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General properties of the symmetric groupoid of a finite set

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Abstract. The aim of this paper is to give some basic properties of the symmetric groupoid of an arbitrary or finite set. The determination of subgroupoids and related topics on the symmetric groupoid of a finite set is discussed.

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1. Introduction

The concept of groupoid was introduced by H. Brandt [1] in a 1926 paper on the composition of quadratic forms in four variables. The concept of groupoid is a generalization of the notion of group. In many aspects a groupoid is like a group with several neutral elements. A groupoid with only one neutral element is a group.

Groupoids also appeared in Galois theory in the description of relations between subfields of a field K via morphisms of K in a paper of A. Loewy [11] around 1927 (the isotropy groups of the constructed groupoid turn out to be the Galois groups).

A generalization of Brandt groupoid has appeared in the work of C. Ehresmann [5] around 1950. There are various definitions for Brandt groupoids, see [2], [3], [14], [15]. In this paper we use the definition of the groupoid given in [14].

A groupoid can be endowed with other algebraic, topological or geometric structures. So we will find Borel groupoids, topological groupoids, measure groupoids, Lie groupoids, symplectic groupoids and so on.

The algebraic, topological and differentiable groupoids play an important role by their applications in algebra, measure theory, harmonic analysis , differential geometry, symplectic geometry and quantum mechanics. For details in these areas, see [3], [4], [6]-[9], [12]-[16].

This paper deals with the groupoids in the sense of Brandt. In the second Section some definitions and results about groupoids are given. The third Section deals with the symmetric groupoid of an arbitrary set. Section 4 is dedicated to establish properties of the symmetric groupoid of degree n.

2. Groupoids and related concepts

Let G be a set endowed with the maps α (*source*) and β (*target*), $\alpha, \beta : G \longrightarrow G$, the *composition law* $\mu : G_{(2)} \longrightarrow G, (x,y) \longrightarrow \mu(x,y)$, where $G_{(2)} = \{ (x,y) \in G \times G \mid \beta(x) = \alpha(y) \}$ and the *inversion map* $i : G \longrightarrow G, x \longrightarrow i(x)$. We write sometimes $x \cdot y$ or xy for $\mu(x,y)$ and x^{-1} for $i(x)$. The elements of $G_{(2)}$ are called *composable pairs* of G.

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The 5- tuple (G,α,β,μ,i) is a (*Brandt*) *groupoid*, if the maps α,β,μ,i satisfy the following axioms :

(1) (*associativity*) $(xy)z = x(yz)$, in the sense that, if one side of the equation is defined so is the other one and then they are equal;

(2) (*identities*) $(\alpha(x),x),\ (x,\beta(x)) \in G_{(2)}$ and $\alpha(x)x = x\beta(x) = x$;

(3) (*inverses*) $(x^{-1},x),\ (x,x^{-1}) \in G_{(2)}$ and $x^{-1}x = \beta(x),\ xx^{-1} = \alpha(x)$.

The element $\alpha(x)$ [resp. $\beta(x)$] is the *left unit* (resp. *right unit*) of $x \in G$. The subset $G_0 = \alpha(G) = \beta(G)$ of G is called the *unit set* of G and we say that G is a G_0 - *groupoid* or *a groupoid over* G_0.

A G_0 -groupoid G will be denoted by $(G,\alpha,\beta,\mu,i;G_0)$ or $(G,\alpha,\beta;G_0)$ or $(G;G_0)$. The maps α,β,μ and i are called the *structural functions* of G.

For each $u \in G_0$, the set $\alpha^{-1}(u)$ (resp. $\beta^{-1}(u)$) is called the α - *fibre* [resp. β - *fibre*] *of* G *over* $u \in G_0$.

A G_0 -groupoid G is said to be *transitive*, if the map $(\alpha,\beta) : G \longrightarrow G_0 \times G_0,\ x \rightarrow (\alpha,\beta)(x) = (\alpha(x),\beta(x))$ is surjective ; (α,β) is called the *anchor* of G.

For the structural functions of a groupoid $(G,\alpha,\beta,\mu,i;G_0)$, the following assertions hold:

$$\alpha(u) = \beta(u) = u \text{ and } u \cdot u = u \text{ for}all\ u \in G_0; \tag{1}$$

$$\alpha(xy) = \alpha(x) \text{ and } \beta(xy) = \beta(y),\ (\forall)\ (x,y) \in G_{(2)}; \tag{2}$$

$$\alpha \circ i = \beta,\ \beta \circ i = \alpha \text{ and } i \circ i = Id_G; \tag{3}$$

$$G(u) = \alpha^{-1}(u) \cap \beta^{-1}(u) = \{x \in G \mid \alpha(x) = \beta(x) = u\ \} \tag{4}$$

is a group with respect to the restriction of μ to $G(u)$, called the **isotropy group of** G **at** u.

Definition 2.1. *A nonempty subset* H *of a* G_0*-groupoid* G *is called* **subgroupoid** *of* G *if it is closed under multiplication (when it is defined) and inversion, i.e. the following conditions hold :*

(i) *for all* $x,y \in H$ *such that* xy *is defined, we have* $xy \in H$*;*

(ii) *for all* $x \in H$, *we have* $x^{-1} \in H$.

Note that from the condition (ii) of Definition 2.1, we deduce that $\alpha(h) \in H$ and $\beta(h) \in H$, for all $h \in H$. If $\alpha(H) = \beta(H) = G_0$, then H is called a *wide subgroupoid* of G.

A group $\mathcal{G}$ having e as unit element is just a $\{e\}$ - groupoid and conversely, every groupoid with one unit element is a group. The wide subgroupoids of $\mathcal{G}$ are just the subgroups of $\mathcal{G}$.

