, algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence, and it provides a general framework to study the concept of a limit. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. Because of this difference in nature, algebra and topology to have a strong tendency to develop independently, not in direct contact with each other. However, in applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory and others, topology and algebra come in contact most naturally. Recently,

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many mathematicians have studied properties of some algebraic structures endowed with a topology (see [13, 17]).

In this paper, we introduce a new family of ideals in an effect algebra $E$, and we construct a topology $\tau$ on $E$. Then we obtain some conditions under which that $(E, \tau)$ is a Hausdorff space. Also, we show that $C(0)$, the connected component of 0, is a closed ideal of $E$ and if the natural map $\pi : E \longrightarrow \frac{E}{C(0)}$ is open, then $C(\frac{x}{C(0)}) = \frac{x}{C(0)}$, for every $x \in E$ and so the quotient topological space $\frac{E}{C(0)}$ is totally disconnected topological effect algebra.

## 2. Preliminaries

In this section, we review the material that we will use in the following sections.

Recall that a set $A$ with a family $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of its subsets is called a topological space, denoted by $(A,\mathcal{U})$, if $A, \emptyset \in \mathcal{U}$, then the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called open sets of $A$ and the complement of $U \in \mathcal{U}$, that is $A \setminus U$, is said to be a closed set. If $B$ is a subset of $A$, the smallest closed set containing $B$ is called the closure of $B$ and denoted by $\bar{B}$. If there is no closed subset of $A$ containing $C$ except itself, then $C$ is called dence in $A$, where $C \subseteq A$. A subset $P$ of $A$ is said to be a neighborhood of $x \in A$ if there exists an open set $U$ such that $x \in U \subseteq P$. A subfamily $\{U_\alpha\}_{\alpha \in I}$ of $\mathcal{U}$ is said to be a base of $\mathcal{U}$ if for each $x \in \mathcal{U}$ there is an $\alpha \in I$ such that $x \in U_\alpha \subseteq \mathcal{U}$, or equivalently, each $U$ in $\mathcal{U}$ is the union of members of $\{U_\alpha\}_{\alpha \in I}$. Let $\mathcal{U}$ and $\mathcal{U}'$ be two topologies on the set $A$. If $\mathcal{U}' \subseteq \mathcal{U}$, then we say that $\mathcal{U}'$ is finer than $\mathcal{U}$. Let $(X,\mathcal{U})$ and $(Y,\mathcal{U}')$ be two topological spaces. A map $f : X \longrightarrow Y$ is called continuous if the inverse image of each open subset of $Y$ is open in $X$. A homomorphism of topological spaces is a continuous function, which is one to one, onto and has a continuous inverse.

Consider the topological space $(A,\mathcal{U})$. We have the following separation axioms:
$T_0$: For each $x, y \in A$, there is an open set that includes one of them and does not include the other, where $x \neq y$.
$T_1$: For each $x, y \in A$, there is an open set $U$ containing $x$ such that $y \notin U$, where $x \neq y$.
$T_2$: For each $x, y \in A$, there are two disjoint open sets $U, V \in \mathcal{U}$ such that $x \in U$ and $y \in V$, where $x \neq y$.
A topological space satisfying $T_i$ is called $T_i$-space, for any $i = 0, 1, 2$. A $T_2$-space is also known as a Hausdorff space.

**Definition 2.1.** [8] An *effect algebra* is a partial algebra $E = (E, \oplus, 0, 1)$ with a partially defined operation "$\oplus$" and two constant elements 0 and 1 such that, for all $a, b, c \in E$, we have:
$(E1)$ Commutative Law: $a \oplus b$ is defined in $E$ if and only if $b \oplus a$ is defined, and in such the case $a \oplus b = b \oplus a$;
$(E2)$ Associative Law: $a \oplus b$ and $(a \oplus b) \oplus c$ are defined in $E$ if and only if $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, and in such the case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
$(E3)$ Orthocomplementation Law: For any $a \in E$, there exists a unique element

$a' \in E$ such that $a \oplus a' = 1$;
$(E4)$ Zero-Unit Law: If $a \oplus 1$ is defined in $E$, then $a = 0$.

Let $E$ be an effect algebra. If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a \oplus c = b$, then $\leq$ is a partial ordering, and we write $c := b \ominus a$. A nonempty subset $I$ of $E$ is said to be an *ideal* of $E$ if the following conditions are satisfied: $(I1)$ If $x \in I$ and $y \leq x$, then $y \in I$, $(I2)$ if $x \ominus y \in I$ and $y \in I$, then $x \in I$, for any $x, y \in E$. The ideal $I$ of $E$ is called a maximal ideal of $E$ if for every ideal $J$ of $E$ that $I \subseteq J$, we have $I = J$. A set $Q \subseteq E$ is called a *sub-effect algebra* of $E$ if the following conditions are satisfied: (1) $1 \in Q$, (2) if two of the elements $a, b$ *and* $c$ in $E$ with $a \oplus b = c$ are in $Q$, then all three are in $Q$. Let $F$ be another effect algebra. A mapping $h : E \longrightarrow F$ is said to be a *homomorphism* of effect algebras (or $E$-homomorphism) if $h(1) = 1$ and $h(a \oplus b) = h(a) \oplus h(b)$, for any $a, b \in E$ whenever $a \oplus b$ is defined in $E$.

**Proposition 2.1.** [14] *A nonempty subset $I$ of effect algebra $E = (E, \oplus, 0, 1)$ is an ideal of $E$ if and only if $x \oplus y \in I$ if and only if $x, y \in I$ where $x \oplus y$ is defined in $E$.*

**Proposition 2.2.** [12] *Consider $E$ is an effect algebra. Then the following properties hold, for every $a, b \in E$:*
*(i)* $a'' = a$,
*(ii)* $1' = 0$ *and* $0' = 1$,
*(iii)* $0 \leq a \leq 1$,
*(iv)* $a \oplus 0 = a$,
*(v) If* $a \oplus b = 0$, *then* $a = b = 0$,
*(vi)* $a \leq a \oplus b$,
*(vii) If* $a \leq b$, *then* $b' \leq a'$,
*(viii)* $b \ominus a = (a \oplus b')'$,
*(ix)* $a \oplus b' = (b \ominus a)'$,
*(x)* $a = a \ominus 0$,
*(xi)* $a \ominus a = 0$,
*(xii)* $a' = 1 \ominus a$ *and* $a = 1 \ominus a'$.

