Existence of solutions to a class of second order differential inclusions

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Abstract. In this paper we prove a existence result for a second order differential inclusion

\[ x'' \in F(x, x'), \ x(0) = x_0, \ x'(0) = y_0, \]

where \( F \) is an upper semicontinuous, compact valued multifunction, such that \( F(x, y) \subset \partial V(y) \), for some convex proper lower semicontinuous function \( V \).

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1. Introduction

The existence of solutions to a Cauchy problem

\[ x' \in F(x), \ x(0) = \xi, \]

where \( F \) is an upper semicontinuous, cyclically monotone multifunction, whose compact values are contained in the subdifferential \( \partial V \) of a proper convex, lower semicontinuous function \( V \), was proved by Bressan, Cellina and Colombo([4]). For some extensions of this result we refer to ([1], [7], [12], [13]).

In this paper we prove a similar existence result for a second order differential inclusion

\[ x'' \in F(x, x'), \ x(0) = x_0, \ x'(0) = y_0, \]

where \( F(., .) \) is an upper semicontinuous, compact valued multifunction, such that \( F(x, y) \subset \partial V(y) \), for some convex proper lower semicontinuous function \( V \).

Second order differential inclusions were studied by many authors, mainly the case when the right-hand side is convex valued. For some existence results we refer to [3], [8], [9], [11].

2. Preliminaries and statement of the main result

Let \( \mathbb{R}^m \) be the m-dimensional euclidean space with norm \( \| \cdot \| \) and scalar product \( \langle \cdot, \cdot \rangle \). For \( x \in \mathbb{R}^m \) and \( \varepsilon > 0 \) let

\[ B_\varepsilon (x) = \{ y \in \mathbb{R}^m : \| y - x \| < \varepsilon \} \]

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be the open ball centered at $x$ and with radius $\varepsilon$, and let $\overline{B}_\varepsilon(x)$ be its closure. For $x \in \mathbb{R}^m$ and for a closed subsets $A \subset \mathbb{R}^m$ we denote by $d(x,A)$ the distance from $x$ to $A$ given by

$$d(x,A) = \inf \{ \|x - y\| : y \in A \}.$$ 

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a proper lower semicontinuous convex function. The multifunction $\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ defined by

$$\partial V(x) = \{ \xi \in \mathbb{R}^m : V(y) - V(x) \geq \langle \xi, y-x \rangle, \forall y \in \mathbb{R}^m \}$$

is called subdifferential (in the sense of convex analysis) of the function $V$.

We say that a multifunction $F : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is upper semicontinuous if for every $x \in \mathbb{R}^m$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(y) \subset F(x) + B_\varepsilon(0), \forall y \in B_\delta(x).$$

For a multifunction $F : \Omega \subset \mathbb{R}^{2m} \to 2^{\mathbb{R}^m}$ and for any $(x_0, y_0) \in \Omega$ we consider Cauchy problem

$$x''(t) \in F(x(t), x'(t)), x(0) = x_0, x'(0) = y_0,$$  \hspace{1cm} (1)

under the following assumptions:

$(H_1)$ $\Omega \subset \mathbb{R}^{2m}$ is an open set and $F : \Omega \to 2^{\mathbb{R}^m}$ is an upper semicontinuous compact valued multifunction;

$(H_2)$ There exists a proper convex and lower semicontinuous function $V : \mathbb{R}^m \to \mathbb{R}$ such that

$$F(x, y) \subset \partial V(y), \forall (x, y) \in \Omega.$$  \hspace{1cm} (2)

**Definition 2.1.** By solution of the problem (1) we mean any absolutely continuous function $x : [0, T] \to \mathbb{R}^m$ with absolutely continuous derivative $x'$ such that $x(0) = x_0$, $x'(0) = y_0$ and

$$x''(t) \in F(x(t), x'(t)), \text{ a.e. on } [0, T].$$

Our main result is the following:

**Theorem 2.1.** If $F : \Omega \to 2^{\mathbb{R}^m}$ and $V : \mathbb{R}^m \to \mathbb{R}$ satisfy assumptions $(H_1)$ and $(H_2)$ then for every $(x_0, y_0) \in \Omega$ there exist $T > 0$ and $x : [0, T] \to \mathbb{R}^m$, a solution of the problem (1).

