Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 30(2), 2003, Pages 126–132
ISSN: 1223-6934

Existence of solutions to a class of second order differential inclusions

Vasile Lupulescu

Abstract. In this paper we prove a existence result for a second order differential inclusion

$$x'' \in F(x, x'),\ x(0) = x_0,\ x'(0) = y_0,$$

where F is an upper semicontinuous, compact valued multifunction, such that $F(x,y) \subset \partial V(y)$, for some convex proper lower semicontinuous function V.

2000 Mathematics Subject Classification. Primary 34A60; Secondary 49J52.
Key words and phrases. differential inclusions, upper semicontinuous, contingent cone.

1. Introduction

The existence of solutions to a Cauchy problem

$$x' \in F(x), x(0) = \xi,$$

where F is an upper semicontinuous, cyclically monotone multifunction, whose compact values are contained in the subdifferential ∂V of a proper convex, lower semicontinuous function V, was proved by Bressan, Cellina and Colombo([4]). For some extensions of this result we refer to ([1], [7], [12], [13]).

In this paper we prove a similar existence result for a second order differential inclusion

$$x'' \in F(x, x'), x(0) = x_0, x'(0) = y_0,$$

where $F(.,)$ is an upper semicontinuous, compact valued multifunction, such that $F(x,y) \subset \partial V(y)$, for some convex proper lower semicontinuous function V.

Second order differential inclusions were studied by many authors, mainly the case when the right-hand side is convex valued. For some existence results we refer to [3], [8], [9], [11].

2. Preliminaries and statement of the main result

Let $\mathbb{R}^m$ be the m-dimensional euclidean space with norm $\|.\|$ and scalar product $\langle .,. \rangle$ .For $x \in \mathbb{R}^m$ and $\varepsilon > 0$ let

$$B_\varepsilon(x) = \{y \in \mathbb{R}^m : \|y - x\| < \varepsilon\}$$

Received: 23 September 2003.
This paper was performed while the author visited the Department of Mathematics of Aveiro University, supported by a Post-doctoral fellowship of the Research unit "Mathematics and Applications"(Control Theory Group).

be the open ball centered at x and with radius ε, and let $\overline{B}_{\varepsilon}(x)$ be its closure. For $x \in \mathbb{R}^m$ and for a closed subsets $A \subset \mathbb{R}^m$ we denote by $d(x, A)$ the distance from x to A given by

$$d(x, A) = \inf \{\|x - y\| ; y \in A\}.$$

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a proper lower semicontinuous convex function. The multifunction $\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ defined by

$$\partial V(x) = \{\xi \in \mathbb{R}^m : V(y) - V(x) \geq \langle \xi, y - x \rangle, \forall y \in \mathbb{R}^m\}$$

is called subdifferential (in the sense of convex analysis) of the function V.

We say that a multifunction $F : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is upper semicontinuous if for every $x \in: \mathbb{R}^m$and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(y) \subset F(x) + B_{\varepsilon}(0), \ \forall y \in B_{\delta}(x).$$

For a multifunction $F : \Omega \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$ and for any $(x_0, y_0) \in \Omega$ we consider Cauchy problem

$$x'' \in F(x, x'), x(0) = x_0, x'(0) = y_0, \tag{1}$$

under the following assumptions:

(H_1) $\Omega \subset \mathbb{R}^{2m}$ is an open set and $F : \Omega \to 2^{\mathbb{R}^m}$ is an upper semicontinuous compact valued multifunction;

(H_2) There exists a proper convex and lower semicontinuous function $V : \mathbb{R}^m \to \mathbb{R}$ such that

$$F(x, y) \subset \partial V(y), \forall (x, y) \in \Omega. \tag{2}$$

Definition 2.1. *By solution of the problem* (1) *we mean any absolutely continuous function* $x : [0, T] \to \mathbb{R}^m$ *with absolutely continuous derivative* x' *such that* $x(0) = x_0$, $x'(0) = y_0$ *and*

$$x''(t) \in F(x(t), x'(t)), \ \textit{a.e. on} \ [0, T].$$

Our main result is the following:

Theorem 2.1. *If* $F : \Omega \to 2^{\mathbb{R}^m}$ *and* $V : \mathbb{R}^m \to \mathbb{R}$ *satisfy assumptions* (H_1) *and* (H_2) *then for every* $(x_0, y_0) \in \Omega$ *there exist* $T > 0$ *and* $x : [0, T] \to \mathbb{R}^m$,*a solution of the problem* (1).

