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Viability for functional differential inclusions without convexity

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Abstract. The aim of this paper is to prove the existence result of viable solutions for the differential inclusion $\dot{x}(t) \in F(t, T(t)x)$ where F is a set-valued map with closed graph. We consider the case when the constraint is moving.

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1. Introduction

The aim of this paper is to prove the existence of solutions for the following nonconvex functional differential inclusions

$$\begin{cases} \dot{x}(t) \in F(t, T(t)x) \text{ a.e. on } [0, \tau], \\ x(t) = \varphi(t), \ \forall t \in [-a, 0], \\ x(t) \in C(t), \ \forall t \in [0, \tau], \end{cases} \tag{1}$$

where F and C are two set-valued maps and φ is a function.

Bressan, Cellina and Colombo, in [7], have first established the existence of local solution, in finite dimensional space and the nonconvex case, for the differential inclusions $\dot{x}(t) \in F(x(t))$, where F is upper semicontinuous and the values of F are contained in the subdifferential (in the sense of analysis convex) of convex lower semicontinuous function. Ancona and Colombo (see [4]), under the same hypotheses, extend this result to the perturbed problem $\dot{x}(t) \in f(t, x(t)) + F(x(t))$ where $f(\cdot, \cdot)$ is a Carathéodory function. In this framework, consult [1, 2, 11, 17] and the references therein for other related results concerning the extension of the main result in [4, 7]. Moreover, in all the above works, the values of the set-valued map is contained in the subdifferential (in the sense of analysis convex or in the sense of Clarke), and the convexity or the uniformly regularity assumption of V were widely used in the proof.

On the other hand, Kannai and Tallos [15] and Cernea [10] proved the existence of solutions to the following differential inclusion $\dot{x}(t) \in F(t, x(t))$, $x(t) \in K$, where K is a convex subset and F is measurable with respect to the first argument and upper semicontinuous with respect to the second argument. The proof in [10, 15] bases on Scorza-Dragoni type results for upper semicontinuous maps and the results are obtained under the following assumption $F(t, x) \cap T_K(x) \cap \partial_c V(x) \neq \emptyset$, where V is lower regular in [15] and is convex in [10]. $T_K(x)$ is the Bouligand tangent cone of K at x and $\partial_c V(x)$ denotes the Clarke subdifferential of V at x.

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In [3], we have established a viable solutions of the problem of Bressan, Cellina and Colombo, but with weaker hypotheses, namely, F is upper semicontinuous with compact values such that

$$F(x) \cap \partial_c V(x) \cap T_K(x) \neq \emptyset, \quad \forall x \in K, \tag{2}$$

where V is regular.

This work extends results which are presented in [3, 7]. Indeed, we get an existence result, in Hilbert space, for functional differential inclusions, with a constraint which depends on time. The right-hand side is not necessary upper semi-continuous with compact values. It remains to notice that the methods used in this paper and in [3, 7] are different.

The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. In Section 3, we prove the existence of solutions for (1).

2. Notations, definitions and the main result

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ and the scalar product $\langle \cdot, \cdot \rangle$. For $x \in H$ and $r > 0$, let $B(x, r)$ be the open ball centered at x with radius r and $\overline{B}(x, r)$ be its closure. Put $B = B(0, 1)$. For I a segment in $\mathbb{R}$, we denote by $\mathcal{C}(I, H)$ the Banach space of continuous functions from I to H equipped with the norm $\|x(.)\|_\infty := \sup\left\{\|x(t)\|; t \in I\right\}$. For any set-valued map F, we denote $Gr(F)$ its graph. For a a positive number, we put $\mathcal{C}_a := \mathcal{C}([-a, 0], H)$ and for any $t \in [0, \tau]$, $\tau > 0$, we define the operator $T(t)$ from $\mathcal{C}([-a, \tau], H)$ to $\mathcal{C}_a$ with $(T(t)(x(.)))(s) := (T(t)x)(s) := x(t + s)$, $s \in [-a, 0]$. For $x \in H$ and for nonempty subset A of H, we denote $d_A(x) := \inf\left\{\|y - x\|; y \in A\right\}$.

We shortly review some notions used in this paper (see [12, 13, 16] as general references).