3. Proof of the main result

Let $(x_0, y_0) \in \Omega$. Since $\Omega$ is open, there exists $r > 0$ such that the compact set $K := \overline{B}_r(x_0, y_0)$ be contained in $\Omega$. Moreover, by the upper semicontinuity of $F$ in $(H_1)$ and by Proposition 1.1.3 in [2], the set

$$F(K) := \bigcup_{(x,y) \in K} F(x,y)$$

is compact, hence there exists $M > 0$ such that

$$\sup \{ \|v\| : v \in F(x,y), (x,y) \in K \} \leq M.$$  

Set

$$T := \min \left\{ r \sqrt{\frac{r}{2M}}, \frac{r}{2(\|y_0\| + 1)} \right\}.$$  

We shall prove the existence of a solution of the problem (1) defined on the interval $[0, T]$. 

For each \( n \geq 1 \) natural and for \( 1 \leq j \leq n \) we set \( t_j^i := \frac{jT}{n}, \quad I_j^i := [t_j^{i-1}, t_j^i] \) and for \( t \in I_j^i \) we define
\[
x_n(t) = x_n^j + (t - t_j^i) y_n^i + \frac{1}{2} (t - t_j^i)^2 v_n^i,
\]
where \( x_n^0 = x_0, \quad y_n^0 = x_0 \), and, for \( 0 \leq j \leq n - 1 \), and \( v_n^i \in F(x_n^j, y_n^j) \),
\[
\begin{cases}
x_n^{j+1} = x_n^j + \frac{T}{n} y_n^j + \frac{1}{2} \left( \frac{T}{n} \right)^2 v_n^j \\
y_n^{j+1} = y_n^j + \frac{T}{n} v_n^j.
\end{cases}
\]

We claim that \((x_n^j, y_n^j) \in K\) for each \( j \in \{1, 2, ..., n\} \). By the choice of \( T \) one has
\[
\|x_n^1 - x_0\| \leq \frac{T}{n} \|y_0^0\| + \frac{1}{2} \left( \frac{T}{n} \right)^2 \|v_0^0\| < T \|y_0\| + \frac{1}{2} MT^2 < r
\]
and
\[
\|y_n^1 - y_0\| \leq T \|y_0\| < r,
\]
hence the claim is true for \( j = 1 \).

We claim that for each \( j > 1 \) one has
\[
\begin{cases}
x_n^j = x_n^{j-1} + t_j^i y_n^0 + \frac{1}{2} \left( \frac{T}{n} \right)^2 [(2j - 1) v_n^0 + (2j - 3) v_n^1 + ... + v_n^{j-1}]
\\y_n^j = y_n^{j-1} + \frac{T}{n} [v_n^0 + v_n^1 + ... + v_n^{j-1}] + v_n^j.
\end{cases}
\]
The statement holds true for \( j = 0 \). Assume it holds for \( j \), with \( 1 \leq j < n \). Then by (4) one obtains that
\[
\begin{align*}
x_n^{j+1} &= x_n^j + \frac{T}{n} y_n^j + \frac{1}{2} \left( \frac{T}{n} \right)^2 v_n^j \\
&= x_n^0 + jT n y_n^0 + \frac{1}{2} \left( \frac{T}{n} \right)^2 [(2j - 1) v_n^0 + (2j - 3) v_n^1 + ... + v_n^{j-1}] + \\
&\quad + \frac{T}{n} y_n^0 + \left( \frac{T}{n} \right)[v_n^0 + v_n^1 + ... + v_n^{j-1}] v_n^j + \frac{1}{2} \left( \frac{T}{n} \right)^2 v_n^j \\
&= x_n^0 + jT n y_n^0 + \frac{1}{2} \left( \frac{T}{n} \right)^2 [(2j + 1) v_n^0 + (2j - 1) v_n^1 + ... + v_n^{j}] \\
\end{align*}
\]
and
\[
y_n^{j+1} = y_n^j + \frac{T}{n} [v_n^0 + v_n^1 + ... + v_n^j].
\]
Therefore the relations in (5) are satisfied for each \( j \), with \( 1 \leq j \leq n \) and our claim was proved.

Now, by (5) it follows easily that
\[
\|x_n^j - x_0\| \leq \frac{jT}{n} \|y_0^0\| + \frac{1}{2} \left( \frac{T}{n} \right)^2 [(2j - 1) + (2j - 3) + ... + 3 + 1] M
\]
\[
= \frac{jT}{n} \|y_0\| + \frac{1}{2} M (\frac{jT}{n})^2 < T \|y_0\| + \frac{1}{2} MT^2 < r.
\]
and
\[
\|y_n^j - y_0\| \leq \frac{jT}{n} M < r,
\]
proving that \((x_n^j, y_n^j) \in K := B_r (x_0, y_0), \) for each \( j \), with \( 1 \leq j \leq n \).