3. Proof of the main result

Let $(x_0, y_0) \in \Omega$. Since Ω is open, there exists $r > 0$ such that the compact set $K := \overline{B}_r(x_0, y_0)$ be contained in Ω. Moreover, by the upper semicontinuity of F in (H_1) and by Proposition 1.1.3 in [2], the set

$$F(K) := \bigcup_{(x,y) \in K} F(x, y)$$

is compact, hence there exists $M > 0$ such that

$$\sup \{\|v\| : v \in F(x, y), \ (x, y) \in K\} \leq M.$$

Set

$$T := \min \left\{ \frac{r}{M}, \sqrt{\frac{r}{M}}, \frac{r}{2(\|y_0\| + 1)} \right\}.$$

We shall prove the existence of a solution of the problem (1) defined on the interval $[0, T]$.

For each $n \geq 1$ natural and for $1 \leq j \leq n$ we set $t_n^j := \frac{jT}{n}$, $I_n^j = [t_n^{j-1}, t_n^j]$ and for $t \in I_n^j$ we define

$$x_n(t) = x_n^j + (t - t_n^j)y_n^j + \frac{1}{2}(t - t_n^j)^2 v_n^j, \tag{3}$$

where $x_n^0 = x_0$, $y_n^0 = x_0$, and, for $0 \leq j \leq n-1$, and $v_n^j \in F(x_n^j, y_n^j)$,

$$\begin{cases} x_n^{j+1} = x_n^j + \frac{T}{n} y_n^j + \frac{1}{2}\left(\frac{T}{n}\right)^2 v_n^j \\ y_n^{j+1} = y_n^j + \frac{T}{n} v_n^j. \end{cases} \tag{4}$$

We claim that $(x_n^j, y_n^j) \in K$ for each $j \in \{1, 2, ..., n\}$. By the choice of T one has

$$\left\|x_n^1 - x_0\right\| \leq \frac{T}{n}\left\|y_0^n\right\| + \frac{1}{2}(\frac{T}{n})^2\left\|v_n^0\right\| < T\left\|y_0\right\| + \frac{1}{2}MT^2 < r$$

and

$$\left\|y_n^1 - y_0\right\| \leq T\left\|y_0\right\| < r,$$

hence the claim is true for $j = 1$.

We claim that for each $j > 1$ one has

$$\begin{cases} x_n^j = x_n^0 + t_n^j y_n^0 + \frac{1}{2}\left(\frac{T}{n}\right)^2 [(2j-1)\, v_n^0 + (2j-3)\, v_n^1 + ... + v_n^j] \\ y_n^j = y_n^0 + \frac{T}{n}[v_n^0 + v_n^1 + ... + v_n^{j-1}]. \end{cases} \tag{5}$$

The statement holds true for $j = 0$. Assume it holds for j, with $1 \leq j < n$. Then by (4) one obtains that

$$\begin{aligned} x_n^{j+1} &= x_n^j + \frac{T}{n} y_n^j + \frac{1}{2}(\frac{T}{n})^2 v_n^j \\ &= x_n^0 + \frac{jT}{n} y_n^0 + \frac{1}{2}(\frac{T}{n})^2[(2j-1)\, v_n^0 + (2j-1)\, v_n^1 + ... + v_n^{j-1}] + \\ &\quad + \frac{T}{n} y_n^0 + (\frac{T}{n})[v_n^0 + v_n^1 + ... + v_n^{j-1}] v_n^j + \frac{1}{2}(\frac{T}{n})^2 v_n^j \\ &= x_n^0 + t_n^{j+1} y_n^0 + \frac{1}{2}(\frac{T}{n})^2[(2j+1)\, v_n^0 + (2j-1)\, v_n^1 + ... + v_n^j], \end{aligned}$$

and

$$y_n^{j+1} = y_n^j + \frac{T}{n} v_n^j + \frac{1}{2}(\frac{T}{n})^2 v_n^j = y_n^0 + \frac{T}{n}[v_n^0 + v_n^1 + ... + v_n^j].$$

Therefore the relations in (5) are satisfied for each j,with $1 \leq j \leq n$ and our claim was proved.