**Theorem 2.3.** [15] *Every finite point set in a Hausdorff space $X$ is closed.*

Let $(A, *)$ be an algebra of type 2 and $\mathcal{U}$ be a topology on $A$. Then $(A, *, \mathcal{U})$ is called a left (right) topological algebra, if for all $a \in A$ the map $* : A \longrightarrow A$ is defined by $x \longrightarrow a * x$ ($x \longrightarrow x * a$) is continuous, or equivalently, for any $x \in A$ and any open subset $V$ containing $a * x$ ($x * a$) there exists an open subset $W$ containing $x$ such that $a * W \subseteq V$ ($W * a \subseteq V$). A right and left topological algebra $(A, *, \mathcal{U})$ is called a semi-topological algebra. Moreover, if for any $x, y \in A$ and any open subset $V$ containing $x * y$, there exists two open subset $V_1$ and $V_2$ containing $x$ and $y$ respectively, such that $V_1 \times V_2 \subseteq V$, then $(A, *, \mathcal{U})$ is called a topological algebra. Clearly, if $(A, *)$ is a topological algebra, then $* : A \times A \longrightarrow A$ is continuous, hence for each $a \in A$, the maps $a * (-) : A \longrightarrow A$ and $(-) * a : A \longrightarrow A$, sending $x$ to $a * x$ and $x * a$, respectively, are continuous. Therefore, $(A, *)$ is a semi-topological algebra, but the converse is not true (see [4, 3]).

Now, let $\mathcal{U}$ be a topology on effect algebra $E$. Then $(E, \oplus, \mathcal{U})$ is called a semi-topological effect algebra if $\oplus : E \times E \longrightarrow E$ is a continuous map. By commutative law, in an effect algebra, right and left topological effect algebra are the same. Therefore, we use the notion "topological effect algebra" instead of "semi-topological effect algebra".

**Note.** From now on, in this paper we let $E = (E, \oplus, 0, 1)$ be an effect algebra.

## 3. Constructing a topology on effect algebras by using ultra ideals

In this section, we present definition of ultra ideal in an effect algebra, and we define a binary relation on $E$ that it is congruence on $E$. Then we find a method to construct a Hausdorff topological effect algebra by using ultra ideals. Also, we will obtain some results about connected components of a topological effect algebra that they hold for the topology induced by ultra ideals.

**Definition 3.1.** Let $I$ be an ideal of $E$. Then $I$ is called an *ultra* ideal of $E$ if for every $x \in E$,

$$x \in I \Longleftrightarrow x' \notin I$$

**Example 3.1.** $(i)$ Let $E = \{0, 1, 2, 3\}$ and the operation " $\oplus$ " be defined on $E$ as follows:

| $\oplus$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 3 | – |
| 2 | 2 | 3 | 2 | – |
| 3 | 3 | – | – | – |

Then $(E; \oplus, 0, 3)$ is an effect algebra such that $I = \{0, 1\}$ and $J = \{0, 2\}$ are two ultra ideals of $E$ and $K = \{0\}$ is not an ultra ideal of $E$.
$(ii)$ Let $E = \{0, 1, 2, 3, 4, 5\}$ and the operation " $\oplus$ " be defined on $E$ as follows:

| $\oplus$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 4 | 3 | 5 | – | – |
| 2 | 2 | 3 | – | – | 5 | – |
| 3 | 3 | 5 | – | – | – | – |
| 4 | 4 | – | 5 | – | – | – |
| 5 | 5 | – | – | – | – | – |

Then $(E; \oplus, 0, 5)$ is an effect algebra such that $I = \{0, 1, 4\}$ is an ultra ideal of $E$ and $K = \{0, 4\}$ is an ideal of $E$. Since $1' = 3$ and $1, 3 \notin K$, $K$ is not an ultra ideal of $E$.

**Note**. Every maximal ideal is not necessarily an ultra ideal.

**Example 3.2.** Let $E = \{0, 1, 2\}$ and the operation " $\oplus$ " be defined on $E$ as follows:

| $\oplus$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | – |
| 2 | 2 | – | – |

Then $(E; \oplus, 0, 2)$ is an effect algebra. It is easy to see that $I = \{0\}$ is a maximal ideal of $E$, but it is not an ultra ideal of $E$. Note that neither $1 \in I$ nor $1' = 1 \in I$.

In the following, we present a congruence relation on $E$, and we will have a quotient effect algebra. Let $I \subseteq E$. We set $\frac{x}{I} = \{y \in E : (x, y) \in \theta_I\}$, where for any $x, y \in E$, relation $\theta_I$ is defined on $E$, as follows:

$$(x, y) \in \theta_I \Longleftrightarrow (x \oplus y')', \ (y \oplus x')' \in I, \text{ where } x \oplus y', \ y \oplus x' \text{ are defined in E.}$$

**Proposition 3.1.** *Let $I$ be an ultra ideal of $E$. Then $\theta_I$ is a congruence relation on $E$ and* $\frac{E}{I} = \{\frac{x}{I} : x \in E\}$ *forms an effect algebra, where* $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, *for every* $x, y \in E$.

*Proof.* The first, we show that $\theta_I$ is an equivalence relation on $E$. Since $0 = (x \oplus x')' = (x' \oplus x)' \in I$, we have $(x, x) \in \theta_I$ and so $\theta_I$ is reflexive. Clearly, $\theta_I$ is a symmetric relation on $E$. Now, let $(x, y), \ (y, z) \in \theta_I$. We prove $(x \oplus z')', \ (x' \oplus z)' \in I$. If $(x' \oplus z)' \notin I$, then $x' \oplus z \in I$ and so by Proposition 2.1, $x', z \in I$. Since $(x, y) \in \theta_I$, we have $(x' \oplus y)' \in I$ and so $x' \oplus y \notin I$. Since $x' \in I$, we have $y \notin I$. Thus $y' \in I$. Since $z, y' \in I$ and $y \oplus z'$ is defined and by Proposition 2.1, we have $y' \oplus z \in I$ and so $(y' \oplus z)' \notin I$. It means that $(y, z) \notin \theta_I$, which is a contradiction. Hence $(x' \oplus z)' \in I$. Similarly, we can show that $(x \oplus z')' \in I$ and so $(x, z) \in \theta_I$. It results that $\theta_I$ is a transitive relation on $E$ and so $\theta_I$ is an equivalence relation on $E$. Now, we show that $\theta_I$ is a congruence relation on $E$. Let $x, y \in E$ such that $(x, y) \in \theta_I$. For every $a \in E$, we have $x' \oplus y \leq x' \oplus a \oplus y$ and so

