General properties of the symmetric groupoid of a finite set

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**Abstract.** The aim of this paper is to give some basic properties of the symmetric groupoid of an arbitrary or finite set. The determination of subgroupoids and related topics on the symmetric groupoid of a finite set is discussed.

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1. Introduction

The concept of groupoid was introduced by H. Brandt [1] in a 1926 paper on the composition of quadratic forms in four variables. The concept of groupoid is a generalization of the notion of group. In many aspects a groupoid is like a group with several neutral elements. A groupoid with only one neutral element is a group.

Groupoids also appeared in Galois theory in the description of relations between subfields of a field $K$ via morphisms of $K$ in a paper of A. Loewy [11] around 1927 (the isotropy groups of the constructed groupoid turn out to be the Galois groups).

A generalization of Brandt groupoid has appeared in the work of C. Ehresmann [5] around 1950. There are various definitions for Brandt groupoids, see [2], [3], [14], [15]. In this paper we use the definition of the groupoid given in [14].

A groupoid can be endowed with other algebraic, topological or geometric structures. So we will find Borel groupoids, topological groupoids, measure groupoids, Lie groupoids, symplectic groupoids and so on.

The algebraic, topological and differentiable groupoids play an important role by their applications in algebra, measure theory, harmonic analysis, differential geometry, symplectic geometry and quantum mechanics. For details in these areas, see [3], [4], [6]-[9], [12]-[16].

This paper deals with the groupoids in the sense of Brandt. In the second Section some definitions and results about groupoids are given. The third Section deals with the symmetric groupoid of an arbitrary set. Section 4 is dedicated to establish properties of the symmetric groupoid of degree $n$.

2. Groupoids and related concepts

Let $G$ be a set endowed with the maps $\alpha$ (source) and $\beta$ (target), $\alpha, \beta : G \to G$, the composition law $\mu : G(2) \to G, (x, y) \mapsto \mu(x, y)$, where $G(2) = \{(x, y) \in G \times G \mid \beta(x) = \alpha(y)\}$ and the inversion map $i : G \to G, x \mapsto i(x)$. We write sometimes $x \cdot y$ or $xy$ for $\mu(x, y)$ and $x^{-1}$ for $i(x)$. The elements of $G(2)$ are called composable pairs of $G$.

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The 5-tuple \((G, \alpha, \beta, \mu, i)\) is a \((\text{Brandt})\) groupoid, if the maps \(\alpha, \beta, \mu, i\) satisfy the following axioms:

1. \((\text{associativity})\) \((xy)z = x(yz)\), in the sense that, if one side of the equation is defined so is the other one and then they are equal;
2. \((\text{identities})\) \((\alpha(x), x), (x, \beta(x)) \in G(2)\) and \(\alpha(x)x = x\beta(x) = x\);
3. \((\text{inverses})\) \((x^{-1}, x), (x, x^{-1}) \in G(2)\) and \(x^{-1}x = \beta(x), xx^{-1} = \alpha(x)\).

The element \(\alpha(x)\) [resp. \(\beta(x)\)] is the left unit (resp. right unit) of \(x \in G\). The subset \(G_0 = \alpha(G) = \beta(G)\) of \(G\) is called the unit set of \(G\) and we say that \(G\) is a \(G_0\)-groupoid or a groupoid over \(G_0\).

A \(G_0\)-groupoid \(G\) will be denoted by \((G, \alpha, \beta, \mu, i; G_0)\) or \((G, \alpha, \beta; G_0)\) or \((G; G_0)\). The maps \(\alpha, \beta, \mu\) and \(i\) are called the structural functions of \(G\).

For each \(u \in G_0\), the set \(\alpha^{-1}(u)\) (resp. \(\beta^{-1}(u)\)) is called the \(\alpha\)- fibre (resp. \(\beta\)- fibre) of \(G\) over \(u \in G_0\).

A \(G_0\)-groupoid \(G\) is said to be transitive, if the map \((\alpha, \beta) : G \rightarrow G_0 \times G_0, x \mapsto (\alpha(x), \beta(x))\) is surjective; \((\alpha, \beta)\) is called the anchor of \(G\).

For the structural functions of a groupoid \((G, \alpha, \beta, \mu, i; G_0)\), the following assertions hold:

\[
\alpha(u) = \beta(u) = u \text{ and } u \cdot u = u \text{ for all } u \in G_0; \quad (1)
\]

\[
\alpha(xy) = \alpha(x) \text{ and } \beta(xy) = \beta(y), \quad (\forall) \ (x, y) \in G(2); \quad (2)
\]

\[
\alpha \circ i = \beta, \quad \beta \circ i = \alpha \text{ and } i \circ i = 1d_G; \quad (3)
\]

\[
G(u) = \alpha^{-1}(u) \cap \beta^{-1}(u) = \{x \in G \mid \alpha(x) = \beta(x) = u\} \quad (4)
\]

is a group with respect to the restriction of \(\mu\) to \(G(u)\), called the isotropy group of \(G\) at \(u\).