Example 2.1. (i) *We give on a nonempty set* X *the following groupoid structure :* $\alpha = \beta = Id_X$, *the elements* $x,y \in X$ *are composable iff* $x = y$ *and we define* $xx = x$. *This groupoid is called the* **nul groupoid** *over* X.

(ii) *The* **pair groupoid over a set**. *Let* X *be a nonempty set. Then* $G = X \times X$ *is a groupoid with respect to the rules:* $\alpha(x,y) = (x,x),\ \beta(x,y) = (y,y)$, *the elements* (x,y) *and* (y',z) *are composable in* G *iff* $y' = y$ *and we take* $(x,y)\cdot(y,z) = (x,z)$ *and the inverse of* (x,y) *is defined by* $(x,y)^{-1} = (y,x)$. *The unit space of the pair groupoid* $X \times X$ *is the diagonal* $\Delta_X = \{(x,x) \mid x \in X\}$ *which can be identified with* X. *The isotropy group* $G(u)$ *at* $u = (x,x)$ *is the nul group* $\{(u,u)\}$.

A subgroupoid H *of the pair groupoid* $X \times X$ *is a relation on* X *which is symmetric and transitive. A wide subgroupoid* H *is an equivalence relation on* X.

(iii) *If* $\{ G_i \mid i \in I \}$ *is a disjoint family of groupoids, let* $G = \cup_{i\in I} G_i$ *and* $G_{(2)} = \cup_{i\in I} G_{i,(2)}$. *Here, two elements* $x, y \in G$ *may be composed iff they lie in the same groupoid* G_i. *This groupoid is called the* **disjoint union of the groupoids** $G_i, i \in I$, *and it is denoted by* $\coprod_{i\in I} G_i$. *The unit set of this groupoid is* $G_0 = \cup_{i\in I} G_{i,0}$, *where* $G_{i,0}$ *is the unit set of* G_i.

In particular, the disjoint union of the groups $\mathcal{G}_i, i \in I$, *is a groupoid,i.e.* $G = \coprod_{i\in I} \mathcal{G}_i$, *which be called the* **groupoid associated to family of groups** $\mathcal{G}_i, i \in I$. *For this groupoid, the isotropy group at* $e_i \in \mathcal{G}_i$ *is the group* $\mathcal{G}_i$ *and* $G_0 = \{e_i \mid i \in I\}$, *where* e_i *is the unit element of* $\mathcal{G}_i$.

Let $(G, \alpha, \beta, \mu, i; G_0)$ and $(G', \alpha', \beta', \mu', i'; G'_0)$ be two groupoids. A *morphism between these groupoids* is a pair $(f, \widetilde{f})$ of maps $f : G \longrightarrow G'$ and $\widetilde{f} : G_0 \to G'_0$ such that the following two conditions are satisfied:

(i) $f(\mu(x, y)) = \mu'(f(x), f(y))$, for all $(x, y) \in G_{(2)}$;

(ii) $\alpha' \circ f = \widetilde{f} \circ \alpha$ and $\beta' \circ f = \widetilde{f} \circ \beta$.

A morphism of groupoids $(f, \widetilde{f}) : (G; G_0) \to (G'; G'_0)$ is said to be *isomorphism of groupoids*, if f and $\widetilde{f}$ are bijective maps.

Example 2.2. (i) *If* $\mathcal{G}$ *is a group, then the map* $\delta : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $\delta(x, y) = xy^{-1}$, *is a morphism of groupoids from the pair groupoid* $\mathcal{G} \times \mathcal{G}$ *into* $\mathcal{G}$.

(ii) *The anchor map* $(\alpha, \beta) : G \longrightarrow G_0 \times G_0$ *of the* G_0 *-groupoid* G *into the pair groupoid* $G_0 \times G_0$ *is a morphism of groupoids.*

In a groupoid $(G, \alpha, \beta; G_0)$ the relation defined on G_0 by:

$$u \sim_G v \iff (\exists) x \in G \text{ with } \alpha(x) = u \text{ and } \beta(x) = v \tag{5}$$

is an equivalence relation. Its equivalence classes are called *orbits* and the orbit of $u \in G_0$ is denoted by $[u]$. The quotient set G_0/G determined by this equivalence relation is called the *orbit space*.

A groupoid (G, G_0) is transitive iff G_0/G is a singleton.

There is a natural decomposition of the unit space G_0 of a groupoid G into orbits. Over each orbit there is a transitive groupoid and the disjoint union of these transitive groupoids is the original groupoid G.

Definition 2.2. *By a* **normal subgroupoid** *of a groupoid* G, *we mean a wide subgroupoid* H *of* G *satisfying the property : for all* $x \in G$ *and* $h \in H$ *such that the product* xhx^{-1} *is defined, we have* $xhx^{-1} \in H$.

Proposition 2.1. ([4]) *A wide subgroupoid* H *of the* G_0 *- groupoid* G *is normal iff* $xH(\beta(x)) = H(\alpha(x))x$ *for all* $x \in G$, *where* $H(u)$ *denotes the isotropy group of the groupoid* H *at* u.

Example 2.3. (i) *If* G *is a* G_0 *- groupoid, then* G_0 *and* $Is(G) = \{x \in G \mid \alpha(x) = \beta(x)\} = \cup_{u\in G_0} G(u)$ *are normal subgroupoids of* G, *called the* **nul subgroupoid** *and the* **isotropy subgroupoid** *of* G, *respectively.*

(ii) *If* $f : G \to G'$ *is a morphism of groupoids, then* $Kerf = \{x \in G \mid f(x) \in G'_0\}$ *is a normal subgroupoid of* G.