$$(x' \oplus (a \oplus y))' \leq (x' \oplus y)' \in I.$$

Thus $(x' \oplus (a \oplus y))' \in I$. Similarly, we have $(x \oplus (a \oplus y)')' \in I$. Then $(x, a \oplus y) \in \theta_I$ and so $(a \oplus y, x) \in \theta_I$. Similarly, we can prove that $(y, a \oplus x) \in \theta_I$. Since $\theta_I$ is an equivalence relation, we have $(a \oplus x, a \oplus y) \in \theta_I$. Similarly, we can show that $(x \oplus a, y \oplus a) \in \theta_I$. Also, it is clear that $(x, y) \in \theta_I$ if and only if $(x', y') \in \theta_I$. Therefore, $\theta_I$ is a congruence relation on $E$ and so $\frac{E}{I} = (\frac{E}{I}, \oplus, \frac{0}{I}, \frac{1}{I})$ is an effect algebra. □

**Definition 3.2.** Let $\tau$ be a topology on $E$. Then $\tau$ is called a *linear topology* on $E$ if there is a base $\beta$ for $\tau$ such that for any element $B$ of $\beta$ containing 0, $B$ is an ideal of $E$.

**Example 3.3.** Consider $E$ is the effect algebra as Example 3.1($i$) and let $\tau = \{\emptyset \ , E, \ I, \ J, \ I \cap J, \ I \cup J\}$. Clearly, $B = \{I, \ J\}$ is a base for $\tau$ and so $\tau$ is a linear topology on $E$.

**Definition 3.3.** Let $\Lambda$ be an upward directed set and $S = \{I_i : i \in \Lambda\}$ be a family of ultra ideals of $E$. Then $S$ is called a system of ultra ideals or briefly a system of $E$ if $i \leq j$ implies $I_j \subseteq I_i$, for any $i, j \in \Lambda$.

**Lemma 3.2.** *Let $S = \{I_i : i \in \Lambda\}$ be a system of $E$. Then*
($i$) *the set* $B = \{\frac{x}{I_i} : x \in E, \ i \in \Lambda\}$ *is a base for a topology on $E$;*
($ii$) *if $\tau$ is the topology induced by $B$, then $\tau$ is a linear topology on $E$.*

*Proof.* ($i$) For any $x \in E$ and $i \in \Lambda$, we have $x \in \frac{x}{I_i}$. Then $E = \bigcup\{\frac{x}{I_i} : x \in E, \ i \in \Lambda\}$. Now, we prove that for every $B_1, B_2 \in B$ and $z \in B_1 \cap B_2$, there exists $B_3 \in B$ such that $z \in B_3 \subseteq B_1 \cap B_2$, where $B_1 = \frac{x}{I_i}, B_2 = \frac{y}{I_j}$, for some $x, y \in E$ and $i, j \in \Lambda$. Let

$z \in \frac{x}{I_i} \cap \frac{y}{I_j}$. Since $\Lambda$ is an upward directed set, there exists $\gamma \in \Lambda$ such that $i \leq \gamma$ and $j \leq \gamma$ and so $I_\gamma \subseteq I_i$ and $I_\gamma \subseteq I_j$. We consider $B_3 = \frac{z}{I_\gamma}$ and prove $\frac{z}{I_\gamma} \subseteq \frac{x}{I_i} \cap \frac{y}{I_j}$. Clearly, $z \in \frac{z}{I_\gamma}$. Let $u \in \frac{z}{I_\gamma}$. Then $(u \oplus z')'$, $(z \oplus u')' \in I_\gamma$. Since $I_\gamma \subseteq I_i$, we get $(u \oplus z')'$, $(z \oplus u')' \in I_i$ and so $u \in \frac{z}{I_i} = \frac{x}{I_i}$. By the similar way, we have $u \in \frac{z}{I_j} = \frac{y}{I_j}$ and so $\frac{z}{I_\gamma} \subseteq \frac{x}{I_i} \cap \frac{x}{I_j}$. Hence $B$ is a base for a topology on $E$.

$(ii)$ Let $\tau$ be the topology induced by $B$ and $\frac{x}{I_i}$ be an element of $B$ containing 0. Then $\frac{x}{I_i} = \frac{0}{I_i}$ and so $(x, 0) \in \theta_{I_i}$. It results that $x \in I_i$. Therefore, $\frac{x}{I_i} = I_i$ and so $\frac{x}{I_i}$ is an ideal of $E$. □

**Theorem 3.3.** *Let $S = \{I_i : i \in \Lambda\}$ be a system of $E$, $B = \{\frac{x}{I_i} : x \in E,\ i \in \Lambda\}$ and $(E, \tau)$ be a linear topological space induced by $B$. Then $(E, \tau)$ is a topological effect algebra.*

*Proof.* By Lemma 3.2, $(E, \tau)$ is a topological space. Let $f : E \times E \longrightarrow E$ be a map which for every $x, y \in E$,

$$f(x, y) = \begin{cases} x \oplus y & x \oplus y \text{ is defined in E} \\ 0 & \text{otherwise} \end{cases}$$

We prove that for any $z \in E$, $f^{-1}(\frac{z}{I_i})$ is an open subset of $E \times E$. Let $(x, y) \in f^{-1}(\frac{z}{I_i})$. Then $x \oplus y = f(x, y) \in \frac{z}{I_i}$. Clearly, $\frac{x}{I_i} \times \frac{y}{I_i}$ is an open subset of $E \times E$ containing $(x, y)$. If $f(x, y) = 0$, then we consider $\frac{0}{I_i} \times \frac{0}{I_i}$ that is an open subset of $E \times E$ containing $(0, 0)$. For any $(u, v) \in \frac{x}{I_i} \times \frac{y}{I_i}$, we have $(x, u)$, $(y, v) \in \theta_{I_i}$. By Proposition 3.1, we have $(x \oplus y, u \oplus v) \in \theta_{I_i}$ and so $u \oplus v \in \frac{z}{I_i}$. It results that $\frac{x}{I_i} \times \frac{y}{I_i} \subseteq \frac{z}{I_i}$. Hence $f^{-1}(\frac{z}{I_i})$ is an open subset of $E$. It means that $f$ is a continuous map. Therefore, $(E, \tau)$ is a topological effect algebra. □