**Definition 2.1.** A nonempty subset \(H\) of a \(G_0\)-groupoid \(G\) is called subgroupoid of \(G\) if it is closed under multiplication (when it is defined) and inversion, i.e. the following conditions hold:

(i) for all \(x, y \in H\) such that \(xy\) is defined, we have \(xy \in H\);

(ii) for all \(x \in H\), we have \(x^{-1} \in H\).

Note that from the condition (ii) of Definition 2.1, we deduce that \(\alpha(h) \in H\) and \(\beta(h) \in H\), for all \(h \in H\). If \(\alpha(H) = \beta(H) = G_0\), then \(H\) is called a wide subgroupoid of \(G\).

A group \(\mathcal{G}\) having \(e\) as unit element is just a \(\{e\}\)-groupoid and conversely, every groupoid with one unit element is a group. The wide subgroupoids of \(\mathcal{G}\) are just the subgroups of \(\mathcal{G}\).

**Example 2.1.** (i) We give on a nonempty set \(X\) the following groupoid structure: \(\alpha = \beta = 1d_X\), the elements \(x, y \in X\) are composable iff \(x = y\) and we define \(xx = x\). This groupoid is called the **null groupoid** over \(X\).

(ii) The pair groupoid over a set. Let \(X\) be a nonempty set. Then \(G = X \times X\) is a groupoid with respect to the rules: \(\alpha(x, y) = (x, y), \beta(x, y) = (y, y)\), the elements \((x, y)\) and \((y, z)\) are composable in \(G\) iff \(y = y\) and we take \((x, y)(y, z) = (x, z)\) and the inverse of \((x, y)\) is defined by \((x, y)^{-1} = (y, x)\). The unit space of the pair groupoid \(X \times X\) is the diagonal \(\Delta_X = \{(x, x) \mid x \in X\}\) which can be identified with \(X\). The isotropy group \(G(u)\) at \(u = (x, x)\) is the null group \(\{(u, u)\}\).
A subgroupoid $H$ of the pair groupoid $X \times X$ is a relation on $X$ which is symmetric and transitive. A wide subgroupoid $H$ is an equivalence relation on $X$.

(iii) If $\{ G_i \mid i \in I \}$ is a disjoint family of groupoids, let $G = \bigcup_{i \in I} G_i$ and $G_{(2)} = \bigcup_{i \in I} G_{i,(2)}$. Here, two elements $x, y \in G$ may be composed iff they lie in the same groupoid $G_i$. This groupoid is called the disjoint union of the groupoids $G_i, i \in I$, and it is denoted by $\coprod_{i \in I} G_i$. The unit set of this groupoid is $G_0 = \bigcup_{i \in I} G_{i,0}$, where $G_{i,0}$ is the unit set of $G_i$.

In particular, the disjoint union of the groups $G_i, i \in I$, is a groupoid, i.e. $G = \coprod_{i \in I} G_i$, which be called the groupoid associated to family of groups $G_i, i \in I$. For this groupoid, the isotropy group at $e_i \in G_i$ is the group $G_i$ and $G_0 = \{ e_i \mid i \in I \}$, where $e_i$ is the unit element of $G_i$.

Let $(G, \alpha, \beta, \mu, \iota; G_0)$ and $(G', \alpha', \beta', \mu', \iota'; G'_0)$ be two groupoids. A morphism between these groupoids is a pair $(f, \tilde{f})$ of maps $f : G \rightarrow G'$ and $\tilde{f} : G_0 \rightarrow G'_0$ such that the following two conditions are satisfied:

(i) $f(\mu(x, y)) = \mu'(f(x), f(y))$, for all $(x, y) \in G_{(2)}$;

(ii) $\alpha' \circ f = \tilde{f} \circ \alpha$ and $\beta' \circ f = \tilde{f} \circ \beta$.

A morphism of groupoids $(f, \tilde{f}) : (G; G_0) \rightarrow (G'; G'_0)$ is said to be isomorphism of groupoids, if $f$ and $\tilde{f}$ are bijective maps.

**Example 2.2.**

(i) If $G$ is a group, then the map $\delta : G \times G \rightarrow G$, $\delta(x, y) = xy^{-1}$, is a morphism of groupoids from the pair groupoid $G \times G$ into $G$.

(ii) The anchor map $(\alpha, \beta) : G \rightarrow G_0 \times G_0$ of the $G_0$-groupoid $G$ into the pair groupoid $G_0 \times G_0$ is a morphism of groupoids.

In a groupoid $(G, \alpha, \beta; G_0)$ the relation defined on $G_0$ by:

$$u \sim_{G_0} v \iff \exists x \in G \text{ with } \alpha(x) = u \text{ and } \beta(x) = v$$

is an equivalence relation. Its equivalence classes are called orbits and the orbit of $u \in G_0$ is denoted by $[u]$. The quotient set $G_0/G$ determined by this equivalence relation is called the orbit space.

A groupoid $(G, G_0)$ is transitive iff $G_0/G$ is a singleton.

There is a natural decomposition of the unit space $G_0$ of a groupoid $G$ into orbits. Over each orbit there is a transitive groupoid and the disjoint union of these transitive groupoids is the original groupoid $G$.