Let $V : H \to \mathbb{R}$ be a lower semi-continuous function and x be any point where V is finite. The generalized Rockafellar directional derivative $V^\uparrow(x, .)$ is

$$V^\uparrow(x, v) := \limsup_{x' \to x, V(x') \to V(x), t \to 0^+} \; \inf_{v' \to v} \frac{V(x' + tv') - V(x')}{t}.$$

The upper generalized Clarke directional derivative $V^o(x, .)$ is

$$V^o(x, v) := \limsup_{h \to 0^+ \; y \to x} \frac{V(y + hv) - V(y)}{h}.$$

Analogously the lower generalized Clarke directional derivative $V_o(x, .)$ is

$$V_o(x, v) := \liminf_{h \to 0^+ \; y \to x} \frac{V(y + hv) - V(y)}{h}.$$

If V is Lipschitz around x, then $V^\uparrow(x, v)$ coincides with $V^o(x, v)$ for all $v \in H$. We also recall that the Clarke subdifferential of V at x is defined by

$$\partial_c V(x) := \left\{ y \in H \, : \, \langle y, v \rangle \, \leq \, V^\uparrow(x, v), \text{ for all } v \in H \right\}.$$

In the following proposition we summarize some useful properties of Clarke generalized directional derivatives.

Proposition 2.1. [12, 13] *Let $V : H \to \mathbb{R}$ be locally Lipschitz. Then the following conditions holds:*

(i) $\partial_c V(x) = \Big\{p \in H : V^o(x,v) \geq \langle p, v\rangle, \forall v \in H\Big\}$
$= \Big\{p \in H : V_o(x,v) \leq \langle p, v\rangle, \forall v \in H\Big\}$;

(ii) $V^o(x,v) = \max\Big\{\langle p, v\rangle, p \in \partial_c V(x)\Big\}$ *and*
$V_o(x,v) = \min\Big\{\langle p, v\rangle, p \in \partial_c V(x)\Big\} = -V^o(x,-v)$.

Let us recall the definition of the concept of the regularity (in the sense of Clarke).

Definition 2.1. [12] Let $V : H \to \mathbb{R}$ be a locally Lipschitz function. We say that V is regular at x if for all $v \in H$, the usual directional derivative $V'(x,v)$ exists and $V'(x,v) = V^o(x,v)$. We say that V is regular over a set S if it is regular at any point in S.

If S is a bounded set of H, then the Kuratowski's measure of noncompactness of S, $\beta(S)$, is defined by

$$\beta(S) = \inf\Big\{d > 0 : S \text{ can be covered by a finite number of sets with diameter less than } d\Big\}.$$

In the following lemma we recall some useful properties for the measure of noncompactness β. For instance see Proposition 9.1 in [14].

Lemma 2.2. *Let X be an infinite dimensional real Banach space and D_1, D_2 be two bounded subsets of X.*

(i) $\beta(D_1) = 0 \Leftrightarrow D_1$ *is relatively compact.*
(ii) $\beta(\lambda D_1) = |\lambda|\beta(D_1)$; $\lambda \in \mathbb{R}$.
(iii) $D_1 \subseteq D_2 \Rightarrow \beta(D_1) \leq \beta(D_2)$.
(iv) $\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$.
(v) *If $x_0 \in X$ and r is a positive real number then $\beta(B(x_0, r)) = 2r$.*

The following lemma is widely used in the sequel.

Lemma 2.3. [8] *Let $\preceq$ be a given preorder on the nonempty set $\mathcal{B}$ and let $\phi : \mathcal{B} \to \mathbb{R} \cup \{+\infty\}$ be an increasing function. Suppose that each increasing sequence in $\mathcal{B}$ is majorated in $\mathcal{B}$. Then, for each $x_0 \in \mathcal{B}$, there exists $x_1 \in \mathcal{B}$ such that $x_0 \preceq x_1$ and $\phi(x_1) = \phi(x)$ if $x_1 \preceq x$.*

The above function ϕ, in [8], is supposed to be finite and bounded from above, but this restriction can be removed by replacing ϕ by the function $x \mapsto \arctan \phi(x)$ (see [9]).

Now let us introduce the following hypotheses which we shall use throughout this paper.

(H1) $C : [0,b] \to 2^H$ is a set-valued map with locally closed graph such that $C([0,b])$ is locally compact, and $\mathcal{K} : [0,b] \to \mathcal{C}_a$ is a set-valued map defined by

$$\mathcal{K}(t) = \Big\{\varphi \in \mathcal{C}_a : \varphi(0) \in C(t)\Big\}, \quad \forall t \in [0,b],$$

(H2) $V : H \to \mathbb{R}$ is a locally Lipschitz function and regular over $C([0,b])$, (H3) $F : Gr(\mathcal{K}) \to 2^H$ is a set-valued map with closed graph satisfying

$$F(t,\varphi) \cap T_{C(t)}(\varphi(0)) \cap \partial_c V(\varphi(0)) \neq \emptyset, \quad \text{for all} \quad (t,\varphi) \in Gr(\mathcal{K}),$$

where

$$T_{C(t)}(\varphi(0)) = \Big\{ v \in H, \quad \liminf_{h \mapsto 0^+} \frac{1}{h} d_{C(t+h)}\left(\varphi(0) + hv\right) = 0 \Big\}.$$

We are now ready to state the main result of this paper.