Let H a wide subgroupoid of the G_0 - groupoid G. The relation " $\equiv$ " defined on the groupoid G by:

$$\begin{cases} x \equiv y(mod\, H) \iff \\ (\exists) h \in H(\alpha(x)), h' \in H(\beta(x)) \text{ such that } y = hxh' \end{cases} \tag{6}$$

is an equivalence relation. We denote by $\widehat{x}$ the equivalence class of $x \in G$ relative to the equivalence relation " $\equiv$ " and let $G/\equiv \, = \{ \widehat{x} \mid x \in G \}$ be the set of the equivalence classes defined on G by " $\equiv$ ".

Proposition 2.2. *([4]) If H is a normal subgroupoid of G, then*

$$\widehat{x} = xH(\beta(x)) = H(\alpha(x))x, \ (\forall) \ x \in G. \tag{7}$$

If H is a normal subgroupoid of G, then

$$G/\equiv = \{xH(\beta(x)) \,|\, x \in G\} = \{ H(\alpha(x))x \,|\, x \in G\} \tag{8}$$

$$G_0/\equiv = \{ \widehat{u} \mid u \in G_0 \} = \{ H(u) \mid u \in G_0 \}. \tag{9}$$

The quotient set $G/\equiv$ has a natural structure of groupoid having $G_0/\equiv$ as unit set with respect to the following rules: $-$ $\widehat{\alpha}, \widehat{\beta} : G/\equiv \to G/\equiv$ are defined by

$$\widehat{\alpha}(\widehat{x}) = H(\alpha(x)), \ \widehat{\beta}(\widehat{x}) = H(\beta(x)), \tag{10}$$

$-$ the multiplication law is defined by

$$\widehat{x} \cdot \widehat{y} \ = \ (xy)H(\beta(xy)) \iff \widehat{\beta}(\widehat{x}) = \widehat{\alpha}(\widehat{y}) \tag{11}$$

$-$ and the inverse of $\widehat{x} = xH(\beta(x))$ is defined by

$$\widehat{i}(\widehat{x}) \ = \ x^{-1}H(\beta(x^{-1})) \ = \ x^{-1}H(\alpha(x)). \tag{12}$$

The groupoid $(G/\equiv, \widehat{\alpha}, \widehat{\beta}, \widehat{\mu}, \widehat{i}; G_0/\equiv)$ is called the *quotient groupoid of G* relative to H and will be denoted by G/H. For more details concerning the groupoids, see [4], [7]-[9], [16].

3. The symmetric groupoid $\mathcal{S}(M)$

Let M be a nonempty set. By *a quasipermutation of the set M* we mean an injective map from a subset of M into M.

We denote by $G = \mathcal{S}(M)$ or $G = Inj(S)$ the set of all quasipermutations of M, i.e. $\mathcal{S}(M) = \{f \mid f : A \to M, f \text{ is injective and } \emptyset \neq A \subseteq M \}$.

For $f \in \mathcal{S}(M)$, let D(f) be the domain of f, $R(f) = f(D(f))$ and $G_{(2)} = \{ (f, g) \mid R(f) = D(g) \}$. For $(f, g) \in G_{(2)}$ we define $\mu(f, g) = g \circ f$.

If Id_A denotes the identity on A, then $G_0 = \{ Id_A \mid A \subseteq M \}$ is the set of units of G, denoted by $\mathcal{S}_0(M)$ and f^{-1} is the inverse function from $R(f)$ to $D(f)$. The maps α, β are defined by $\alpha(f) = Id_{D(f)}$, $\beta(f) = Id_{R(f)}$. Thus $\mathcal{S}(M)$ is a groupoid over $\mathcal{S}_0(M)$. $\mathcal{S}(M)$ is called the *symmetric groupoid of M* or the *groupoid of quasipermutations of M*.

Theorem 3.1. *If M and N are equipotent sets, then the symmetric groupoids $\mathcal{S}(M)$ and $\mathcal{S}(N)$ are isomorphic.*

Proof. Let $\varphi : M \to N$ be a bijective map. For each $f \in \mathcal{S}(M)$ we have that $\varphi \circ f \circ \varphi^{-1}$ is an injective mapping from a subset of N into N. Hence $\varphi \circ f \circ \varphi^{-1} \in \mathcal{S}(N)$. It is easy to check that the map $\widetilde{\varphi} : \mathcal{S}(M) \to \mathcal{S}(N)$ defined by $\widetilde{\varphi}(f) = \varphi \circ f \circ \varphi^{-1}$, $(\forall)\ f \in \mathcal{S}(M)$, is a bijective morphism of groupoids. Therefore, the groupoids $\mathcal{S}(M)$ and $\mathcal{S}(N)$ are isomorphic. □

For a given groupoid $(G; G_0)$, let $(\mathcal{S}(G); \mathcal{S}_0(G))$ be the symmetric groupoid of the set G, where $\mathcal{S}_0(G) = \{Id_A \mid A \subseteq G\}$.

We consider now the set $\mathcal{L}(G) = \{\ L_a \mid a \in G\ \}$ of all left translations $L_a : G \longrightarrow G,\ x \to L_a(x) = ax$, whenever $(a, x) \in G_{(2)}$.

We have $D(L_a) = \{\ x \in G \mid (a, x) \in G_{(2)}\ \} \neq \emptyset$, since $(a, \beta(a)) \in G_{(2)}$ and so $L_a \in \mathcal{S}(G)$. Hence $\mathcal{L}(G)$ is a subset of $\mathcal{S}(G)$.