**Definition 2.2.** By a normal subgroupoid of a groupoid $G$, we mean a wide subgroupoid $H$ of $G$ satisfying the property: for all $x \in G$ and $h \in H$ such that the product $xhx^{-1}$ is defined, we have $xhx^{-1} \in H$.

**Proposition 2.1.** ([4]) A wide subgroupoid $H$ of the $G_0$-groupoid $G$ is normal iff $xH(\beta(x)) = H(\alpha(x))x$ for all $x \in G$, where $H(u)$ denotes the isotropy group of the groupoid $H$ at $u$.

**Example 2.3.**

(i) If $G$ is a $G_0$-groupoid, then $G_0$ and $Is(G) = \{ x \in G \mid \alpha(x) = \beta(x) \}$ are normal subgroupoids of $G$, called the null subgroupoid and the isotropy subgroupoid of $G$, respectively.

(ii) If $f : G \rightarrow G'$ is a morphism of groupoids, then $\text{Ker} f = \{ x \in G \mid f(x) \in G_0 \}$ is a normal subgroupoid of $G$. 
Let $H$ be a wide subgroupoid of the $G_0$-groupoid $G$. The relation $\equiv$ defined on the groupoid $G$ by:

\[
\{ (\exists) h \in H(\alpha(x)), h' \in H(\beta(x)) \text{ such that } y = hxh' \}
\]

is an equivalence relation. We denote by $\hat{x}$ the equivalence class of $x \in G$ relative to the equivalence relation $\equiv$ and let $G/\equiv = \{ \hat{x} \mid x \in G \}$ be the set of the equivalence classes defined on $G$ by $\equiv$.

**Theorem 3.1.** If $\alpha, \beta$ are quasipermutations of $M$, the maps $\alpha, \beta$ on the groupoid $S$ of quasipermutations of $M$. The maps $\alpha, \beta$ are defined by $\alpha(f, g) = f \circ g$. For $f, g \in G(2)$ we define $\mu(f, g) = g \circ f$.

Let $R(f)$ be the domain of $f$, $R(f) = f(D(f))$ and $G(2) = \{ (f, g) \mid R(f) = D(g) \}$. For $f, g \in G(2)$ we define $\mu(f, g) = g \circ f$. If $I_{D_A}$ denotes the identity on $A$, then $G_0 = \{ I_{D_A} \mid A \subseteq M \}$ is the set of units of $G$, denoted by $S_0(M)$ and $f^{-1}$ is the inverse function from $R(f)$ to $D(f)$. The maps $\alpha, \beta$ are defined by $\alpha(f) = I_{D(f)}$, $\beta(f) = I_{R(f)}$. Thus $S(M)$ is a groupoid over $S_0(M)$. $S(M)$ is called the symmetric groupoid of $M$ or the groupoid of quasipermutations of $M$.

**Proposition 2.2.** ([4]) If $H$ is a normal subgroupoid of $G$, then

\[
\hat{x} = xH(\beta(x)) = H(\alpha(x))x, \forall x \in G.
\]

If $H$ is a normal subgroupoid of $G$, then

\[
G/\equiv = \{ xH(\beta(x)) \mid x \in G \} = \{ H(\alpha(x))x \mid x \in G \}
\]

$G_0/\equiv = \{ \hat{u} \mid u \in G_0 \} = \{ H(u) \mid u \in G_0 \}$.

The quotient set $G/\equiv$ has a natural structure of groupoid having $G_0/\equiv$ as unit set with respect to the following rules: $\hat{\alpha}, \hat{\beta} : G/\equiv \to G/\equiv$ are defined by

\[
\hat{\alpha}(\hat{x}) = H(\alpha(x)), \hat{\beta}(\hat{x}) = H(\beta(x)),
\]

the multiplication law is defined by

\[
\hat{x} \cdot \hat{y} = (xy)H(\beta(yx)) \iff \hat{\beta}(\hat{x}) = \hat{\alpha}(\hat{y})
\]

and the inverse of $\hat{x} = xH(\beta(x))$ is defined by

\[
\hat{i}(\hat{x}) = x^{-1}H(\beta(x^{-1})) = x^{-1}H(\alpha(x)).
\]

The groupoid $(G/\equiv, \hat{\alpha}, \hat{\beta}, \hat{\mu}, \hat{i}; G_0/\equiv)$ is called the **quotient groupoid** of $G$ relative to $H$ and will be denoted by $G/H$. For more details concerning the groupoids, see [4], [7]-[9], [16].

### 3. The symmetric groupoid $S(M)$

Let $M$ be a nonempty set. By a quasipermutation of the set $M$ we mean an injective map from a subset of $M$ into $M$.

We denote by $G = S(M)$ or $G = \text{Inj}(S)$ the set of all quasipermutations of $M$, i.e. $S(M) = \{ f \mid f : A \rightarrow M, f \text{ is injective and } \emptyset \neq A \subseteq M \}$.

For $f \in S(M)$, let $D(f)$ be the domain of $f$, $R(f) = f(D(f))$ and $G(2) = \{ (f, g) \mid R(f) = D(g) \}$. For $f, g \in G(2)$ we define $\mu(f, g) = g \circ f$.