**Proposition 3.4.** *Let $S = \{I_i : i \in \Lambda\}$ and $T = \{J_i : i \in \Gamma\}$ be two systems of $E$ and $\tau$, $\upsilon$ be the topologies induced by them, respectively. Then $\tau$ is finer than $\upsilon$ if for any $\gamma \in \Gamma$, there exists $\lambda \in \Lambda$ such that $I_\lambda \subseteq J_\gamma$.*

*Proof.* The proof is straightforward. □

**Note**. From now on, in this paper, $S = \{I_i : i \in \Lambda\}$ is a system of ultra ideals of $E$ and $\tau$ is a linear topology on $E$ induced by $B = \{\frac{x}{I_i} : i \in \Lambda\}$, unless otherwise stated.

**Proposition 3.5.** *Let $J$ be a nonempty subset of $E$. Then $J$ is an ideal of $E$ if and only if $(a' \oplus b)', b \in J$ imply $a \in J$, for any $a, b \in E$.*

*Proof.* $(\Leftarrow)$ Let for $a, b \in E$ such that $a \ominus b$, $b \in J$. Then by Proposition 2.2 $(viii)$, we have $(a' \oplus b)'$, $b \in J$ and so $a \in J$. Now, let $b \leq a$ and $a \in J$. Since $b \leq a$, there

exists $c \in E$ such that $b \oplus c = a \in J$ and so by Proposition 2.1, we have $b \in J$. Hence $J$ is an ideal of $E$.
($\Rightarrow$) Let $J$ be an ideal of $E$ and $(a' \oplus b)'$, $b \in J$, for any $a, b \in E$. By Proposition 2.2(*viii*), $a \ominus b \in J$ and so $a \in J$. □

**Lemma 3.6.** *Let $\emptyset \neq K \subseteq E$. If for any $I_i \in S$, $\frac{K}{I_i} = \bigcup\{\frac{x}{I_i} : x \in K\}$, then $\bar{K} = \bigcap\{\frac{K}{I_i} : I_i \in S\}$, where $\bar{K}$ is the topological closure of $K$.*

*Proof.* Let $x \in \bigcap\{\frac{K}{I_i} : I_i \in S\}$. Then $x \in \frac{K}{I_i}$, for every $i \in \Lambda$. Thus for every $\frac{a}{I_i} \in B$, $x \in \frac{a}{I_i}$ implies $\frac{a}{I_i} \cap K \neq \emptyset$. Hence for every $U \in \tau$, $x \in U$ implies $U \cap K \neq \emptyset$. It results that $x \in \bar{K}$ and so $\bigcap\{\frac{K}{I_i} : I_i \in S\} \subseteq \bar{K}$. Similarly, we have $\bar{K} \subseteq \bigcap\{\frac{K}{I_i} : I_i \in S\}$. Therefore, $\bar{K} = \bigcap\{\frac{K}{I_i} : I_i \in S\}$. □

**Theorem 3.7.** *If $I$ is an ideal of $E$, then $\bar{I}$ is an ideal of $E$, too.*

*Proof.* Let $I$ be an ideal of $E$ and $(x' \oplus y)'$ $, y \in \bar{I}$, for some $x, y \in E$. Then by Lemma 3.6, $\bar{I} = \bigcap\{\frac{I}{I_i} : I_i \in S\}$ and so for every $I_i \in S$, we have $(x' \oplus y)'$ $, y \in \frac{I}{I_i} = \bigcup\{\frac{t}{I_i} : t \in I\}$. Hence there exist $t_1, t_2 \in I$ such that $(x' \oplus y)' \in \frac{t_1}{I_i}$ $, y \in \frac{t_2}{I_i}$ and so $(t_1, (x' \oplus y)')$, $(t_2, y) \in \theta_{I_i}$. It means that $(t_1 \oplus (x' \oplus y))'$, $(t_2 \oplus y')' \in I_i \subseteq I$. Since $I$ is an ideal of $E$ and $t_1, t_2 \in I$, by Proposition 3.5, we have $(x' \oplus y)'$, $y \in I$ and so $x \in I$. Since $(x, x) \in \theta_{I_i}$, for every $I_i \in S$, we have $x \in \frac{I}{I_i}$ and so $x \in \bar{I}$. Therefore, $\bar{I}$ is an ideal of $E$. □

**Corollary 3.8.** *Let $I$ be an ideal of $E$. Then $I$ is dense in $E$ if and only if $I' \cap I_i \neq \emptyset$, for every $I_i \in S$, where $I' = \{x' : x \in I\}$.*

*Proof.* Let $I$ be dense in $E$. Then $\bar{I} = E$ and so by Lemma 3.6, $\frac{I}{I_i} = E$, for every $I_i \in S$. Let $I_i \in S$. Since $1 \in \frac{I}{I_i} = E$, there is $a \in I$ such that $(1, a) \in \theta_{I_i}$ and so $a' = (1' \oplus a)' \in I_i$. Hence $I_i \cap I' \neq \emptyset$.
Conversely, let $I_i \in S$ and $x$ be an arbitrary element of $E$. Since $I_i \cap I' \neq \emptyset$, there exists $a \in I_i \cap I'$ such that $a \in I_i$. Since $(a' \oplus 0)'$, $(0' \oplus a)' \in I_i$, we have $(a, 0) \in \theta_{I_i}$ and so $(0, a) \in \theta_{I_i}$. Hence by Proposition 3.1, we have $(x, (a \oplus x')') \in \theta_{I_i}$. Since $(a \oplus x')' \leq a' \in I_i$, we get $(a \oplus x')' \in I_i$. Also, since $(x, (a \oplus x')') \in \theta_{I_i}$, we have $x \in \frac{(a \oplus x')'}{I_i} \subseteq \frac{I}{I_i}$. Hence $I = \frac{I}{I_i}$, for every $I_i \in S$. Now, by Lemma 3.6, $\bar{I} = \bigcap\{\frac{I}{I_i} : I_i \in S\} = E$. Therefore, $I$ is dense in $E$.