For all $a, b, x \in G$ such that $\beta(a) = \alpha(b)$ and $\beta(b) = \alpha(x)$ we have $L_a(L_b(x)) = L_a(bx) = a(bx) = (ab)x = L_{ab}(x)$ and we note that $L_a \circ L_b = L_{ab}$ if $(a, b) \in G_{(2)}$. Consequently, we have $L_{\alpha(x)} \circ L_x = L_x \circ L_{\beta(x)} = L_x$, $(\forall)\ x \in G$.

For all $u \in G_0$ we have $L_u = Id_{D(L_u)}$, hence $L_u \in \mathcal{S}_0(G)$ and $\mathcal{L}_0(G) = \{L_u \mid u \in G_0\}$ is a subset of $\mathcal{S}_0(G)$. Since $\mathcal{L}(G) \subseteq \mathcal{S}(G)$ and the conditions (i) and (ii) from Definition 2.1 are satisfied, it follows that $\mathcal{L}(G)$ is a subgroupoid of $\mathcal{S}(G)$.

This groupoid is called the *groupoid of left translations of* G.

Theorem 3.2. *(***Cayley theorem for groupoids***.) Every groupoid G is isomorphic to a subgroupoid of the symmetric groupoid $\mathcal{S}(G)$.*

Proof. Let $(\mathcal{L}(G); \mathcal{L}_0(G))$ be the groupoid of left translations of G. We have that $\mathcal{L}(G)$ is a subgroupoid of $\mathcal{S}(G)$. It is easy to verify that $\varphi : G \longrightarrow \mathcal{L}(G),\ \varphi(a) = L_a$, $(\forall)\ a \in G$, is an isomorphism of groupoids. □

Remark 3.1. *In view of Cayley's theorem for groupoids, many groupoids occur naturally as subgroupoids of some symmetric groupoid.*

4. The symmetric groupoid of a finite set

When $M = \{1, 2, ..., n\}$, we write $\mathcal{S}_n$ for $\mathcal{S}(M)$ and call $\mathcal{S}_n$ the *symmetric groupoid of degree* n.

The symmetric groupoid of a finite set play an important role in the study of finite groupoids, since by Cayley's theorem every finite groupoid of degree n is isomorphic to some subgroupoid of $\mathcal{S}_n$.

Theorem 4.1. *([10]) Let n be a fixed number such that $n \geq 1$. The symmetric groupoid $\mathcal{S}_n$ contains $|\mathcal{S}_n|$ elements, where*

$$\mid \mathcal{S}_n \mid \ = \sum_{k=1}^{n} k!(\tbinom{n}{k})^2.$$

Proof. For each $k, 1 \leq k \leq n$, we denote by $X_k = \{\ i_1, i_2, \ldots, i_k\ \}$ a subset of $M = \{1, 2, \ldots, n\}$ such that $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

If X_k is a fixed subset of M, let $f_k : X_k \to M$ be an injective mapping. We write the function f_k in the following form $\begin{pmatrix} i_1 & i_2 & \ldots & i_k \\ f_k(i_1) & f_k(i_2) & \ldots & f_k(i_k) \end{pmatrix}$, where $f_k(i_j) \in M$, for $j = \overline{1, k}$ and $f_k(i_j) \neq f_k(i_s)$ for $j, s = \overline{1, k}$.

Using the fact that the set M contains $\binom{n}{k}$ subsets with k elements of the form $\{f_k(i_1), f_k(i_2), \ldots, f_k(i_k)\}$, it follows that there exist $\binom{n}{k}$ injective mappings having the domain $\{i_1, i_2, \ldots, i_k\}$ and with values into M, where $\binom{n}{k}$ is the k -th binomial coefficient.

For each injective mapping

$$f_k = \begin{pmatrix} i_1 & i_2 & \ldots & i_k \\ f_k(i_1) & f_k(i_2) & \ldots & f_k(i_k) \end{pmatrix}$$

having the image $\{f_k(i_1), f_k(i_2), \ldots, f_k(i_k)\}$ we obtain $k!$ injective mappings taking on 2 -th arrow an arbitrary permutation of elements $f_k(i_1), f_k(i_2), \ldots, f_k(i_k)$. Hence, for a fixed subset X_k of M we have $k!\binom{n}{k}$ injective mappings defined on X_k with values into M.

The set M contains $\binom{n}{k}$ subsets of the form $X_k = \{i_1, i_2, \ldots, i_k\}$ and for each X_k there exist $k!\binom{n}{k}$ injective mappings. This implies that we have $(k!\binom{n}{k})\binom{n}{k} = k!(\binom{n}{k})^2$ injective mappings having the domain that contains k elements of M.

Since $1 \leq k \leq n$ we obtain that $\mathcal{S}_n$ contains $\sum\limits_{k=1}^{n} k!(\binom{n}{k})^2$ injective mappings defined on the subsets of M and with values into M. $\square$

Theorem 4.2. *Let $\mathcal{S}_n$ be the symmetric groupoid of degree n. Then the normal subgroupoids $\mathcal{S}_{n,0}$ and $Is(\mathcal{S}_n)$ contain $|\mathcal{S}_{n,0}|$ resp. $|Is(\mathcal{S}_n)|$ elements, where*

$$|\ \mathcal{S}_{n,0}\ | \ = \ 2^n - 1, \quad |\ Is(\mathcal{S}_n)| = \sum_{k=1}^{n} k!\binom{n}{k}.$$

Proof. For $1 \leq k \leq n$, we denote by $X_k = \{\ i_1, i_2, \ldots, i_k\ \}$ a subset of $M = \{1, 2, \ldots, n\}$ such that $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

If X_k is a fixed subset of M, then $Id_{X_k} = \begin{pmatrix} i_1 & i_2 & \ldots & i_k \\ i_1 & i_2 & \ldots & i_k \end{pmatrix}$ is a unity of $\mathcal{S}_n$.