Theorem 2.4. *If assumptions (H1)-(H3) are satisfied, for all $\varphi \in \mathcal{K}(0)$, there exist $\tau > 0$ and a map $x(.) \in \mathcal{C}([-a,\tau], H)$ which is absolutely continuous on $[0,\tau]$ such that $x(.)$ is a solution of (1).*

3. Proof of the main result Theorem 2.4

Let $\varphi \in \mathcal{K}(0)$ and set $x_0 = \varphi(0)$. By assumptions, there exists $r > 0$ such that $C([0,b]) \cap \overline{B}(x_0, r)$ is compact, $Gr(C) \cap \overline{B}((0,x_0), r)$ is closed and V is Lipschitz continuous on $\overline{B}(x_0, r)$ with Lipschitz constant $\lambda > 0$. Then $\partial_c V(x) \subset \lambda \overline{B}$ for every $x \in \overline{B}(x_0, r)$. Put

$$\tau = \inf\Big\{ \frac{r}{2(2+\lambda)}, 1, b \Big\}.$$

For all $0 < \varepsilon < 1$, set $\mathcal{B}(\varepsilon)$ the set of all $(x,\theta)_d$ where $d \in]0,\tau]$, $x(.) : [-a,d] \to H$ is a continuous function and $\theta(.) : [0,d[\to [0,d[$ is a step function such that

(i) $x(\theta(t)) \in C(\theta(t))$, $0 \leq t - \theta(t) \leq \varepsilon$, for all $t \in [0,d[$;

(ii) $x(d) \in C(d)$, $x(.) = \varphi(.)$ on $[-a,0]$ and $x(t) \in \overline{B}(x_0, r)$ for all $t \in [0,d]$;

(iii) $\dot{x}(t) \in [F(\theta(t), T(\theta(t))x) + \varepsilon B] \cap [\partial_c V(x(\theta(t))) + \varepsilon B]$ for almost all $t \in [0,d]$;

(iv) $V(x(d)) - V(\varphi(0)) \geq \int_0^d \|\dot{x}(s)\|^2 ds - d\varepsilon(3\lambda + 4)$.

Proposition 3.1. *If assumptions (H1)-(H3) are satisfied, then for all $0 < \varepsilon < 1$, there exists at least one $(x,\theta)_\tau \in \mathcal{B}(\varepsilon)$.*

Proof. Let $0 < \varepsilon < 1$. Put

$$x(t) = \varphi(t), \ \forall t \in [-a, 0].$$

Select $u_0 \in F(0,\varphi) \cap T_{C(0)}(\varphi(0)) \cap \partial_c V(\varphi(0))$. There exists $0 < \rho < \inf\{\tau, \varepsilon\}$ such that for all $0 < h < \rho$

$$V(\varphi(0) + hu_0) - V(\varphi(0)) \ \geq \ hV'(\varphi(0), u_0) - \varepsilon h.$$

By the regularity of V, we rewrite this last inequality as

$$V(\varphi(0) + hu_0) - V(\varphi(0)) \ \geq \ h\langle u_0, w\rangle - \varepsilon h, \quad \forall w \in \partial_c V(\varphi(0)).$$

Hence

$$V(\varphi(0) + hu_0) - V(\varphi(0)) \ \geq \ h\langle u_0, u_0\rangle - \varepsilon h, \quad \forall h \in]0, \rho[. \tag{3}$$

Moreover, there exists $h_0 \in]0,\rho[$ satisfying

$$\frac{1}{h_0} d_{C(h_0)}\Big(\varphi(0) + h_0 u_0\Big) \leq \frac{\varepsilon}{2}.$$