□

**Theorem 3.9.** *The topological space $(E, \tau)$ is a Hausdorff space if and only if $\bigcap\{I_i : I_i \in S\} = \{0\}$.*

*Proof.* Let $(E,\tau)$ be a Hausdorff space. Then by Theorem 2.3, the set $\{0\}$ is closed. Hence by Lemma 3.6, we have

$$\{0\} = \{\bar{0}\} = \bigcap\{\frac{0}{I_i} : I_i \in S\} = \bigcap\{I_i : I_i \in S\}.$$

Conversely, let $\bigcap\{I_i : I_i \in S\} = \{0\}$ and $x, y$ be incomparable elements of $E$. Then $x \not\leq y$ or $y \not\leq x$ and so $(x' \oplus y)' \neq 0$ or $(x \oplus y')' \neq 0$. Let $(x' \oplus y)' \neq 0$. Then by the assumption there is $\gamma \in \Lambda$ such that $(x' \oplus y)' \notin I_\gamma$ and so $(x, y) \notin \theta_{I_\gamma}$. It means that $x \notin \frac{y}{I_\gamma}$. Since $x \in \frac{x}{I_\gamma}$ and $\frac{x}{I_\gamma} \cap \frac{y}{I_\gamma} = \emptyset$, we conclude that $(E,\tau)$ is a Hausdorff space. □

**Theorem 3.10.** *For topological space $(E,\tau)$, the following statements are equivalent:*
*(i) $(E,\tau)$ is a Hausdorff space.*
*(ii) $(E,\tau)$ is a $T_1$-space.*
*(iii) $(E,\tau)$ is a $T_0$-space.*

*Proof.* The proofs of $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.
$(iii) \Rightarrow (i)$ Let $(E,\tau)$ be a $T_0$-space and $x \in \bigcap\{I_i : I_i \in S\}$. Then $(x, 0) \in \theta_{I_i}$ and so $x \in \frac{0}{I_i}$, for every $I_i \in S$. If $U$ is an open subset of $E$ containing $x$, then there exist $y \in E$ and $I_j \in S$ such that $x \in \frac{y}{I_j} \subseteq U$. Hence $x \in \frac{y}{I_j} \cap \frac{0}{I_j}$. It follows that $\frac{0}{I_j} = \frac{y}{I_j} \subseteq U$ and so $0 \in U$. Then there is no disjoint open neighborhood for $x$ or $0$ containing the other and so by the assumption $x = 0$. Therefore, by Theorem 3.9, we conclude that $(E,\tau)$ is a Hausdorff space. □

**Theorem 3.11.** *Let $(E,\tau)$ be a topological effect algebra, $I$ be an ideal of $E$ and $\pi : E \longrightarrow \frac{E}{I}$ be the natural homomorphism. If $\frac{E}{I}$ is a Hausdorff space, then $I$ is a closed ideal of $E$. The converse is true, when $\pi$ is an open map.*

*Proof.* Let $\frac{E}{I}$ be a Hausdorff space and $x \in E \setminus I$. Then $\frac{x}{I} \neq \frac{0}{I}$ and by assumption, there are two open subsets $V, W$ of $\frac{E}{I}$ such that $\frac{x}{I} \in W$, $\frac{0}{I} \in V$ and $V \cap W = \emptyset$. Hence

$$x \in \pi^{-1}(W) \in \tau \quad , \quad I \subseteq \pi^{-1}(V) \in \tau \quad , \quad I \cap \pi^{-1}(W) = \emptyset$$

Since $\pi^{-1}(W)$ is an open subset of $E$ such that $x \in \pi^{-1}(W)$ and $\pi^{-1}(W) \cap I = \emptyset$, we have $x \notin \bar{I}$ and so $I = \bar{I}$.

Conversely, let $\pi$ be open and $\frac{x}{I} \neq \frac{y}{I}$, for some $x, y \in E$. Then $(x' \oplus y)' \in E \setminus I$ or $(x \oplus y')' \in E \setminus I$. Let $(x' \oplus y)' \in E \setminus I$. Then by assumption, $(x' \oplus y)' \notin \bar{I}$ and so there is an open subset $V$ of $E$ such that $(x' \oplus y)' \in V$ and $V \cap I = \emptyset$. Hence $\frac{0}{I} \notin \frac{V}{I}$. Consider the map $f : E \times E \longrightarrow E$ is defined by $f(a, b) = (a' \oplus b)'$, for every $a, b \in E$. Since $(E,\tau)$ is a topological effect algebra, $f$ is a continuous map and so $\pi \circ f$ is continuous map, too. Since $\pi$ is an open map, $\frac{V}{I}$ is an open subset of $\frac{E}{I}$. Hence $(\pi \circ f)^{-1}(\frac{V}{I})$ is an open subset of $E \times E$ and so there are $W, A \in \tau$ such that $(\pi\circ f)^{-1}(\frac{V}{I})=W\times A$. Since $(x'\oplus y)'\in V$, we get $(\pi\circ f)(x,y)\in\frac{V}{I}$ and so $(x,y)\in(\pi\circ f)^{-1}(\frac{V}{I})$. Hence $x\in W$, $y\in A$ and so $\frac{x}{I}\in\frac{W}{I}$ and $\frac{y}{I}\in\frac{A}{I}$. Since $\pi$ is an open map, $\frac{W}{I}$ and $\frac{A}{I}$ are open subsets of $\frac{E}{I}$. Now, we show that $\frac{W}{I}\cap\frac{A}{I}=\emptyset$. Let $\frac{z}{I}\in\frac{W}{I}\cap\frac{A}{I}$. Then there are $a\in W$ and $b\in A$ such that $\frac{a}{I}=\frac{b}{I}=\frac{z}{I}$. It results that $\frac{V}{I}=\pi\circ f(W\times A)\ni\pi(f(a,b))=\pi((a'\oplus b)')=\frac{(a'\oplus b)'}{I}=(\frac{a}{I}'\oplus\frac{b}{I})'=(\frac{z}{I}'\oplus\frac{z}{I})'=\frac{0}{I}$. It means that $\frac{0}{I}\in\frac{V}{I}$, which is a contradiction. Therefore, $\frac{W}{I}\cap\frac{A}{I}=\emptyset$ and so $\frac{E}{I}$ is a Hausdorff space. □