If $I_{D_A}$ denotes the identity on $A$, then $G_0 = \{ I_{D_A} \mid A \subseteq M \}$ is the set of units of $G$, denoted by $S_0(M)$ and $f^{-1}$ is the inverse function from $R(f)$ to $D(f)$. The maps $\alpha, \beta$ are defined by $\alpha(f) = I_{D(f)}$, $\beta(f) = I_{R(f)}$. Thus $S(M)$ is a groupoid over $S_0(M)$. $S(M)$ is called the symmetric groupoid of $M$ or the groupoid of quasipermutations of $M$.

**Theorem 3.1.** If $M$ and $N$ are equipotent sets, then the symmetric groupoids $S(M)$ and $S(N)$ are isomorphic.
Proof. Let \( \varphi : M \to N \) be a bijective map. For each \( f \in \mathcal{S}(M) \) we have that \( \varphi \circ f \circ \varphi^{-1} \) is an injective mapping from a subset of \( N \) into \( N \). Hence \( \varphi \circ f \circ \varphi^{-1} \in \mathcal{S}(N) \). It is easy to check that the map \( \tilde{\varphi} : \mathcal{S}(M) \to \mathcal{S}(N) \) defined by \( \tilde{\varphi}(f) = \varphi \circ f \circ \varphi^{-1} \), \( \forall f \in \mathcal{S}(M) \), is a bijective morphism of groupoids. Therefore, the groupoids \( \mathcal{S}(M) \) and \( \mathcal{S}(N) \) are isomorphic. \( \square \)

For a given groupoid \( (G; G_0) \), let \( \{ \mathcal{S}(G); S_0(G) \} \) be the symmetric groupoid of the set \( G \), where \( S_0(G) = \{Id_A \mid A \subseteq G \} \). We consider now the set \( \mathcal{L}(G) = \{ L_a \mid a \in G \} \) of all left translations \( L_a : G \to G, x \to L_a(x) = ax \), whenever \( (a, x) \in G_{(2)} \).

We have \( D(L_a) = \{ x \in G \mid (a, x) \in G_{(2)} \} \neq \emptyset \), since \( (a, \beta(a)) \in G_{(2)} \) and so \( L_a \in \mathcal{S}(G) \). Hence \( \mathcal{L}(G) \) is a subset of \( \mathcal{S}(G) \).

For all \( a, b, x \in G \) such that \( \beta(a) = \alpha(b) \) and \( \beta(b) = \alpha(x) \) we have \( L_aL_b(x) = L_a(bx) = a(bx) = (ab)x = L_{ab}(x) \) and we note that \( L_a \circ L_b = L_{ab} \) if \( (a, b) \in G_{(2)} \). Consequently, we have \( L_{a(x)} \circ L_a = L_x \circ L_{\beta(x)} = L_x, \forall x \in G \).

For all \( u \in G_0 \) we have \( L_u = Id_{D(L_u)} \), hence \( L_a \in S_0(G) \) and \( \mathcal{L}_0(G) = \{ L_u \mid u \in G_0 \} \) is a subset of \( S_0(G) \). Since \( \mathcal{L}(G) \subseteq \mathcal{S}(G) \) and the conditions (i) and (ii) from Definition 2.1 are satisfied, it follows that \( \mathcal{L}(G) \) is a subgroupoid of \( \mathcal{S}(G) \).

This groupoid is called the groupoid of left translations of \( G \).

Theorem 3.2. (Cayley theorem for groupoids.) Every groupoid \( G \) is isomorphic to a subgroupoid of the symmetric groupoid \( \mathcal{S}(G) \).

Proof. Let \( \{ \mathcal{L}(G); \mathcal{L}_0(G) \} \) be the groupoid of left translations of \( G \). We have that \( \mathcal{L}(G) \) is a subgroupoid of \( \mathcal{S}(G) \). It is easy to verify that \( \varphi : G \to \mathcal{L}(G), \varphi(a) = L_a, \forall a \in G, \) is an isomorphism of groupoids. \( \square \)

Remark 3.1. In view of Cayley’s theorem for groupoids, many groupoids occur naturally as subgroupoids of some symmetric groupoid.

4. The symmetric groupoid of a finite set

When \( M = \{1, 2, \ldots, n\} \), we write \( S_n \) for \( \mathcal{S}(M) \) and call \( S_n \) the symmetric groupoid of degree \( n \).

The symmetric groupoid of a finite set play an important role in the study of finite groupoids, since by Cayley’s theorem every finite groupoid of degree \( n \) is isomorphic to some subgroupoid of \( S_n \).

Theorem 4.1. ([10]) Let \( n \) be a fixed number such that \( n \geq 1 \). The symmetric groupoid \( S_n \) contains \( |\mathcal{S}_n| \) elements, where

\[
\left| \mathcal{S}_n \right| = \sum_{k=1}^{n} k! \binom{n}{k}^2.
\]

Proof. For each \( k, 1 \leq k \leq n \), we denote by \( X_k = \{ i_1, i_2, \ldots, i_k \} \) a subset of \( M = \{1, 2, \ldots, n\} \) such that \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \).