Then there exists $x_1 \in C(h_0)$ such that

$$\frac{1}{h_0}\Big\| x_1 - \varphi(0) - h_0 u_0 \Big\| \leq \varepsilon.$$

Set
$$u_1 = \frac{x_1 - \varphi(0)}{h_0}.$$
So, we get $x_1 = \varphi(0)+h_0u_1$ and $\|u_1-u_0\| \le \varepsilon$. Set $d_0 = h_0$, $\theta_0(t) = 0$ for all $t \in [0, d_0[$ and $x(t) = x_0 + tu_1$ for all $t \in [0, d_0]$. Remark that, for all $t \in [0, d_0]$
$$\|x(t) - x_0\| \le h_0\|u_1\| \le h_0(\varepsilon + \|u_0\|) \le \tau(1 + \lambda) \le r,$$
then $x(t) \in \overline{B}(x_0, r)$ for all $t \in [0, d_0]$. On the other hand, set $u_0 = u_1 + \varepsilon b$ where $b \in \overline{B}$. By (3), one has
$$\begin{aligned} V(\varphi(0) + h_0u_1 + h_0b\varepsilon) - V(\varphi(0)) &\ge h_0\langle u_1 + b\varepsilon, u_1 + b\varepsilon\rangle - \varepsilon h_0 \\ &\ge \int_0^{d_0} \|\dot{x}(s)\|^2 ds - (2\lambda + 4)d_0\varepsilon \end{aligned}$$
because $|\langle u_1, b\varepsilon\rangle| \le (\lambda + 1)\varepsilon$ and $|\langle b\varepsilon, b\varepsilon\rangle| \le \varepsilon$. We deduce that
$$\begin{aligned} & V(\varphi(0) + h_0u_1) - V(\varphi(0)) \\ = \; & V(\varphi(0) + h_0u_1) - V(\varphi(0) + h_0u_1 + h_0b\varepsilon) + V(\varphi(0) + h_0u_1 + h_0b\varepsilon) - V(\varphi(0)) \\ \ge \; & V(\varphi(0) + h_0u_1) - V(\varphi(0) + h_0u_1 + h_0b\varepsilon) + \int_0^{d_0} \|\dot{x}(s)\|^2 ds - (2\lambda + 4)d_0\varepsilon \end{aligned}$$
Since
$$\|\varphi(0) + h_0u_1 - x_0\| \;\le\; h_0(\lambda + 1) \;\le\; r$$
and
$$\|\varphi(0) + h_0u_1 + h_0b\varepsilon - x_0\| \;\le\; h_0(\lambda + 2) \;\le\; r,$$
we get
$$\Big|V(\varphi(0) + h_0u_1) - V(\varphi(0) + h_0u_1 + h_0b\varepsilon)\Big| \;\le\; d_0\lambda\varepsilon.$$
Using the above inequality, we obtain
$$V(x(d_0)) - V(\varphi(0)) \;\ge\; \int_0^{d_0} \|\dot{x}(s)\|^2 ds - (3\lambda + 4)d_0\varepsilon.$$
We conclude that $(x, \theta)_{d_0} \in \mathcal{B}(\varepsilon)$ and hence $\mathcal{B}(\varepsilon) \ne \emptyset$. Now, consider the following preorder:
$$(x_1, \theta_1)_{d_1} \preceq (x_2, \theta_2)_{d_2} \Leftrightarrow d_1 \le d_2,\ x_1 = x_2|_{[0,d_1]} \text{ and } \theta_1 = \theta_2|_{[0,d_1[}.$$
Let $\phi : \mathcal{B}(\varepsilon) \to \mathbb{R}$ be the function defined by
$$\phi((x, \theta)_d) = d, \quad \forall (x, \theta)_d \in \mathcal{B}(\varepsilon).$$
By definition, ϕ is increasing on $\mathcal{B}(\varepsilon)$. On the other hand, if $\Big((x_i, \theta_i)_{d_i}\Big)_{i\in\mathbb{N}}$ is an increasing sequence in $\mathcal{B}(\varepsilon)$, we construct a majorant $(x, \theta)_d$ of $\Big((x_i, \theta_i)_{d_i}\Big)_{i\in\mathbb{N}}$ as follows:
$$d = \lim_i d_i,\ \theta(t) = \theta_i(t) \text{ for all } t \in [0, d_i[,$$
$$x(.) = \varphi(.) \text{ on } [-a, 0],\ x(t) = x_i(t) \text{ for all } t \in [0, d_i].$$
We claim that $(x, \theta)_d \in \mathcal{B}(\varepsilon)$. Indeed, for all $i \in \mathbb{N}$, we have $x(d_i) = x_i(d_i) \in C(d_i)$. Then $(d_i, x(d_i)) \in Gr(C) \cap \overline{B}((0, x_0), r)$, for all $i \in \mathbb{N}$. Since $Gr(C) \cap \overline{B}((0, x_0), r)$ is closed, we conclude that $(d, x(d)) \in Gr(C) \cap \overline{B}((0, x_0), r)$. The other assertions are obvious. Next, for applying Lemma 2.3, we need the following Claim.

Claim: For all $(x,\theta)_d \in \mathcal{B}(\varepsilon)$ with $d<\tau$, there exists $(\bar{x},\bar{\theta})_{\bar{d}} \in \mathcal{B}(\varepsilon)$ such that $(x,\theta)_d \preceq (\bar{x},\bar{\theta})_{\bar{d}}$ and $\phi((x,\theta)_d) < \phi((\bar{x},\bar{\theta})_{\bar{d}})$.