Let $I$ be an ideal of $E$, $\frac{E}{I}$ be the quotient effect algebra with respect to $I$, $\pi:E\longrightarrow\frac{E}{I}$ be the natural epimorphism and $\Omega$ be a topology on $E$. Then we present definition of topology on $\frac{E}{I}$ as follows:
A subset $U$ of $\frac{E}{I}$ is open if $\pi_I^{-1}(U)$ is an open subset of $E$. This topology on $\frac{E}{I}$ is called the quotient topology induced by $\pi_I$. Let $\bar{\Omega}$ be the quotient topology on $\frac{E}{I}$. If $V\in\bar{\Omega}$, then there exists $U\in\Omega$ such that $\pi_I(U)=V$.

In the next theorem, we will find a method to construct a Hausdorff topological effect algebra, using a system of ultra ideals. In fact, we will show that if $I$ is a system of $E$ and $I=\bigcap_{i\in\Lambda}I_i$, then $\frac{E}{I}$ is a Hausdorff topological effect algebra, with respect to quotient topology induced by $I$.

**Proposition 3.12.** *Let $I$ be an ideal of $E$, $\tau$ be a topology on $E$ and $\bar{\tau}$ be the quotient topology on $\frac{E}{I}$.*
*(i) If $(E,\tau)$ is a topological effect algebra, then $(\frac{E}{I},\bar{\tau})$ is a topological effect algebra, too,*
*(ii) If the canonical epimorphism $\pi_I$ is open and $(E,\tau)$ is a topological effect algebra, then $(\frac{E}{I},\bar{\tau})$ is a topological effect algebra, too.*

*Proof.* The proof is straightforward. □

**Lemma 3.13.** *Let $I$ be an ideal of topological effect algebra $(E,\tau)$ and $I\subseteq\bigcap\{I_i : I_i\in S\}$. Then the natural homomorphism $\pi:E\longrightarrow\frac{E}{I}$ is an open map and the topological space $(\frac{E}{I},\bar{\tau})$ is a topological effect algebra.*

*Proof.* Since the set $\{\frac{x}{I_i}:x\in E,\ I_i\in S\}$ is a base for $\tau$, it is sufficient to show that $\pi(\frac{x}{I_i})$ is an open subset in $\frac{E}{I}$, for any $x\in E$ and $I_i\in S$. We show that $\pi^{-1}(\pi(\frac{x}{I_i}))\in\tau$.

Let $a$ be an arbitrary element of $\pi^{-1}(\pi(\frac{x}{I_i}))$. Then $\pi(a) \in \pi(\frac{x}{I_i})$ and so $\frac{a}{I} \in \frac{(\frac{x}{I_i})}{I}$. Hence there exists $b \in \frac{x}{I_i}$ such that $(a,b) \in \theta_I$. Since $(b,x) \in \theta_{I_i}$ and $I \subseteq I_i$, we get $(a,b),\ (b,x) \in \theta_{I_i}$ and so $(a,x) \in \theta_{I_i}$. Hence $a \in \frac{x}{I_i}$ and so $\pi^{-1}(\pi(\frac{x}{I_i})) \subseteq \frac{x}{I_i}$. Clearly, $\frac{x}{I_i} \subseteq \pi^{-1}(\pi(\frac{x}{I_i}))$ and so $\frac{x}{I_i} = \pi^{-1}(\pi(\frac{x}{I_i})) \in \tau$. Now, by Proposition 3.12 $(ii)$, we conclude that $(\frac{E}{I}, \bar{\tau})$ is a topological effect algebra. □

**Proposition 3.14.** *If $I \subseteq J$, then $\frac{x}{I} \subseteq \frac{x}{J}$, where $I, J \subseteq E$.*

*Proof.* Let $y \in \frac{x}{I}$. Then $(x,y) \in \theta_I$ and so $(x' \oplus y)', (x \oplus y')' \in I \subseteq J$. Hence $(x' \oplus y)', (x \oplus y')' \in J$ and so $(x,y) \in \theta_J$. Therefore, $y \in \frac{x}{J}$ and therefore, $\frac{x}{I} \subseteq \frac{x}{J}$. □

**Theorem 3.15.** *Let $I$ be an ideal of topological effect algebra $(E, \tau)$ and $I \subseteq \bigcap\{I_i : I_i \in S\}$. Then $(\frac{E}{\bar{I}}, \bar{\tau})$ is a Hausdorff topological effect algebra, where $\bar{\tau}$ is the quotient topology on $\frac{E}{\bar{I}}$.*

*Proof.* The first, we show that the natural homomorphism $\pi' : E \longrightarrow \frac{E}{\bar{I}}$ is open. Since $B = \{\frac{x}{I_i} : x \in E,\ I_i \in S\}$ is a base for $\tau$, it is sufficient to show that $\pi'(\frac{x}{I_i})$ is an open subset in $\frac{E}{\bar{I}}$. Let $x \in E$ and $I_i \in S$. Since $I \subseteq \bar{I}$, by Proposition 3.14, we get $\frac{(\frac{x}{I_i})}{I} \subseteq \frac{(\frac{x}{I_i})}{\bar{I}}$. Let $a \in \frac{(\frac{x}{I_i})}{\bar{I}}$. Then there is $b \in \frac{x}{I}$ such that $(a' \oplus b)',\ (a \oplus b')' \in \bar{I}$. By Lemma 3.6, $(a' \oplus b)',\ (a \oplus b')' \in \frac{I}{I_i}$ and so there are $z, y \in I$ such that $(z' \oplus (a' \oplus b)')',\ (y' \oplus (a \oplus b')')' \in I_i$. Since $I \subseteq I_i$, by Proposition 3.5, we get $(a' \oplus b)',\ (a \oplus b')' \in I_i$ and so $a \in \frac{(\frac{x}{I_i})}{I}$. Hence $\frac{(\frac{x}{I_i})}{I} = \frac{(\frac{x}{I_i})}{\bar{I}}$. By Lemma 3.13, the natural homomorphism $\pi : E \longrightarrow \frac{E}{\bar{I}}$ is open and so $\pi(\frac{x}{I_i})$ is an open subset of $E$. Since $\pi(\frac{x}{I_i}) = \frac{(\frac{x}{I_i})}{I}$, we get $\frac{(\frac{x}{I_i})}{\bar{I}}$ is an open subset of $E$. It follows that $\pi'(\frac{x}{I_i}) = \frac{(\frac{x}{I_i})}{\bar{I}}$ is open and so $\pi'$ is an open map. Therefore, by Theorem 3.11 and Lemma 3.13, $(\frac{E}{\bar{I}}, \bar{\tau})$ is a Hausdorff topological effect algebra. □