If \( X_k \) is a fixed subset of \( M \), let \( f_k : X_k \to M \) be an injective mapping. We write the function \( f_k \) in the following form

\[
\left( \begin{array}{cccc}
  i_1 & i_2 & \ldots & i_k \\
  f_k(i_1) & f_k(i_2) & \ldots & f_k(i_k)
\end{array} \right),
\]

where \( f_k(i_j) \in M, \) for \( j = 1, k \) and \( f_k(i_j) \neq f_k(i_s) \) for \( j, s = 1, k \).

Using the fact that the set \( M \) contains \( \binom{n}{k} \) subsets with \( k \) elements of the form \( \{f_k(i_1), f_k(i_2), \ldots, f_k(i_k)\} \), it follows that there exist \( \binom{n}{k} \) injective mappings.
having the domain \( \{i_1, i_2, \ldots, i_k\} \) and with values into \( M \), where \( \binom{n}{k} \) is the \( k \)-th binomial coefficient.

For each injective mapping

\[
f_k = \begin{pmatrix}
i_1 & i_2 & \cdots & i_k \\
f_k(i_1) & f_k(i_2) & \cdots & f_k(i_k)
\end{pmatrix}
\]

having the image \( \{f_k(i_1), f_k(i_2), \ldots, f_k(i_k)\} \) we obtain \( k! \) injective mappings taking on \( 2^k \) th arrow an arbitrary permutation of elements \( f_k(i_1), f_k(i_2), \ldots, f_k(i_k) \). Hence, for a fixed subset \( X_k \) of \( M \) we have \( k! \binom{n}{k} \) injective mappings defined on \( X_k \) with values into \( M \).

The set \( M \) contains \( \binom{n}{k} \) subsets of the form \( X_k = \{i_1, i_2, \ldots, i_k\} \) and for each \( X_k \) there exist \( k! \binom{n}{k} \) injective mappings. This implies that we have \( (k!)^{\binom{n}{k}} \) injective mappings having the domain that contains \( k \) elements of \( M \).

Since \( 1 \leq k \leq n \) we obtain that \( S_n \) contains \( \sum_{k=1}^{n} k! \binom{n}{k}^2 \) injective mappings defined on the subsets of \( M \) and with values into \( M \).

\[ \square \]

**Theorem 4.2.** Let \( S_n \) be the symmetric groupoid of degree \( n \). Then the normal subgroups of \( S_{n,0} \) and \( Is(S_{n}) \) contain \( |S_{n,0}| \) resp. \( |Is(S_{n})| \) elements, where

\[ |S_{n,0}| = 2^n - 1, \quad |Is(S_{n})| = \sum_{k=1}^{n} k! \binom{n}{k}. \]

**Proof.** For \( 1 \leq k \leq n \), we denote by \( X_k = \{i_1, i_2, \ldots, i_k\} \) a subset of \( M = \{1, 2, \ldots, n\} \) such that \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \).

If \( X_k \) is a fixed subset of \( M \), then \( Id_{X_k} = \begin{pmatrix}
i_1 & i_2 & \cdots & i_k \\
1 & 2 & \cdots & k
\end{pmatrix} \) is a unity of \( S_n \).

Using the fact that the set \( M \) contains \( \binom{n}{k} \) subsets with \( k \) elements of the form \( \{f_k(i_1), f_k(i_2), \ldots, f_k(i_k)\} \), it follows that there exist \( \binom{n}{k} \) units having the domain \( \{i_1, i_2, \ldots, i_k\} \).

Since \( 1 \leq k \leq n \) we obtain that \( S_{n,0} \) contains \( \sum_{k=1}^{n} \binom{n}{k} \) identity mappings defined on the subsets of \( M \) with values into \( M \). Hence \( |S_{n,0}| = 2^n - 1 \).

Using the fact that the isotropy groups \( G(f_j) \), \( f_j \in G_0 = S_{n,0} \) of \( S_n \) are disjoint sets, we have \( |Is(S_n)| = \sum_{f_j \in G_0} |G(f_j)| \).

For a fixed unity \( Id_{X_k} = \begin{pmatrix}
i_1 & i_2 & \cdots & i_k \\
1 & 2 & \cdots & k
\end{pmatrix}, \) denoted by \( f_{0,k} \), the isotropy group \( G(f_{0,k}) \) is a group of order \( k! \).

Using the fact that the set \( G_0 \) contains \( \binom{n}{k} \) units with the domain \( \{i_1, i_2, \ldots, i_k\} \), it follows that there exist \( \binom{n}{k} \) isotropy groups having \( k! \) elements. Therefore, we have \( |Is(S_n)| = \sum_{k=1}^{n} k! \binom{n}{k}. \)

\[ \square \]

Let us illustrate the concepts of Section 2 in the case of the symmetric groupoid of degree 2 or 3.