By (3) we have that
\[
x_n^j(t) = y_n^j + (t-j^i) v_n^j, \quad x_n^m(t) = v_n^j \in F(x_n^j, y_n^j), \forall t \in I_j^i.
\]
Therefore, by (9) and (10) and passing to the limit for \( n \to \infty \), we obtain that \((x_n')_n \) converge pointwise to \( x' \). However, since \((x_n'')_n \) are bounded in \( L^2([0,T],\mathbb{R}^m) \) and equi-Lipschitzian, hence, by Theorem 0.3.4 in [2] there exist a subsequence (again denoted by) \((x_n')_n \) and an absolutely continuous function \( x : [0,T] \to \mathbb{R}^m \) such that

(i) \((x_n')_n \) converges uniformly to \( x \);
(ii) \((x_n'')_n \) converges uniformly to \( x'' \);
(iii) \((x_n''')_n \) converges weakly in \( L^2([0,T],\mathbb{R}^m) \) to \( x''' \).

By (11) and Theorem 1.4.1 in [2] we get then that

\[
x''(t) \in \text{co} F(x'(t),x'(t)) \subseteq \partial V(x'(t)), \text{ a.e. in } [0,T],
\]

where \( \text{co} \) stands for the closed convex hull.

By (8) and Lemma 3.3 in [5] we obtain that

\[
\frac{d}{dt} V(x'(t)) = \|x''(t)\|^2,
\]

hence,

\[
V(x'(T)) - V(x'(0)) = \int_0^T \|x''(t)\|^2 dt.
\]

On the other hand, since

\[
x_n'(t) = v_n(t) \in F(x_n',y_n') \subseteq \partial V(x_n'(t_n^j)), \forall t \in I_n^j,
\]

it follows that

\[
V(x_n'(t_{n+1}^j)) - V(x_n'(t_{n}^j)) \geq \langle x_n''(t_{n}^j), x_n'(t_{n+1}^j) - x_n'(t_{n}^j) \rangle
\]

\[
= \langle x_n''(t_n^j), \int_{t_n^j}^{t_{n+1}^j} x_n(s) ds \rangle = \int_{t_n^j}^{t_{n+1}^j} \|x''(t_n^j)\|^2 dt.
\]

By adding the \( n \) inequalities from above, we get

\[
V(x_n'(T)) - V(y_0) \geq \int_0^T \|x''(t_n^j)\|^2 dt,
\]

and passing to the limit for \( n \to \infty \), we obtain

\[
V(x'(T)) - V(y_0) \geq \limsup_{n \to \infty} \int_0^T \|x''(t_n^j)\|^2 dt.
\]

Therefore, by (9) and (10),

\[
\int_0^T \|x''(t)\|^2 dt \geq \limsup_{n \to \infty} \int_0^T \|x''(t_n^j)\|^2 dt
\]

and, since \((x''')_n \) converge weakly in \( L^2([0,T],\mathbb{R}^m) \) to \( x''' \), by applying Proposition III.30 in [6], we obtain that \((x''')_n \) converge strongly in \( L^2([0,T],\mathbb{R}^m) \) to \( x''' \), hence a subsequence again denoted by \((x''')_n \) converge pointwise to \( x''' \).
Since by \((H_1)\) the graph of \(F\) is closed and, by \((7)\),
\[
\lim_{n \to \infty} d((x_n(t), x'_n(t), x''_n(t)), graph(F)) = 0,
\]
we obtain that
\[
x''(t) \in F(x(t), x'(t)), \text{ a.e. on } [0,T].
\]
Since \(x\) satisfies obviously the initial conditions, it is a solution of the problem \((1)\).

4. An application

For \(D \subset R^n\) and \(x \in D\) denote by \(T_D(x)\) the Bouligand’s contingent cone of \(D\) at \(x\), defined by
\[
T_D(x) = \{ v \in R^n; \liminf_{h \to 0} \frac{d(x + hv, D)}{h} = 0 \}.
\]
Also, \(N_D(x)\) is the normal cone of \(D\) at \(x\), defined by
\[
N_D(x) = \{ v \in R^n; \langle y, v \rangle \leq 0, (\forall) v \in T_D(x) \}.
\]

In what follows we consider that \(D\) is closed subset such that \(\theta \in D\) and \(\theta \notin int(D)\), where \(\theta\) is the zero element of \(R^n\).