Now, by (5) it follows easily that

$$\begin{aligned} \|x_n^j - x_0\| &\leq \frac{jT}{n}\|y_0^n\| + \frac{1}{2}(\frac{T}{n})^2\,[(2j-1) + (2j-3) + ... + 3 + 1]\, M \\ &= \frac{jT}{n}\|y_0\| + \frac{1}{2}M(\frac{jT}{n})^2 < T\|y_0\| + \frac{1}{2}MT^2 < r. \end{aligned}$$

and

$$\|y_n^j - y_0\| \leq \frac{jT}{n} M < r,$$

proving that $(x_n^j, y_n^j) \in K := B_r(x_0, y_0)$,for each j, with $1 \leq j \leq n$.

By (3) we have that

$$x_n'(t) = y_n^j + (t - t_n^j) v_n^j,\ \ x_n''(t) = v_n^j \in F(x_n^j, y_n^j), \forall t \in I_n^j,$$

hence

$$\begin{cases} \left\| x_n''(t) \right\| \le M, \forall t \in [0,T], \\ \left\| x_n'(t) \right\| \le \left\| y_0 \right\| + 2r, \forall t \in [0,T] \\ \left\| x_n(t) \right\| \le \left\| x_0 \right\| + (T+2)\, r, \forall t \in [0,T] \end{cases} \tag{6}$$

Moreover, for all t$\in [0,T]$ we have that

$$d\left(\left(x_n(t), x_n'(t), x_n''(t)\right), graph(F)\right) \le \frac{r(T+2)\,M}{n}. \tag{7}$$

Then, by (6) we obtain that $(x_n'')_n$ is bounded in $L^2([0,T],\mathbb{R}^m)$, $(x_n)_n$ and $(x_n')_n$ are bounded in $C([0,T],\mathbb{R}^m)$ and equi-Lipschitzian, hence, by Theorem 0.3.4 in [2] there exist a subsequence (again denoted by) $(x_n)_n$ and an absolutely continuous function $x:[0,T]\to\mathbb{R}^m$ such that

(i) $(x_n)_n$ converges uniformly to x;
(ii) $(x_n')_n$ converges uniformly to x';
(iii) $(x_n'')_n$ converges weakly in $L^2([0,T],\mathbb{R}^m)$ to x''.

By (H_2) and Theorem 1.4.1 in [2] we get then that

$$x''(t) \in coF\left(x(t), x'(t)\right) \subset \partial V\left(x'(t)\right),\ a.e. \text{ in } [0,T], \tag{8}$$

where co stands for the closed convex hull.

By (8) and Lemma 3.3 in [5] we obtain that

$$\frac{d}{dt}V\left(x'(t)\right) = \left\| x''(t) \right\|^2,$$

hence,

$$V\left(x'(T)\right) - V\left(x'(0)\right) = \int_0^T \left\| x''(t) \right\|^2 dt. \tag{9}$$

On the other hand, since

$$x_n''(t) = v_n^j \in F(x_n^j, y_n^j) \subset \partial V(x_n'(t_n^j)), \forall t \in I_n^j,$$

it follows that

$$\begin{aligned} V(x_n'(t_n^{j+1})) - V(x_n'(t_n^j)) &\ge \langle x_n''(t), x_n'(t_n^{j+1}) - x_n'(t_n^j)\rangle \\ &= \langle x_n''(t), \textstyle\int_{t_n^j}^{t_n^{j+1}} x_n''(s)\,ds\rangle = \textstyle\int_{t_n^j}^{t_n^{j+1}} \left\| x''(t) \right\|^2 dt. \end{aligned}$$

By adding the n inequalities from above, we get

$$V\left(x_m'(T)\right) - V\left(y_0\right) \ge \int_0^T \left\| x_n''(t) \right\|^2 dt,$$

and passing to the limit for $n\to\infty$,we obtain

$$V\left(x'(T)\right) - V\left(y_0\right) \ge \limsup_{n\to\infty} \int_0^T \left\| x_n''(t) \right\|^2 dt. \tag{10}$$

Therefore, by (9) and (10),

$$\int_0^T \left\| x''(t) \right\|^2 dt \ge \limsup_{n\to\infty} \int_0^T \left\| x_n''(t) \right\|^2 dt \tag{11}$$

and, since $(x_n'')_n$ converge weakly in $L^2([0,T],\mathbb{R}^m)$ to x'', by applying Proposition III.30 in [6], we obtain that $(x_n'')_n$ converge strongly in $L^2([0,T],\mathbb{R}^m)$ to x'', hence a subsequence again denoted by $(x_n'')_n$ converge pointwise to x''.