Proof. Let $(x,\theta)_d \in \mathcal{B}(\varepsilon)$ with $d<\tau$. Select

$$u_0 \in F(d, T(d)x) \cap T_{C(d)}(x(d)) \cap \partial_c V(x(d)).$$

There exists $0<\rho<\inf\{\tau-d,\varepsilon\}$ such that for all $0<h<\rho$

$$V(x(d)+hu_0) - V(x(d)) \;\geq\; hV'(x(d),u_0) - \varepsilon h.$$

By the regularity of V, as above, we get

$$V(x(d)+hu_0) - V(x(d)) \;\geq\; h\langle u_0,u_0\rangle - \varepsilon h, \quad \forall h \in]0,\rho[. \tag{4}$$

Moreover, there exist $h_0 \in]0,\rho[$ and $x_1 \in C(d+h_0)$ satisfying

$$\frac{1}{h_0}\Big\| x_1 - x(d) - h_0 u_0 \Big\| \leq \varepsilon.$$

Set

$$u_1 = \frac{x_1 - x(d)}{h_0}.$$

So, we get $x_1 = x(d) + h_0 u_1$ and $\|u_1 - u_0\| \leq \varepsilon$. Set $\bar{d} = d + h_0$, $\tilde{\theta}(t) = d$ for all $t \in [d,\bar{d}[$ and $\tilde{x}(t) = x(d) + (t-d)u_1$ for all $t \in [d,\bar{d}]$. We have for all $t \in [d,\bar{d}]$

$$\begin{aligned}
\|\tilde{x}(t) - x_0\| &\leq \|x(d) - x_0\| + (t-d)\|u_1\| \\
&\leq \int_0^d \|\dot{x}(s)\|ds + (t-d)(\lambda+1) \\
&\leq d(\lambda+1) + (t-d)(\lambda+1) \\
&\leq \tau(\lambda+1) \\
&\leq r
\end{aligned}$$

then $\tilde{x}(t) \in \overline{B}(x_0,r)$ for all $t \in [d,\bar{d}]$. On the other hand, set $u_0 = u_1 + \varepsilon b$ where $b \in \overline{B}$. Since

$$\begin{aligned}
\|x(d) + h_0 u_1 - x_0\| &\leq \int_0^d \|\dot{x}(s)\|ds + (\bar{d}-d)(\lambda+1) \\
&\leq d(\lambda+1) + (\bar{d}-d)(\lambda+1) \\
&\leq \tau(\lambda+1) \\
&\leq r
\end{aligned}$$

and

$$\begin{aligned}
\|x(d) + h_0 u_1 + h_0 b\varepsilon - x_0\| &\leq \int_0^d \|\dot{x}(s)\|ds + (\bar{d}-d)(\lambda+2) \\
&\leq d(\lambda+1) + (\bar{d}-d)(\lambda+2) \\
&\leq \tau(\lambda+2) \\
&\leq r
\end{aligned}$$

we get

$$\Big| V(x(d)+h_0u_1) - V(x(d)+h_0u_1+h_0b\varepsilon) \Big| \;\leq\; h_0\lambda\varepsilon.$$

Using the last inequality and (4), as above, we obtain

$$V(\tilde{x}(\bar{d})) - V(\tilde{x}(d)) \geq \int_d^{\bar{d}} \|\dot{\tilde{x}}(s)\|^2 ds - (3\lambda + 4)(\bar{d} - d)\varepsilon.$$

Now, we define $\bar{\theta}$ and $\bar{x}$ as follows:

$$\bar{\theta}(t) = \theta(t), \text{ for all } t \in [0, d[, \ \bar{\theta}(t) = \tilde{\theta}(t) \text{ for all } t \in [d, \bar{d}[,$$

$$\bar{x}(.) = \varphi(.) \text{ on } [-a, 0], \ \bar{x}(t) = x(t) \text{ for all } t \in [0, d] \text{ and } \bar{x}(t) = \tilde{x}(t) \text{ for all } t \in [d, \bar{d}].$$

Finally, it is clair that $(\bar{x}, \bar{\theta})_{\bar{d}} \in \mathcal{B}(\varepsilon)$, $(x, \theta)_d \preceq (\bar{x}, \bar{\theta})_{\bar{d}}$ and $\phi((x, \theta)_d) < \phi((\bar{x}, \bar{\theta})_{\bar{d}})$. □