Using the fact that the set M contains $\binom{n}{k}$ subsets with k elements of the form $\{f_k(i_1), f_k(i_2), \ldots, f_k(i_k)\}$, it follows that there exist $\binom{n}{k}$ units having the domain $\{i_1, i_2, \ldots, i_k\}$.

Since $1 \leq k \leq n$ we obtain that $\mathcal{S}_{n,0}$ contains $\sum\limits_{k=1}^{n} \binom{n}{k}$ identity mappings defined on the subsets of M with values into M. Hence $|\mathcal{S}_{n,0}| = 2^n - 1$.

Using the fact that the isotropy groups $G(f_j)$, $f_j \in G_0 = \mathcal{S}_{n,0}$ of $\mathcal{S}_n$ are disjoint sets, we have $|Is(\mathcal{S}_n)| = |\bigcup_{f_j \in G_0} G(f_j)| = \sum\limits_{f_j \in G_0} |G(f_j)|$.

For a fixed unity $Id_{X_k} = \begin{pmatrix} i_1 & i_2 & \ldots & i_k \\ i_1 & i_2 & \ldots & i_k \end{pmatrix}$, denoted by $f_{0,k}$, the isotropy group $G(f_{0,k})$ is a group of order $k!$.

Using the fact that the set G_0 contains $\binom{n}{k}$ units with the domain $\{i_1, i_2, \ldots, i_k\}$, it follows that there exist $\binom{n}{k}$ isotropy groups having $k!$ elements. Therefore, we have $|\ Is(\mathcal{S}_n)| = \sum\limits_{k=1}^{n} k!\binom{n}{k}$. $\square$

Let us illustrate the concepts of Section 2 in the case of the symmetric groupoid of degree 2 or 3.

By Theorem 4.1, the symmetric groupoid $\mathcal{S}_2$ is $\{f_i \mid i = \overline{1,6}\ \}$, where $f_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $f_3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $f_4 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $f_5 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $f_6 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

The composition law $\mu : G_{(2)} \to G$ defined on $G = \mathcal{S}_2$ is given in the table

$\cdot$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1			f_4		
f_2		f_2			f_5	
f_3			f_3			f_6
f_4		f_4			f_1	
f_5	f_5			f_2		
f_6			f_6			f_3

The absence of the element from the arrow "i" and the column "j" in the table of composition law indicates the fact that the pair $(f_i, f_j) \in \mathcal{S}_2 \times \mathcal{S}_2$ is not composable. Indeed, for example we have that $f_1 \cdot f_2$ is not defined, since $R(f_1) = \{1\} \neq D(f_2) = \{2\}$; $f_1 \cdot f_4 = f_4 \circ f_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = f_4$.

The set of unit elements of G is $G_0 = \{f_1, f_2, f_3\}$. The structural functions α, β and i, the α - fibres, β - fibres and isotropy groups of $G = \mathcal{S}_2$ are given by

f	f_1	f_2	f_3	f_4	f_5	f_6
$\alpha(f)$	f_1	f_2	f_3	f_1	f_2	f_3
$\beta(f)$	f_1	f_2	f_3	f_2	f_1	f_3
$i(f)$	f_1	f_2	f_3	f_5	f_4	f_6

u	f_1	f_2	f_3
$\alpha^{-1}(u)$	$\{f_1, f_4\}$	$\{f_2, f_5\}$	$\{f_3, f_6\}$
$\beta^{-1}(u)$	$\{f_1, f_5\}$	$\{f_2, f_4\}$	$\{f_3, f_6\}$
$G(u)$	$\{f_1\}$	$\{f_2\}$	$\{f_3, f_6\}$

We calculate now the orbits of the symmetric groupoid $G = \mathcal{S}_2$.

We have $f_2 \sim f_1$, since $(\exists)\ f_5 \in G$ such that $\alpha(f_5) = f_2$ and $\beta(f_5) = f_1$. We obtain that $[f_1] = [f_2] = \{\ f_1, f_2\ \}$ and $[f_3] = \{f_3\}$. Therefore, the orbit space of $G = \mathcal{S}_2$ is $G_0/G = \{\ \{f_1, f_2\ \},\ \{\ f_3\ \}\ \}$. We have that $G = \mathcal{S}_2$ is not a transitive groupoid.

Proposition 4.1. *The symmetric groupoid $G = \mathcal{S}_2$ contains a transitive subgroupoid $H_4 = \{\ f_1, f_2, f_4, f_5\ \}$ of order 4.*

Proof. Using the tables for the composition law defined on $\mathcal{S}_2$ and for the structural functions, it is easy to verify that the conditions from the definition of a subgroupoid are satisfied for H_4. The set of unit elements of H_4 is $H_{4,0} = \{f_1, f_2\}$ and H_4 is not a wide subgroupoid of $\mathcal{S}_2$, since $H_{4,0} \subset G_0$.

The map $(\alpha, \beta) : H_4 \to H_{4,0} \times H_{4,0}$ given by $(\alpha, \beta)(f) = (\alpha(f), \beta(f))$, $(\forall)\ f \in H_4$, is surjective. Indeed, we have that: for $(f_i, f_i) \in H_{4,0}$, $(\exists)\ f_i \in H_4$ such that $\alpha(f_i) = \beta(f_i) = f_i$, $i = 1, 2$;
for $(f_1, f_2) \in H_{4,0}$, $(\exists)\ f_4 \in H_4$ such that $\alpha(f_4) = f_1$, $\beta(f_4) = f_2$;
for $(f_2, f_1) \in H_{4,0}$, $(\exists)\ f_5 \in H_4$ such that $\alpha(f_5) = f_2$, $\beta(f_5) = f_1$.