Since by (H_1) the graph of F is closed and, by (7),

$$\lim_{n\to\infty} d\left(\left(x_n(t), x_n'(t), x_n''(t)\right), graph(F)\right) = 0,$$

we obtain that

$$x''(t) \in F\left(x(t), x'(t)\right), \text{ a.e. on } [0,T].$$

Since x satisfies obviously the initial conditions, it is a solution of the problem (1).

4. An application

For $D \subset R^n$ and $x \in D$ denote by $T_D(x)$ the Bouligand's contingent cone of D at x, defined by

$$T_D(x) = \{v \in R^m; \liminf_{h\to 0_+} \frac{d(x+hv, D)}{h} = 0\}.$$

Also, $N_D(x)$ is the normal cone of D at x, defined by

$$N_D(x) = \{v \in R^m; \langle y, v\rangle \le 0, (\forall)\, v \in T_D(x)\}.$$

In what follows we consider that D is closed subset such that $\theta \in D$ and $\theta \notin int(D)$, where θ is the zero element of R^m.

Suppose that the following condition is satisfied

(H) $int\left(N_D(x)\right) \neq \emptyset$

We set $K = T_D(\theta)$, $Q = int\left(N_D(\theta)\right)$, $\Omega = B_1(\theta) \times Q$ and denote by $\pi_K(y)$ the projection a best approximation on K from y, defined by

$$\pi_K(y) = \{u \in K; d(y,u) = d(y,K)\}.$$

Lemma 4.1. *Suppose that (H) is satisfied. Then there exists a convex function $V : R^m \to R$ such that*

$$(1 - \|x\|)\, \pi_K(y) \subset \partial V(y), \ \ (\forall)\, (x,y) \in \Omega.$$

Proof. By Proposition 2 in [4] there exists a convex function V such that

$$\pi_K(y) \subset \partial V(y), \ \ (\forall)\, y \in Q.$$

We recall (see [4]) that the function V is defined by

$$V(y) = \sup\{\varphi_u(y); u \in K\},$$

where

$$\varphi_u(y) = \langle u, y\rangle - \frac{1}{2}\|u\|^2, \ y \in Q.$$

Also, we observe that the following assertions are equivalent:

$$\begin{cases} (i)\ u \in \pi_K(y); \\ (ii)\ \|y-u\| \le \|y-v\|, \ (\forall)\, v \in K; \\ (iii)\ \varphi_u(y) \ge \varphi_v(y), \ (\forall)\, v \in K. \end{cases} \tag{12}$$

Let $(x,y) \in \Omega$ by $z \in F(x,y)$. Then there exists $u \in \pi_K(y)$ such that $z = (1 - \|x\|)\, u$. We have that

$$\begin{aligned} \varphi_{(1-\|x\|)u}(y) &= \langle (1-\|x\|)\, u, y\rangle - \frac{1}{2}(1-\|x\|)^2 \|u\|^2 \\ &\ge \langle (1-\|x\|)\, u, y\rangle - \frac{1}{2}(1-\|x\|)\, \|u\|^2 \\ &= \langle u, y\rangle - \frac{1}{2}\|u\|^2 - \|x\|(\langle u, y\rangle - \frac{1}{2}\|u\|^2), \end{aligned}$$

hence

$$\varphi_{(1-\|x\|)u}(y) \geq (1-\|x\|)\,\varphi_u(y). \tag{13}$$

Since $u \in \pi_K(y)$ then $\varphi_u(y) \geq \varphi_v(y)$, for every $v \in K$, by (13) we have

$$\begin{aligned}\varphi_{(1-\|x\|)u}(y) - \varphi_v(y) &\geq (1-\|x\|)\,\varphi_u(y) - \varphi_v(y)\\ &\geq (1-\|x\|)\,\varphi_v(y) - \varphi_v(y) = -\|x\|\varphi_v(y),\end{aligned}$$

hence

$$\varphi_{(1-\|x\|)u}(y) - \varphi_v(y) \geq -\|x\|\varphi_v(y) \tag{14}$$

for every $v \in K$.