Let $(E, U)$ be a topological effect algebra and $A$ be a connected subset of $E$. Then $A$ is called a connected component of $E$ if for every connected subset $B$ of $E$ with $A \subseteq B$, we have $A = B$. Moreover, $C(x)$ is showed the connected component of $E$ containing $x$, for every $x \in E$.

In the following, we want to give some results about connected components of a topological effect algebra $(E, U)$. It is clear that, they are hold for topology induced by a system $S$ of an effect algebra $E$.

**Lemma 3.16.** *Let $(E, U)$ be a topological effect algebra and $C(x)$ be the connected component of $x$, for any $x \in E$. Then*
*(i) $C(0)$ is the greatest closed ideal of $E$, which is connected.*
*(ii) $C(0) = \{0\}$ if and only if $C(x) = \{x\}$, for any $x \in E$.*

*Proof.* $(i)$ It is enough to prove that $C(0)$ is an ideal of $E$. Clearly, $C(0) \neq \emptyset$. Let $x, y \in C(0)$. We prove $x \oplus y \in C(0)$. Since $(E, \tau)$ is a topological effect algebra, the map $f_x : E \longrightarrow E$ is a continuous map, where

$$f_x(y) = \begin{cases} x \oplus y & \text{where } x \oplus y \text{ is defined in E} \\ 0 & \text{otherwise} \end{cases}$$

So $x \oplus C(0) = \{x \oplus c \mid c \in C(0)\} = f_x(C(0))$ is a connected subset of $E$. Since $0 \in C(0)$, we have

$$x = x \oplus 0 \in ((x \oplus C(0)) \cap C(0))$$

and so $C(0) \cup (x \oplus C(0))$ is a connected subset of $E$ containing 0. Hence

$$C(0) \cup (x \oplus C(0)) \subseteq C(0)$$

where $(x \oplus C(0)) \subseteq C(0)$. It follows that $x \oplus y \in C(0)$. Hence by Proposition 2.1, $C(0)$ is an ideal of $E$. Since $\overline{C(x)} = C(x)$, for any $x \in E$, $C(0)$ is the greatest closed ideal of $E$, which is connected.
$(ii)$ Let $C(0) = \{0\}$ and $x \in E$. Similar to the proof of $(i)$, we can show that $(x' \oplus C(x))'$ is a connected subset of $E$. Since $x \in C(x)$, we have $0 \in (x' \oplus C(x))'$ and so

$$(x' \oplus C(x))' \subseteq C(0) = \{0\}$$

Hence $(x' \oplus a)' = 0$, for any $a \in C(x)$. By similar way, $(a' \oplus x)' = 0$, for any $a \in C(x)$ and so $C(x) = \{x\}$. The proof of converse is clear. □

**Theorem 3.17.** *Let $(E, U)$ be a topological effect algebra and $C(0)$ be the connected component of 0 such that the natural homomorphism $\pi : E \longrightarrow \frac{E}{C(0)}$ be open. Then $C(\frac{x}{C(0)}) = \frac{x}{C(0)}$, for any $x \in E$.*

*Proof.* Since $(E, U)$ is a topological effect algebra, it is routine to see that $(\frac{E}{C(0)}, \bar{U})$ is a topological effect algebra, too. Hence by Lemma 3.16$(ii)$, it suffices to show that $C(\frac{0}{C(0)}) = \{\frac{0}{C(0)}\}$. Let $C(\frac{0}{C(0)}) \neq \{\frac{0}{C(0)}\}$ and $S$ be a connected subset of $\frac{E}{C(0)}$ containing $\{\frac{0}{C(0)}\}$ such that there exists $\frac{x}{C(0)} \in S - \{\frac{0}{C(0)}\}$. Then $\bar{S}$ is a connected subset of $\frac{E}{C(0)}$, which is closed. Clearly, $\pi^{-1}(\bar{S})$ is a closed subset of $E$ such that $C(0) \subsetneq \pi^{-1}(\bar{S})$, where $\pi : E \longrightarrow \frac{E}{C(0)}$ is the natural homomorphism. Since $C(0)$ is the greatest connected subset of $E$ containing 0, so $\pi^{-1}(\bar{S})$ is not connected. Hence there exists $A \subseteq \pi^{-1}(\bar{S})$, which is both closed and open in $\pi^{-1}(\bar{S})$. Let

$B = \pi^{-1}(\bar{S}) - A$. Since $\pi^{-1}(\bar{S})$ is a closed subset of $(E, U)$ and $A$ and $B$ are closed subsets of $\pi^{-1}(\bar{S})$, we get $A, B$ are closed subsets of $(E, U)$. Since $\pi$ is onto, we have

$$\bar{S} = \pi(\pi^{-1}(\bar{S})) = \pi(A \cup B) = \pi(A) \cup \pi(B).$$

Since $\pi^{-1}(\pi(A)) = A$ and $\pi^{-1}(\pi(B)) = B$, we have

$$\pi^{-1}(\pi(A) \cap \pi(B)) = \pi^{-1}(\pi(A)) \cap \pi^{-1}(\pi(B)) = A \cap B = \emptyset$$

and so $\pi(A) \cap \pi(B) = \emptyset$. Now, we show that $\frac{E}{C(0)} - \pi(A) = \pi(E - A)$ and $\frac{E}{C(0)} - \pi(B) = \pi(E - B)$. Clearly,