Therefore, H_4 is a transitive subgroupoid of $G = \mathcal{S}_2$. □

Remark 4.1. *(i) The isotropy subgroupoid of $\mathcal{S}_2$ is $H = Is(\mathcal{S}_2) = G(f_1) \cup G(f_2) \cup G(f_3) = \{f_1, f_2, f_3, f_6\}$. We have $|H_4| = 4$, $|Is(\mathcal{S}_2)| = 4$ and the groupoids H_4 and $Is(\mathcal{S}_2)$ are not isomorphic.*

(ii) We have that $|\mathcal{S}_2| = 6$, $|Is(\mathcal{S}_2)| = 4$ and the order of $Is(\mathcal{S}_3)$ is not a divisor of $|\mathcal{S}_2|$. Hence, Lagrange' s theorem for finite groups is not valid for finite groupoids.

Proposition 4.2. *Let H and K be two subgroupoids of a groupoid G such that $H \cap K = \emptyset$. If the products $x \cdot z$ and $t \cdot y$ are not defined in G for all $x, y \in H$ and $z, t \in K$, then $H \cup K$ is a subgroupoid of G.*

Proof. It is easy to verify that the conditions from Definition 2.1 are satisfied. □

Proposition 4.3. *The symmetric groupoid* $G = \mathcal{S}_2$ *contains a subgroupoid* H_j *of order* j, *for* $1 \leq j \leq 4$.

Proof. Using the above subgroupoids and Proposition 4.2, we obtain the following list of subgroupoids of the symmetric groupoid $\mathcal{S}_2$: (1) *subgroupoids of order* 1 :
$H_1^1 = \{f_1\} = G(f_1),\ H_1^2 = \{f_2\} = G(f_2),\ H_1^3 = \{f_3\} = H_1.$
(2) *subgroupoids of order* 2 :
$H_2^1 = \{f_3, f_6\} = G(f_3),\ H_2^2 = G(f_1) \cup G(f_2) = \{f_1, f_2\} = H_2.$
(3) *subgroupoids of order* 3 and 4 :
$H_3^1 = \mathcal{S}_{3,0} = \{f_1, f_2, f_3\}$ (the nul subgroupoid);
$H_4^1 = \{f_1, f_2, f_3, f_6\} = Is(\mathcal{S}_2),\ H_4^2 = \{\ f_1, f_2, f_4, f_5\ \} = H_4.$
Hence, the symmetric groupoid $G = \mathcal{S}_2$ contains only two normal subgroupoids, namely $\mathcal{S}_{2,0}$ and $Is(\mathcal{S}_2)$ and one transitive groupoid, namely H_4. □

Proposition 4.4. *The quotient groupoid* $G/H = \mathcal{S}_2/Is(\mathcal{S}_2)$ *of* $\mathcal{S}_2$ *determined by the isotropy subgroupoid* $Is(\mathcal{S}_2)$ *is a groupoid of order* 5 .

Proof. We point out the left cosets determined by the subgroupoid H in G, i.e. the sets $\widehat{f_i} = f_i H(\beta(f_i))$, for all $i = \overline{1,6}$. We have: $\widehat{f_1} = \{f_1\}$, $\widehat{f_2} = \{f_2\}$,
$\widehat{f_3} = f_3 H(\beta(f_3)) = f_3 H(f_3) = f_3\{f_3, f_6\} = \{f_3 \cdot f_3, f_3 \cdot f_6\} = \{f_3, f_6\},$
$\widehat{f_4} = \{f_4\},\ \widehat{f_5} = \{f_5\}$ and $\widehat{f_6} = \{f_3, f_6\} = \widehat{f_3}.$
Then $G/H = \{\widehat{f_j} \mid j = \overline{1,5}\ \}$ and $G_0/H = \{\widehat{f_k} \mid k = \overline{1,3}\ \}$.
Applying the relations $(10)-(12)$, we calculate the products of elements in G/H and the images of its elements by the maps $\widehat{\alpha}, \widehat{\beta}, \widehat{i}$. These are given by the tables:

$\cdot$	$\widehat{f_1}$	$\widehat{f_2}$	$\widehat{f_3}$	$\widehat{f_4}$	$\widehat{f_5}$
$\widehat{f_1}$	$\widehat{f_1}$			$\widehat{f_4}$	
$\widehat{f_2}$		$\widehat{f_2}$			$\widehat{f_5}$
$\widehat{f_3}$			$\widehat{f_3}$		
$\widehat{f_4}$		$\widehat{f_4}$			$\widehat{f_1}$
$\widehat{f_5}$	$\widehat{f_5}$			$\widehat{f_2}$	

$\widehat{f}$	$\widehat{f_1}$	$\widehat{f_2}$	$\widehat{f_3}$	$\widehat{f_4}$	$\widehat{f_5}$
$\widehat{\alpha}(\widehat{f})$	$\widehat{f_1}$	$\widehat{f_2}$	$\widehat{f_3}$	$\widehat{f_1}$	$\widehat{f_2}$
$\widehat{\beta}(\widehat{f})$	$\widehat{f_1}$	$\widehat{f_2}$	$\widehat{f_3}$	$\widehat{f_2}$	$\widehat{f_1}$
$\widehat{i}(\widehat{f})$	$\widehat{f_1}$	$\widehat{f_2}$	$\widehat{f_3}$	$\widehat{f_5}$	$\widehat{f_4}$

.