Since $y \in Q = int\,(N_D(\theta))$ we have that

$$\langle y, v\rangle \leq 0 \text{ for every } v \in K = T_D(\theta),$$

hence

$$\varphi_v(y) = \langle y, v\rangle - \frac{1}{2}\|v\|^2 \leq 0 \text{ for every } v \in K. \tag{15}$$

By (14) and (15) follows that

$$\varphi_{(1-\|x\|)u}(y) \geq \varphi_v(y) \text{ for every } v \in K. \tag{16}$$

Then (16) and the equivalent assertions in (12) imply that

$$z = (1-\|x\|)\,u \in \pi_K(y) \subset \partial V(y).$$

□

Proposition 4.1. *Suppose that (H) is satisfied. Then there exist $T > 0$ and $x(.) : [0,T] \to R^m$ a solution for the following Cauchy problem*

$$x'' \in (1-\|x\|)\,\pi_K(x'), \quad (x(0), x'(0)) = (x_0, y_0).$$

Proof. If we define the multifunction $F : \Omega \to 2^{R^m}$ by

$$F(x,y) = (1-\|x\|)\,\pi_K(y),$$

then F is with compact valued and upper semicontinuous and there exists a convex function $V : R^m \to R$ such that

$$F(x,y) \subset \partial V(y), \quad (\forall)\,(x,y) \in \Omega.$$

Therefore, F satisfies assumptions (H_1), (H_2) and Proposition is proved. □

References

[1] F. Ancona, G. Colombo, Existence of solutions for a class of nonconvex differential inclusions, *Rend . Se Mat. Univ. Padova*, **83**, 71-76 (1990).

[2] J.P. Aubin, A. Cellina, *Differential Inclusions*, Berlin, Springer-Verlag, 1984.

[3] A. Auslender, J. Mechler, Second order viability problems for differential inclusions, *J. Math. Anal. Appl.*, **181**, 205-218 (1984).

[4] A. Bressan, A. Cellina, G. Colombo, Upper semicontinuous differential inclusions without convexity, *Proc. Amer. Math. Soc.*, **106**, 771-775 (1989).

[5] H. Brezis, *Operateurs Maximaux Monotones et Semigroups de Contractions Dans Les Espaces de Hilbert,* Amsterdam, North-Holland, 1973.

[6] H. Brezis, *Analyse Fonctionelle, Théorie et Applications*, Paris, Masson, 1983.

[7] A. Cellina, V. Staicu, On evolution equations having monotonicities of opposite sign, *J. Differential Equations*, **90**, 71-80 (1991).

[8] B. Cornet, B. Haddad, Théoreme de viabilité pour inclusions differentielles du second order, *Israel J. Math.*, **57**, 225-238 (1987).

[9] T.X.D. Ha, M. Marques, Nonconvex second order differential inclusions with memory, *Set-Valued Analysis*, **3**, 71-86 (1995).

[10] V. Lupulescu, Existence of solutions to a class of second order differential inclusions, Technial report, CM 01/I-11, Department of Mathematics of Aveiro University, 2001.
[11] L. Marco, J. A. Murillo, Viability theorems for higher-order differential inclusions, *Set-Valued Anal.*, **6**, 21-37 (1998).
[12] F. Papalini, Existence of solutions for differential inclusions without convexity, *Rend. Inst. Mat. Univ. Trieste*, **24**, 193-206 (1992).
[13] A. Syam, *Contributions Aux Inclusions Differentielles*, PhD Thesis, Universite Montpellier II, 1993.

(Vasile Lupulescu) UNIVERSITATEA "CONSTANTIN BRÂNCUSI"
BULEVARDUL REPUBLICII NR.1,
1400 TÂRGU-JIU, ROMANIA
E-mail address: vasile@mail.utgjiu.ro