Now, we are ready to complete the proof of Proposition 3.1. From Lemma 2.3, there exists $(x, \theta)_d \in \mathcal{B}(\varepsilon)$ such that $\phi((x, \theta)_d) = \phi((\bar{x}, \bar{\theta})_{\bar{d}})$ and $(x, \theta)_d \preceq (\bar{x}, \bar{\theta})_{\bar{d}}$ for all $(\bar{x}, \bar{\theta})_{\bar{d}} \in \mathcal{B}(\varepsilon)$. Moreover, if $\phi((x, \theta)_d) < \tau$, by the last Claim, there exists $(\bar{x}, \bar{\theta})_{\bar{d}} \in \mathcal{B}(\varepsilon)$ such that $(x, \theta)_d \preceq (\bar{x}, \bar{\theta})_{\bar{d}}$ and $\phi((x, \theta)_d) < \phi((\bar{x}, \bar{\theta})_{\bar{d}})$. Hence $\phi((x, \theta)_d) = \tau$. The proof is complete. □

In the next, we will prove our Theorem 2.4. Let $(\varepsilon_n)_{n\geq 1}$ be a strictly decreasing sequence of positive scalars such that $0 < \varepsilon_n < 1$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. In view of Proposition 3.1, we can define inductively sequences $(x_n(.))_{n\geq 1} \subset \mathcal{C}([-a, \tau], H)$, and $(\theta_n(.))_{n\geq 1}, \subset S([0, \tau[, [0, \tau[)$ where $S([0, \tau[, [0, \tau[)$ denotes the space of step functions from $[0, \tau[$ into $[0, \tau[$ such that

(a) $x_n(\theta_n(t)) \in C(\theta_n(t))$, $0 \leq t - \theta_n(t) \leq \varepsilon_n$, for all $t \in [0, \tau[$;
(b) $x_n(\tau) \in C(\tau)$, $x_n(.) = \varphi(.)$ on $[-a, 0]$ and $x_n(t) \in \overline{B}(x_0, r)$ for all $t \in [0, \tau]$;
(c) $\dot{x}_n(t) \in [F(\theta_n(t), T(\theta_n(t))x_n) + \varepsilon_n B] \cap [\partial_c V(x_n(\theta_n(t))) + \varepsilon_n B]$ for almost all $t \in [0, \tau]$;
(d) $V(x_n(\tau)) - V(\varphi(0)) \geq \int_0^\tau \|\dot{x}_n(s)\|^2 ds - \tau\varepsilon_n(3\lambda + 4)$.

In the rest of this paper, we take $\theta_n(\tau) = \tau$ for all $n \geq 1$. By (c), for a.e. $t \in [0, \tau]$, we have

$$\|\dot{x}_n(t)\| \leq \lambda + 1. \tag{5}$$

Hence the sequence $(x_n(.))_n$ is equicontinuous. In order to apply Arzelà-Ascoli's theorem, we are going to show that for every $t \in [0, \tau]$, the set

$$S(t) = \{x_n(t) : n \geq 1\},$$

is relatively compact in H. By (a), for all $t \in [0, \tau]$, $x_n(\theta_n(t)) \in C([0, b]) \cap \overline{B}(x_0, r)$. Thus for all $t \in [0, \tau]$, the set $\{x_n(\theta_n(t)) : n \geq 1\}$ is relatively compact in H, hence by Lemma 2.2,

$$\beta\Big(\{x_n(\theta_n(t)) : n \geq 1\}\Big) = 0.$$

Next, for all $t \in [0, \tau]$

$$\begin{aligned} \beta(S(t)) &= \beta\Big(\{x_n(t) : n \geq 1\}\Big) \\ &= \beta\Big(\{x_n(t) - x_n(\theta_n(t)) + x_n(\theta_n(t)) : n \geq 1\}\Big). \end{aligned}$$

Then by Lemma 2.2 and Relation (5), we obtain

$$\begin{aligned}\beta(S(t)) &\leq \beta\Big(\{x_n(t)-x_n(\theta_n(t)) : n\geq 1\}\Big)+\beta\Big(\{x_n(\theta_n(t)) : n\geq 1\}\Big)\\ &\leq \beta\Big(\{x_n(t)-x_n(\theta_n(t)) : n\geq 1\}\Big)\\ &= \beta\left(\left\{\int_{\theta_n(t)}^{t}\dot{x}_n(s)ds : n\geq 1\right\}\right)\\ &\leq \beta\left(B\left(0,\int_{\theta_n(t)}^{t}(\lambda+1)ds\right)\right)\\ &= 2\int_{\theta_n(t)}^{t}(\lambda+1)ds.\end{aligned}$$