□

Remark 4.2. *If we consider the nul subgroupoid* $H' = \{f_1, f_2, f_3\}$ *and we compute the left cosets determined by* H' *in* G, *we obtain* $\widetilde{f_i} = f_i H'(\beta(f_i))$, *for all* $i = \overline{1,6}$, *where*
$\widetilde{f_1} = \{f_1\},\ \widetilde{f_2} = \{f_2\},\ \widetilde{f_3} = \{f_3\},\ \widetilde{f_4} = \{f_4\},\ \widetilde{f_5} = \{f_5\},\ \widetilde{f_6} = \{f_6\}.$
The quotient groupoid $\mathcal{S}_2/\mathcal{S}_{2,0} = \{\widetilde{f_j} \mid j = \overline{1,6}\ \}$ *is a groupoid with* 6 *elements. We have* $|\mathcal{S}_2| = |G/H'| = 6$ *and* $\mathcal{S}_2 \not\cong \mathcal{S}_2/\mathcal{S}_{2,0}$.

By Theorem 4.1, the symmetric groupoid $\mathcal{S}_3$ is $\{g_i \mid i = \overline{1,33}\ \}$, where

$$g_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ g_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix},\ g_3 = \begin{pmatrix} 3 \\ 3 \end{pmatrix},\ g_4 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$$
$$g_5 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix},\ g_6 = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix},\ g_7 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$
$$g_8 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ g_9 = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ g_{10} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ g_{11} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$
$$g_{12} = \begin{pmatrix} 3 \\ 1 \end{pmatrix},\ g_{13} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},\ g_{14} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},\ g_{15} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix},$$

$$g_{16} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix},\ g_{17} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix},\ g_{18} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix},$$
$$g_{19} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix},\ g_{20} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix},\ g_{21} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix},$$
$$g_{22} = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix},\ g_{23} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix},\ g_{24} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix},$$
$$g_{25} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix},\ g_{26} = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix},\ g_{27} = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix},$$
$$g_{28} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix},\ g_{29} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ g_{30} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$
$$g_{31} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ g_{32} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},\ g_{33} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

The nul subgoupoid of $\Gamma = \mathcal{S}_3$ is $\Gamma_0 = \mathcal{S}_{3,0} = \{\ g_i \mid i = \overline{1,7}\ \}$. The isotropy groups of the groupoid Γ are the following ones :

$\Gamma(g_j) = \{g_j\}$ for $j = \overline{1,3}$, $\Gamma(g_4) = \{g_4, g_{14}\}$, $\Gamma(g_5) = \{g_5, g_{21}\}$, $\Gamma(g_6) = \{g_6, g_{28}\}$ and $\Gamma(g_7) = \{g_7, g_{29}, g_{30}, g_{31}, g_{32}, g_{33}\ \}$.

The isotropy subgroupoid of $\mathcal{S}_3$ is $Is(\mathcal{S}_3) = \bigcup\limits_{i=1}^{7} \Gamma_{g_i}$ and it is a normal subgroupoid of order 15.

Remark 4.3. *(i) The symmetric groupoid* $\mathcal{S}_3$ *contains :*

(a) a transitive subgroupoid $K_9 = \{g_1, g_2, g_3, g_8, g_9, g_{11}, g_{12}, g_{13}\}$ *of order* 9*;*

(b) a subgroupoid $K_6 = \Gamma(g_4) \cup \Gamma(g_5) \cup \Gamma(g_6)$ *of order* 6 *such that* $K_6 \ncong \mathcal{S}_2$*.*
(ii) $\Gamma/K = \mathcal{S}_3/Is(\mathcal{S}_3)$ *is a groupoid of order* 19 *(see* [10]*).*

Let $(G, \alpha, \beta, \mu, i; G_0)$ be a groupoid and $\psi : G \to G'$ a bijective map from G into a set G'. We consider the maps $\alpha', \beta',\ i' : G' \to G'$ given by $\alpha' = \psi \circ \alpha$, $\beta' = \psi \circ \beta$, $i' = \psi \circ i$ and we take $G'_0 = \psi(G_0)$.

For $(x', y') \in G' \times G'$ we define the multiplication law on G' by $\mu'(x', y') = \psi(\mu(\psi^{-1}(x'), \psi^{-1}(y'))) \iff (\psi^{-1}(x'), \psi^{-1}(y')) \in G_{(2)}$, i.e.

$$x' \cdot y' = \psi(\psi^{-1}(x') \cdot \psi^{-1}(y')) \iff \psi^{-1}(x') \cdot \psi^{-1}(y') \text{ is defined in } G. \tag{13}$$

We have $G'_0 = \alpha'(G') = \beta'(G')$ and

$$(x, y) \in G_{(2)} \iff (x', y') \in G'_{(2)},\ \text{where } x' = \psi(x),\ y' = \psi(y). \tag{14}$$

It is easy to prove the following

Proposition 4.5. *Let* $(G, \alpha, \beta, \mu, i; G_0)$ *be a groupoid and* $\psi : G \to G'$ *a bijective map. Then*
(i) $(G', \alpha', \beta', \mu', i'; G'_0)$ *is a groupoid, where*
$\alpha' = \psi \circ \alpha$, $\beta' = \psi \circ \beta$, $i' = \psi \circ i$, $G'_0 = \psi(G_0)$ *and the multiplication* μ' *is given by (3.1).*
(ii) $\psi : (G; G_0) \to (G'; G'_0)$ *is an isomorphism of groupoids.*

The groupoid $(G', \alpha', \beta', \mu', i'; G'_0)$ is called the *direct image of the groupoid* $(G, \alpha, \beta, \mu, i; G_0)$ *by the bijection* $\psi : G \to G'$. Also, we say that the groupoid G' *is obtained by transport of the structure from the groupoid* G *via the bijection* $\psi : G \to G'$.