$$\frac{E}{C(0)} - \pi(A) \subseteq \frac{E - A}{C(0)} = \pi(E - A)$$

Let $\frac{x}{C(0)} \in \frac{E - A}{C(0)}$. Then there is $y \in E - A$ such that $\frac{x}{C(0)} = \frac{y}{C(0)}$. Let $y \in \pi^{-1}(\bar{S})$. Since $y \in A \cup B$ and $y \in E - A$, we get $y \in B$. Hence $\frac{y}{C(0)} \in \frac{B}{C(0)} = \pi(B)$. Since $\pi(A) \cap \pi(B) = \emptyset$, we have $\frac{y}{C(0)} \notin \frac{A}{C(0)}$ and so

$$\frac{x}{C(0)} = \frac{y}{C(0)} \in \frac{E}{C(0)} - \frac{A}{C(0)}$$

If $y \notin \pi^{-1}(\bar{S})$, then $\frac{y}{C(0)} \notin \bar{S}$. Since $\bar{S} = \pi(A) \cup \pi(B)$, we have $\frac{y}{C(0)} \notin \pi(A) = \frac{A}{C(0)}$ and so

$$\frac{x}{C(0)} = \frac{y}{C(0)} \in \frac{E}{C(0)} - \frac{A}{C(0)}$$

Then $\frac{E}{C(0)} - \pi(A) = \pi(E - A)$. By the similar way, we have $\frac{E}{C(0)} - \pi(B) = \pi(E - B)$. Since $A, B$ are closed subsets of $(E, U)$ and $\pi$ is an open map, we get $\frac{E}{C(0)} - \pi(A)$ and $\frac{E}{C(0)} - \pi(B)$ are open subset of $(\frac{E}{C(0)}, \bar{U})$ and so $\pi(A)$ and $\pi(B)$ are closed subsets of $(\frac{E}{C(0)}, \bar{U})$. Hence $\pi(A)$ and $\pi(B)$ are closed subsets of $\bar{S}$. By $\bar{S} = \pi(A) \cup \pi(B)$ and $\pi(A) \cap \pi(B) = \emptyset$, we get $\pi(A) = \bar{S} - \pi(B)$ and so $\pi(A)$ is an open subset of $\bar{S}$. It follows that $\bar{S}$ is not connected, which is a contradiction. Therefore, $C(\frac{0}{C(0)}) = \{\frac{0}{C(0)}\}$. □

## 4. Conclusion

Effect algebras generalized many examples of quantum structures, like Boolean algebras, orthomodular lattices or posets, orthoalgebras, $MV$-algebras and etc. In this paper, a new family of ideals in an effect algebra $E$ is introduced, and a topology $\tau$ on $E$ by those ideals is constructed. Some conditions under which that $(E, \tau)$ is a Hausdorff space are investigated. Also, some results are obtained about connected components of this topological space, and a quotient topological effect algebra is constructed. In future works, we will present definitions of cauchy sequence and convergence sequence in a linear topological effect algebra. Then we intend to construct a complete topology on an effect algebra. Also, we shall try to investigate tensor product of effect algebras and topological structures of them.

## References


[1] M.K. Bennett and D.J. Foulis, Phi-symmetric effect algebras, *Foundations of Physics* **25** (1995), 1699–1722.

[2] R.A. Borzooei, A. Dvurečenskij, and A.H. Sharafi, Material implications in lattice effect algebras, *Information Sciences* **433–434** (2018), 233–240. DOI: 10.1016/j.ins.2017.12.049

[3] R.A. Borzooei, G.R. Rezaei, and N. Kouhestani, On (semi)topological $BL$-algebra, *Iranian Journal Mathematical Science and Informatics* **6** (2001), no. 1, 59–77.

[4] R.A. Borzooei and O. Zahiri, Topology on $BL$-algebras, *Fuzzy Sets and Systems* **289** (2016), 137–150. DOI: 10.1016/j.fss.2014.11.014

[5] B. Busneag, D. Piciu, and M. Istrata, The Belluce lattice associated with a bounded $BCK$-algebra, *Journal of Algebraic Hyperstructures and Logical Algebras* **1** (2021), no. 1, 1–16. DOI: 10.52547/HATEF.JAHLA.2.1.1

[6] R. Cignoli, I.M.L. D'Ottaviano, and D. Mundici, *Algebraic foundations of many-valued reasoning*, Springer, Dordrecht, 2000.

[7] A. Dvurečenskij and S. Pulmannová, *New trends in quantum structures*, Springer Dordrecht, 2000.

[8] A. Dvurečenskij, Product effect algebras, *International Journal of Theoretical Physics* **10** (2002), 193–215. DOI: 10.1023/A:1021017905403

[9] A. Dvurečenskij, States on pseudo effect algebras and integrals, *Foundations of Physics* **41** (2011), 1143–1162. DOI: 10.1007/s10701-011-9537-4

[10] A. Dvurečenskij and T. Vetterlein, Pseudo effect algebras. I. Basic properties, *International Journal of Theoretical Physics* **40** (2001), 685–701. DOI: 10.1023/A:1004192715509

[11] A. Dvurečenskij and T. Vetterlein, Pseudo effect algebras. II. Group represetation, *International Journal of Theoretical Physics* **40** (2001), 703–726. DOI: 10.1023/A:1004144832348

[12] D.J. Foulis and M.K. Bennett, Effect algebras and unsharp quantum logics, *Foundations of Physics* **24** (1994), no. 10, 1331–1352.

[13] D.J. Foulis, S. Pulmannova, and E. Vincekova, Unitizations of generalized pseudo effect algebras and their ideals, *Order* **33** (2016), 311–332. DOI: 10.1007/s11083-015-9368-6

[14] Z. Ma, Note on ideals of effect algebras, *Information Sciences* **5** (2009), 505–507. DOI: 10.1016/j.ins.2008.07.018

[15] J. R. Munkres, *Topology a first course*, Prentice-Hall, 1974.

[16] M.R. Rakhshani, R.A. Borzooei, and G.R. Rezaei, On topological effect algebras, *Italian Journal of Pure and Applied Mathematics* **39** (2018), 312–325.

[17] S. Saidi Goraghani and R.A. Borzooei, Module structures on effect algebras, *Bulletin of the Section of Logic* **49**(3) (2020), 269–290. DOI: 10.18778/0138-0680.2020.17



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