Since

$$\int\limits_{\theta_n(t)}^{t}(\lambda+1)ds \text{ converges to } 0 \text{ as } n\to\infty,$$

we get $\beta(S(t))=0$. Hence $S(t)$ is relatively compact in H. Therefore, by Arzelà-Ascoli's theorem (see [5]), we can select a subsequence, again denoted by $(x_n(.))_n$ which converges uniformly to an absolutely continuous function $x(.)$ on $[0,\tau]$, moreover $\dot{x}_n(.)$ converges weakly to $\dot{x}(.)$ in $L^2([0,\tau],H)$. Since all functions $x_n(.)$ agree with φ on $[-a,0]$, we can obviously say that $x_n(.)$ converges uniformly to $x(.)$ on $[-a,\tau]$, if we extend $x(.)$ in such a way that $x(.)\equiv\varphi$ on $[-a,0]$. Also, by the following inequality

$$\begin{aligned}\|x_n(\theta_n(t))-x(t)\| &\leq \|x_n(\theta_n(t))-x_n(t)\|+\|x_n(t)-x(t)\|\\ &\leq \int_{\theta_n(t)}^{t}\|\dot{x}_n(s)\|ds+\|x_n(t)-x(t)\|\\ &\leq (t-\theta_n(t))(\lambda+1)+\|x_n(t)-x(t)\|\\ &\leq (\lambda+1)\varepsilon_n+\|x_n(t)-x(t)\|\end{aligned}$$

we deduce that $x_n(\theta_n(.))$ converges uniformly to $x(.)$ on $[0,\tau]$. By construction, we have $(\theta_n(t),x_n(\theta_n(t)))\in Gr(C)\cap\overline{B}((0,x_0),r)$ for every $t\in[0,\tau]$, then $x(t)\in C(t)$ for all $t\in[0,\tau]$. Now, let $t\in[0,\tau]$. We have

$$\begin{aligned}&\|T(\theta_n(t))x_n-T(t)x\|_\infty\\ =&\sup_{-a\leq s\leq 0}\|x_n(\theta_n(t)+s)-x(t+s)\|\\ \leq&\sup_{-a\leq s\leq 0}\|x_n(\theta_n(t)+s)-x(\theta_n(t)+s)\|+\sup_{-a\leq s\leq 0}\|x(\theta_n(t)+s)-x(t+s)\|\\ \leq&\sup_{0\leq s\leq\tau}\|x_n(s)-x(s)\|+\sup_{-a\leq s\leq 0}\|x(\theta_n(t)+s)-x(t+s)\|\end{aligned}$$

Using the fact that $x(.)$ is absolutely continuous on $[0,\tau]$ and $x_n(.)$ converges uniformly to $x(.)$ on $[-a,\tau]$, we deduce that $T(\theta_n(t))x_n$ converges to $T(t)x$ in $\mathcal{C}_a$.

Proposition 3.2. *For almost every $t\in[0,\tau]$, we have $\dot{x}(t)\in\partial_c V(x(t))$.*

Proof. The weak convergence of $\dot{x}_n(.)$ to $\dot{x}(.)$ in $L^2([0,\tau],H)$ and the Mazur's Lemma entail

$$\dot{x}(t) \in \bigcap_n \bar{co}\{\dot{x}_m(t) : m \geq n\},$$

for almost every $t \in [0,\tau]$. Then for all $y \in H$ and for almost every $t \in [0,\tau]$,

$$\langle y, \dot{x}(t)\rangle \;\leq\; \inf_m \sup_{n\geq m} \langle y, \dot{x}_n(t)\rangle$$

which together with $\dot{x}_n(t) \in \partial_c V(x_n(\theta_n(t))) + \varepsilon_n B$ gives for all m

$$\langle y, \dot{x}(t)\rangle \;\leq\; \sup_{n\geq m} \sigma\Big(y, \partial_c V(x_n(\theta_n(t))) + \varepsilon_n B\Big),$$

from which we deduce that

$$\langle y, \dot{x}(t)\rangle \;\leq\; \limsup_{n\to+\infty} \sigma\Big(y, \partial_c V(x_n(\theta_n(t))) + \varepsilon_n B\Big).$$

Next, by Proposition 6.4.9 in [6], the function $x \mapsto \sigma\Big(y, \partial_c V(x)\Big)$ is upper semi-continuous and hence we get

$$\langle y, \dot{x}(t)\rangle \;\leq\; \sigma\Big(y, \partial_c V(x(t))\Big).$$

So, the convexity and the closedness of the set $\partial_c V(x(t))$ ensure $\dot{x}(t) \in \partial_c V(x(t))$. ☐

Now, we use the regularity of the function V to prove the following proposition.