By Theorem 4.1, the symmetric groupoid \( S_2 \) is \( \{f_i \mid i = 1, 2, 6 \} \), where \( f_1 = \begin{pmatrix}1 & 2 \\ 1 & 2 \end{pmatrix}, \ f_2 = \begin{pmatrix}2 & 1 \\ 2 & 1 \end{pmatrix}, \ f_3 = \begin{pmatrix}1 & 2 \\ 1 & 2 \end{pmatrix}, \ f_4 = \begin{pmatrix}2 & 1 \\ 1 & 2 \end{pmatrix}, \ f_5 = \begin{pmatrix}1 & 2 \\ 2 & 1 \end{pmatrix}. \)

The composition law \( \mu : G_{(2)} \to G \) defined on \( G = S_2 \) is given in the table...
\[ \begin{array}{cccccc} \cdot & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\
1 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\
2 & f_2 & f_3 & f_4 & f_5 & f_6 & f_1 \\
3 & f_4 & f_5 & f_6 & f_1 & f_2 & f_3 \\
4 & f_5 & f_6 & f_1 & f_2 & f_3 & f_4 \\
5 & f_6 & f_1 & f_2 & f_3 & f_4 & f_5 \\
\end{array} \]

The absence of the element from the arrow "\(i\)" and the column "\(j\)" in the table of composition law indicates the fact that the pair \((f_i, f_j) \in S_2 \times S_2\) is not composable. Indeed, for example we have that \(f_1 \cdot f_2\) is not defined, since \(R(f_1) = \{1\} \neq D(f_2) = \{2\}\); \(f_1 \cdot f_4 = f_4 \circ f_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix} = f_4\).

The set of unit elements of \(G\) is \(G_0 = \{f_1, f_2, f_3\}\). The structural functions \(\alpha, \beta\) and \(i\), the \(\alpha\)-fibres, \(\beta\)-fibres and isotropy groups of \(G = S_2\) are given by

\[
\begin{array}{cccccc}
f & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\
\alpha(f) & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\
\beta(f) & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\
i(f) & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\
\end{array}
\]

We calculate now the orbits of the symmetric groupoid \(G = S_2\).

We have \(f_2 \sim f_1\), since \((\exists) f_5 \in G\) such that \(\alpha(f_5) = f_2\) and \(\beta(f_5) = f_1\). We obtain that \([f_1] = [f_2] = \{f_1, f_2\}\) and \([f_3] = \{f_3\}\). Therefore, the orbit space of \(G = S_2\) is \(G_0/G = \{f_1, f_2, \{f_3\}\}\). We have that \(G = S_2\) is not a transitive groupoid.

**Proposition 4.1.** The symmetric groupoid \(G = S_2\) contains a transitive subgroupoid \(H_4 = \{f_1, f_2, f_4, f_5\}\) of order 4.

**Proof.** Using the tables for the composition law defined on \(S_2\) and for the structural functions, it is easy to verify that the conditions from the definition of a subgroupoid are satisfied for \(H_4\). The set of unit elements of \(H_4\) is \(H_{4,0} = \{f_1, f_2\}\) and \(H_4\) is not a wide subgroupoid of \(S_2\), since \(H_{4,0} \subset G_0\).

The map \((\alpha, \beta) : H_4 \rightarrow H_{4,0} \times H_{4,0}\) given by \((\alpha, \beta)(f) = (\alpha(f), \beta(f))\), \((\forall) f \in H_4\), is surjective. Indeed, we have that: for \((f_i, f_i) \in H_{4,0}\), \((\exists) f_i \in H_4\) such that \(\alpha(f_i) = \beta(f_i) = f_i\), \(i = 1, 2, 3\); for \((f_2, f_2) \in H_{4,0}\), \((\exists) f_2 \in H_4\) such that \(\alpha(f_2) = f_1\), \(\beta(f_2) = f_2\); for \((f_2, f_1) \in H_{4,0}\), \((\exists) f_5 \in H_4\) such that \(\alpha(f_5) = f_2\), \(\beta(f_5) = f_1\).

Therefore, \(H_4\) is a transitive subgroupoid of \(G = S_2\). \(\square\)

**Remark 4.1.** (i) The isotropy subgroupoid of \(S_2\) is \(H = Is(S_2) = G(f_1) \cup G(f_2) \cup G(f_3) = \{f_1, f_2, f_3, f_6\}\). We have \(|H_4| = 4, |Is(S_2)| = 4\) and the groupoids \(H_4\) and \(Is(S_2)\) are not isomorphic.

(ii) We have that \(|S_2| = 6, |Is(S_2)| = 4\) and the order of \(Is(S_2)\) is not a divisor of \(|S_2|\). Hence, Lagrange’s theorem for finite groups is not valid for finite groupoids.

**Proposition 4.2.** Let \(H\) and \(K\) be two subgroupoids of a groupoid \(G\) such that \(H \cap K = \emptyset\). If the products \(x \cdot z\) and \(t \cdot y\) are not defined in \(G\) for all \(x, y \in H\) and \(z, t \in K\), then \(H \cup K\) is a subgroupoid of \(G\).

**Proof.** It is easy to verify that the conditions from Definition 2.1 are satisfied. \(\square\)
Proposition 4.3. The symmetric groupoid $G = S_2$ contains a subgroupoid $H_j$ of order $j$, for $1 \leq j \leq 4$.