In the sequel we give a method for construction of finite groupoids. For instance, we will construct a groupoid of order 9. For this, we consider the transitive groupoid $H_4 = \{ f_1, f_2, f_4, f_5 \}$ of $\mathcal{S}_2$ and the bijection $\varphi : H_4 \to M_4$, where $M_4 = \{ a_1, a_2, a_3, a_4 \}$, $\varphi(f_1) = a_1$, $\varphi(f_2) = a_2$, $\varphi(f_4) = a_3$, $\varphi(f_5) = a_4$.

Applying Proposition 4.5 we introduce on M_4 a structure of groupoid over $M_{4,0} = \{ a_1, a_2 \}$ obtained by transporting the structure of H_4 via φ and we notice that M_4 is a transitive groupoid such that $M_4 \cong H_4$.

We now consider the set $M_5 = \{a_5, a_6, a_7, a_8, a_9 \}$ and the bijection $\psi : \mathcal{S}_2/Is(\mathcal{S}_2) \to M_5$, $\psi(\widehat{f_1}) = a_5$, $\psi(\widehat{f_2}) = a_6$, $\psi(\widehat{f_3}) = a_7$, $\psi(\widehat{f_4}) = a_8$, $\psi(\widehat{f_5}) = a_9$.

Applying Proposition 4.5, the set M_5 has a groupoid structure over $M_{5,0} = \{ a_5, a_6, a_7 \}$ obtained by transporting the structure of $\mathcal{S}_2/Is(\mathcal{S}_2)$ via ψ. We notice that M_5 is a groupoid of order 5 and $M_5 \cong \mathcal{S}_2/Is(\mathcal{S}_2)$.

Using now the groupoids M_4 and M_5, we consider the disjoint union of these groupoids and we denote $M = M_4 \coprod M_5$.

The composition law and the structural functions of the groupoid $M = \{ a_i \mid i = \overline{1, 9} \}$ are given by the following tables:

$\cdot$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	a_1		a_3						
a_2		a_2		a_4					
a_3		a_3		a_1					
a_4	a_4		a_2						
a_5					a_5			a_8	
a_6						a_6			a_9
a_7							a_7		
a_8						a_8			a_5
a_9					a_9			a_6	

a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
$\overline{\alpha}(a)$	a_1	a_2	a_1	a_2	a_5	a_6	a_7	a_5	a_6
$\overline{\beta}(a)$	a_1	a_2	a_2	a_1	a_5	a_6	a_7	a_6	a_5
$\overline{i}(a)$	a_1	a_2	a_4	a_3	a_5	a_6	a_7	a_9	a_8

We have that M is a groupoid of order 9 and $M \ncong K_9$.

Remark 4.4. *Applying the above method we can construct several finite groupoids of any order. For example, if we want to construct a groupoid of order* 72, *we consider the groupoids* Ω_i, $i = \overline{1, 6}$, *such that* $\Omega_1 \cong H_4$, $\Omega_2 \cong \mathcal{S}_2/Is(\mathcal{S}_2)$, $\Omega_3 \cong \mathcal{S}_2$, $\Omega_4 \cong K_9$, $\Omega_5 \cong Is(\mathcal{S}_3)$, $\Omega_6 \cong \mathcal{S}_3$ *and we take the disjoint union* $\Omega = \coprod_{i=1}^{6} \Omega_i$. *We obtain that* Ω *is a groupoid of order* 72.

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References

[1] H. Brandt, Über eine Verallgemeinerung der Gruppen-Begriffes, *Math. Ann.*, **96**, 360-366 (1926).
[2] R. Brown, Fibrations of groupoids, *J. Algebra*, **15**, 103-132 (1970).
[3] A. Coste, P. Dazord, A. Weinstein, Groupoidés symplectiques, *Publ. Dept. Math. Lyon*, **2/A**, 1-62 (1987).
[4] B. Dumons, Gh. Ivan, *Introduction à la théorie des groupoides*, Dept.Math. Univ. Poitiers(France), URA, C.N.R.S. D1322, nr.86, 1994.

[5] C. Ehresmann, *Oeuvres complètes, Parties I.1, I.2. Topologie algébrique et géométrie différentielle*, Paris, Dunod, 1950.
[6] P. Hahn, Haar measure for measure groupoids, *Trans. Amer. Math. Soc.*, 242, 1-33 (1978).
[7] P. J. Higgins, *Notes on categories and groupoids*, Von Nostrand Reinhold, London,1971.
[8] Gh. Ivan, Strong morphisms of groupoids, *Balkan Journal Geometry and Its Appl.*, **4**(1), 91-102 (1999).
[9] Gh. Ivan, Algebraic constructions of Brandt groupoids, *Proceedings of the Algebra Symposium, " Babeş- Bolyai" University, Cluj*, 69-90 (2002).
[10] M. Ivan, On the symmetric groupoid, *Seminar Arghiriade, West University of Timişoara*, **32**, 1-23 (2002).
[11] A. Loewy, Neue elementare Begründung und Erweiterung der Galoisschen Theorie, *S.-B.Heidelberger Akad. Wiss Math. Nat Kl.*, **1925**(7), 1927.
[12] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc., Lecture Notes Series, 124, Cambridge Univ.Press., 1987.
[13] G. W. Mackey, Ergodic theory and virtual groups, *Math. Ann.*, **166**, 187-207 (1966).
[14] K. Mikami, A. Weinstein, Moments and reduction for symplectic groupoid actions, *Publ. RIMS Kyoto Univ.*, **42**, 121-140 (1988).
[15] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes Series, 793, Springer,1980.
[16] A. Weinstein, Groupoids: Unifying internal and external symmetries, *Notices Amer. Math. Soc.*, **43**, 744-752 (1996).

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