Suppose that the following condition is satisfied \((H)\)
\[
\text{int}(N_D(x)) \neq \emptyset
\]
We set \(K = T_D(\theta), Q = int(N_D(\theta)), \Omega = B_1(\theta) \times Q\) and denote by \(\pi_K(y)\) the projection a best approximation on \(K\) from \(y\), defined by
\[
\pi_K(y) = \{ u \in K; d(y, u) = d(y, K) \}.
\]

**Lemma 4.1.** Suppose that \((H)\) is satisfied. Then there exists a convex function \(V: R^n \to R\) such that
\[
(1 - ||x||) \pi_K(y) \subset \partial V(y), \quad (\forall) (x, y) \in \Omega.
\]

*Proof.* By Proposition 2 in [4] there exists a convex function \(V\) such that
\[
\pi_K(y) \subset \partial V(y), \quad (\forall) y \in Q.
\]

We recall (see [4]) that the function \(V\) is defined by
\[
V(y) = \sup \{ \varphi_u(y); u \in K \},
\]
where
\[
\varphi_u(y) = \langle u, y \rangle - \frac{1}{2} ||u||^2, \quad y \in Q.
\]

Also, we observe that the following assertions are equivalent:
\[
\begin{align*}
(i) & \quad u \in \pi_K(y); \\
(ii) & \quad ||y - u|| \leq ||y - v||, \quad (\forall) v \in K; \\
(iii) & \quad \varphi_u(y) \geq \varphi_v(y), \quad (\forall) v \in K.
\end{align*}
\]

Let \((x, y) \in \Omega\) by \(z \in F(x, y)\). Then there exists \(u \in \pi_K(y)\) such that \(z = (1 - ||x||)u\). We have that
\[
\varphi_{(1-||x||)u}(y) = \langle (1 - ||x||)u, y \rangle - \frac{1}{2} (1 - ||x||)^2 ||u||^2 \\
\geq \langle (1 - ||x||)u, y \rangle - \frac{1}{2} (1 - ||x||) ||u||^2 \\
= \langle u, y \rangle - \frac{1}{2} ||u||^2 - ||x||((u, y) - \frac{1}{2} ||u||^2),
\]
hence
\[ \varphi(1-\|x\|)u(y) \geq (1-\|x\|)\varphi_u(y). \]  \hspace{1cm} (13)

Since \( u \in \pi_K(y) \) then \( \varphi_u(y) \geq \varphi_v(y) \), for every \( v \in K \), by (13) we have
\[ \varphi(1-\|x\|)u(y) = \varphi_u(y) \geq (1-\|x\|)\varphi_u(y) - \varphi_v(y) \geq (1-\|x\|)\varphi_u(y) - \varphi_v(y) = -\|x\|\varphi_v(y), \]
hence
\[ \varphi(1-\|x\|)u(y) \geq -\|x\|\varphi_v(y) \]  \hspace{1cm} (14)
for every \( v \in K \).

Since \( y \in Q = \text{int}(N_D(\theta)) \) we have that
\[ \langle y, v \rangle \leq 0 \]  \hspace{1cm} (15)
for every \( v \in K \).

By (14) and (15) follows that
\[ \varphi(1-\|x\|)u(y) \geq \varphi_v(y) \]  \hspace{1cm} (16)
for every \( v \in K \).

Proposition 4.1. Suppose that \((H)\) is satisfied. Then there exist \( T > 0 \) and \( x(.) : [0, T] \to \mathbb{R}^m \) a solution for the following Cauchy problem
\[ x'' \in (1-\|x\|)\pi_K(x'), \quad (x(0), x'(0)) = (x_0, y_0). \]

Proof. If we define the multifunction \( F : \Omega \to 2^{\mathbb{R}^m} \) by
\[ F(x, y) = (1-\|x\|)\pi_K(y), \]
then \( F \) is with compact valued and upper semicontinuous and there exists a convex function \( V : \mathbb{R}^m \to \mathbb{R} \) such that
\[ F(x, y) \subset \partial V(y), \quad \forall (x, y) \in \Omega. \]

Therefore, \( F \) satisfies assumptions \((H_1), (H_2)\) and Proposition is proved. \( \square \)

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