Proof. Using the above subgroupoids and Proposition 4.2, we obtain the following list of subgroupoids of the symmetric groupoid $S_2$:

1. subgroupoids of order 1:
   - $H_1^1 = \{f_1 \}$
   - $H_2^1 = \{f_2 \}$
   - $H_3^1 = \{f_3 \}$
   - $H_4^1$ determined by $S_{2,0}$ (the nul subgroupoid);

2. subgroupoids of order 2:
   - $H_2^2 = \{f_2, f_3 \}$
   - $H_3^2 = G(f_3) \cup G(f_2) = \{f_1, f_2, f_3 \}$

3. subgroupoids of order 3 and 4:
   - $H_3^3 = \{f_1, f_2, f_3, f_6 \} = Is(S_2)$
   - $H_4^4 = \{f_1, f_2, f_3, f_5 \}$

Hence, the symmetric groupoid $G = S_2$ contains only two normal subgroupoids, namely $S_{2,0}$ and $Is(S_2)$ and one transitive groupoid, namely $H_4$.

Proposition 4.4. The quotient groupoid $G/H = S_2/Is(S_2)$ of $S_2$ determined by the isotropy subgroupoid $Is(S_2)$ is a groupoid of order 5.

Proof. We point out the left cosets determined by the subgroupoid $H$ in $G$, i.e. the sets $\hat{f}_i = f_i H = f_i(H(f_i))$, for all $i = 1, 6$. We have: $\hat{f}_1 = \{f_1 \}, \hat{f}_2 = \{f_2 \}$,

- $\hat{f}_3 = f_3 H = f_3 H(f_3)$
- $\hat{f}_4 = f_4 H = f_4 H(f_4)$
- $\hat{f}_5 = f_5 H = f_5 H(f_5)$

Then $G/H = \{\hat{f}_i \}$ and $G_0/H = \{\hat{f}_k \}$.

Applying the relations (10)–(12), we calculate the products of elements in $G/H$ and the images of its elements by the maps $\alpha, \beta, \iota$. These are given by the tables:

\[
\begin{array}{cccccccc}
 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\
f_1 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\
f_2 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\
f_3 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 \\
f_4 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} \\
f_5 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} \\
f_6 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{12} \\
\end{array}
\]

Remark 4.2. If we consider the nul subgroupoid $H' = \{f_1, f_2, f_3 \}$ and we compute the left cosets determined by $H'$ in $G$, we obtain $\hat{f}_i = f_i H'(\beta(f_i))$, for all $i = 1, 6$, where

- $\hat{f}_1 = \{f_1 \}, \hat{f}_2 = \{f_2 \}, \hat{f}_3 = \{f_3 \}, \hat{f}_4 = \{f_4 \}, \hat{f}_5 = \{f_5 \}, \hat{f}_6 = \{f_6 \}$.

The quotient groupoid $S_2/S_{2,0} = \{\hat{f}_j \ | \ j = 1, 6 \}$ is a groupoid with 6 elements. We have $|S_2| = |G/H'| = 6$ and $S_2 \neq S_2/S_{2,0}$.

By Theorem 4.1, the symmetric groupoid $S_3$ is $\{g_i \ | \ i = 1, 33 \}$, where

- $g_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
- $g_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$
- $g_3 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$
- $g_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
- $g_5 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$
- $g_6 = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$
- $g_7 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
- $g_8 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$
- $g_9 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$
- $g_{10} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$
- $g_{11} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$
- $g_{12} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$
- $g_{13} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$
- $g_{14} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$
- $g_{15} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$


The symmetric groupoid $\Gamma$ is called the groupoid of $\mathcal{S}$.

The nul subgroupoid of $\Gamma = S_3$ is $\Gamma_0 = S_{3;0} = \{ g_i \mid i = 1,7 \}$. The isotropy groups of the groupoid $\Gamma$ are the following ones:

\[ \Gamma(g_j) = \{ g_j \} \text{ for } j = 1,3, \Gamma(g_4) = \{ g_4, g_14 \}, \Gamma(g_5) = \{ g_5, g_21 \}, \Gamma(g_6) = \{ g_6, g_28 \} \]

and $\Gamma(g_7) = \{ g_7, g_29, g_30, g_31, g_32, g_33 \}$.

The isotropy subgroupoid of $S_3$ is $I_s(S_3) = \bigcup_{i=1}^{7} \Gamma_{g_i}$ and it is a normal subgroupoid of order 15.

**Remark 4.3.** (i) The symmetric groupoid $S_4$ contains:

(a) a transitive subgroupoid $K_9 = \{ g_1, g_2, g_3, g_8, g_9, g_{11}, g_{12}, g_{13} \}$ of order 9;

(b) a subgroupoid $K_6 = \Gamma(g_1) \cup \Gamma(g_5) \cup \Gamma(g_6)$ of order 6 such that $K_6 \neq S_2$.

(ii) $\Gamma/K = S_3/I_s(S_3)$ is a groupoid of order 7 (see [10]).

Let $(G, \alpha, \beta, \mu, i; G_0)$ be a groupoid and $\psi : G \to G'$ a bijective map from $G$ into a set $G'$. We consider the maps $\alpha', \beta', i' : G' \to G'$ given by $\alpha' = \psi \circ \alpha$, $\beta' = \psi \circ \beta$, $i' = \psi \circ i$ and we take $G'_0 = \psi(G_0)$.

For $(x', y') \in G' \times G'$ we define the multiplication law on $G'$ by $\mu'(x', y') = \psi(\mu(\psi^{-1}(x'), \psi^{-1}(y')))$. 

\[ x' \cdot y' = \psi(\psi^{-1}(x') \cdot \psi^{-1}(y')) \iff \psi^{-1}(x') \cdot \psi^{-1}(y') \text{ is defined in } G. \quad (13) \]

We have $G'_0 = \alpha'(G'_0) = \beta'(G'_0)$ and

\[ (x, y) \in G(2) \iff (x', y') \in G'(2), \text{ where } x' = \psi(x), \ y' = \psi(y). \quad (14) \]

It is easy to prove the following

**Proposition 4.5.** Let $(G, \alpha, \beta, \mu, i; G_0)$ be a groupoid and $\psi : G \to G'$ a bijective map. Then

(i) $(G', \alpha', \beta', \mu', i'; G'_0)$ is a groupoid, where $\alpha' = \psi \circ \alpha$, $\beta' = \psi \circ \beta$, $i' = \psi \circ i$, $G'_0 = \psi(G_0)$ and the multiplication $\mu'$ is given by (3.1).

(ii) $\psi : (G; G_0) \to (G'; G'_0)$ is an isomorphism of groupoids.

The groupoid $(G', \alpha', \beta', \mu', i'; G'_0)$ is called the direct image of the groupoid $(G, \alpha, \beta, \mu, i; G_0)$ by the bijection $\psi : G \to G'$. Also, we say that the groupoid $G'$ is obtained by transport of the structure from the groupoid $G$ via the bijection $\psi : G \to G'$. 
In the sequel we give a method for construction of finite groupoids. For instance, we will construct a groupoid of order 9. For this, we consider the transitive groupoid $H_4 = \{ f_1, f_2, f_4, f_5 \}$ of $S_2$ and the bijection $\varphi : H_4 \rightarrow M_4$, where $M_4 = \{ a_1, a_2, a_3, a_4 \}$, $\varphi(f_1) = a_1$, $\varphi(f_2) = a_2$, $\varphi(f_4) = a_3$, $\varphi(f_5) = a_4$.

Applying Proposition 4.5 we introduce on $M_4$ a structure of groupoid over $M_{4,0} = \{ a_1, a_2 \}$ obtained by transporting the structure of $H_4$ via $\varphi$ and we notice that $M_4$ is a transitive groupoid such that $M_4 \cong H_4$.

We now consider the set $M_5 = \{ a_5, a_6, a_7, a_8, a_9 \}$ and the bijection $\psi : S_2/\text{Is}(S_2) \rightarrow M_5$, $\psi(f_1) = a_5$, $\psi(f_2) = a_6$, $\psi(f_4) = a_7$, $\psi(f_5) = a_8$, $\psi(f_5) = a_9$.

Applying Proposition 4.5, the set $M_5$ has a groupoid structure over $M_{5,0} = \{ a_5, a_6, a_7 \}$ obtained by transporting the structure of $S_2/\text{Is}(S_2)$ via $\psi$. We notice that $M_5$ is a groupoid of order 5 and $M_5 \cong S_2/\text{Is}(S_2)$.

Using now the groupoids $M_4$ and $M_5$, we consider the disjoint union of these groupoids and we denote $M = M_4 \coprod M_5$.

The composition law and the structural functions of the groupoid $M = \{ a_i | i = 1, 9 \}$ are given by the following tables:

\begin{align*}
\cdot & | a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\
\hline
a_1 & a_1 & a_3 & & & & & & & \\
a_2 & a_2 & a_4 & & & & & & & \\
a_3 & a_3 & a_1 & & & & & & & \\
a_4 & a_4 & a_2 & & & & & & & \\
a_5 & a_5 & a_3 & & & & & & & \\
a_6 & a_6 & a_5 & & & & & & & \\
a_7 & a_7 & a_6 & & & & & & & \\
a_8 & a_8 & a_7 & & & & & & & \\
a_9 & a_9 & a_8 & & & & & & &
\end{align*}

\[\alpha(a) = a_1, \quad \beta(a) = a_1, \quad \gamma(a) = a_1.\]

We have that $M$ is a groupoid of order 9 and $M \not\cong K_9$.

**Remark 4.4.** Applying the above method we can construct several finite groupoids of any order. For example, if we want to construct a groupoid of order 72, we consider the groupoids $\Omega_i, i = 1, 9$, such that $\Omega_1 \cong H_4, \Omega_2 \cong S_2/\text{Is}(S_2), \Omega_3 \cong S_2, \Omega_4 \cong K_9, \Omega_5 \cong \text{Is}(S_5), \Omega_6 \cong S_3$ and take the disjoint union $\Omega = \bigsqcup_{i=1}^{6} \Omega_i$. We obtain that $\Omega$ is a groupoid of order 72.

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