Proposition 3.3. *The set* $\Big\{\langle p, \dot{x}(t)\rangle, p \in \partial_c V(x(t))\Big\}$ *is reduced to the singleton* $\Big\{\frac{d}{dt}V(x(t))\Big\}$ *for almost every* $t \in [0,\tau]$.

Proof. Since $x(.)$ is absolutely continuous function and V is locally Lipschitz continuous. The function $Vox(.)$ is absolutely continuous and then for almost all t there exists $\frac{d}{dt}V(x(t))$. Let $t \in [0,\tau]$ be such that there exists both $\dot{x}(t)$ and $\frac{d}{dt}V(x(t))$. There is $\delta > 0$ such that for every $|h| < \delta$

$$x(t+h) \in \overline{B}(x_0, r),\; (x(t) + h\dot{x}(t)) \in \overline{B}(x_0, r)$$

and

$$x(t+h) - x(t) - h\dot{x}(t) = r(h) \text{ where } \lim_{h\to 0} \|r(h)\|/h = 0.$$

Since V is Lipschitz continuous on $\overline{B}(x_0, r)$ with Lipschitz constant $\lambda > 0$, we have

$$|V(x(t+h)) - V(x(t) + h\dot{x}(t))| \leq \lambda\|r(h)\|,$$

whenever $|h| < \delta$. Consequently, the function $h \to V(x(t) + h\dot{x}(t))$ is differentiable at $h = 0$, and its derivative is the same as the derivative of $h \to V(x(t+h))$ at $h = 0$. Hence

$$\frac{d}{dt}V(x(t)) = \lim_{h\to 0} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}. \tag{6}$$

Since V is regular over $C([0,b])$ and $x(t) \in C([0,b])$, we obtain

$$V^o(x(t), \dot{x}(t)) = \lim_{h\to 0} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}. \tag{7}$$

In addition, one has

$$\begin{aligned} V^o(x(t), -\dot{x}(t)) &= \lim_{h\to 0} \frac{V(x(t)+h(-\dot{x}(t))) - V(x(t))}{h} \\ &= -\lim_{h\to 0} \frac{V(x(t)+h\dot{x}(t)) - V(x(t))}{h}. \end{aligned}$$

By Proposition 2.1, $V^o(x(t), -\dot{x}(t)) = -V_o(x(t), \dot{x}(t))$, then

$$V_o(x(t), \dot{x}(t)) = \lim_{h\to 0} \frac{V(x(t)+h\dot{x}(t)) - V(x(t))}{h}. \tag{8}$$

By (6), (7) and (8), we deduce that

$$V^o(x(t), \dot{x}(t)) = \frac{d}{dt}V(x(t)) = V_o(x(t), \dot{x}(t)).$$

This means, by Proposition 2.1, that for almost all t the set

$$\Big\{\langle p, \dot{x}(t)\rangle, p \in \partial_c V(x(t))\Big\}$$

reduces to the singleton $\Big\{\frac{d}{dt}V(x(t))\Big\}$. □

Proposition 3.4. *The application $x(.)$ is a solution of the problem (1).*

Proof. First by using Proposition 3.2 and Proposition 3.3, we obtain

$$\frac{d}{dt}V(x(t)) = \langle \dot{x}(t), \dot{x}(t)\rangle, \quad \text{a.e. on } [0,\tau].$$

Therefore by integrating on $[0,\tau]$, we get

$$V(x(\tau)) - V(x_0) = \int_0^\tau \| \dot{x}(s) \|^2 \, ds. \tag{9}$$

Now, by passing to the limit for $n \to \infty$ in (d) and using the continuity of the function V on the ball $\overline{B}(x_0, r)$, we obtain

$$V(x(\tau)) - V(x_0) \geq \limsup_{n\to+\infty} \int_0^\tau \|\dot{x}_n(s)\|^2 ds$$

Moreover, by (9), we have $\|\dot{x}\|_2^2 \geq \limsup_{n\to+\infty} \|\dot{x}_n\|_2^2$ and by the weak lower semi-continuity of the norm $\|\dot{x}\|_2^2 \leq \liminf_{n\to+\infty} \|\dot{x}_n\|_2^2$. Hence we get $\|\dot{x}\|_2^2 = \lim_{n\to+\infty} \|\dot{x}_n\|_2^2$. Finally, there exists a subsequence of $(\dot{x}_n(.))_n$ (still denoted $(\dot{x}_n(.))_n$) converges pointwisely to $\dot{x}(.)$. Since $\dot{x}_n(t) \in F(\theta_n(t), T(\theta_n(t))x_n) + \varepsilon_n B$, we have

$$d_{Gr(F)}(\theta_n(t), T(\theta_n(t))x_n, \dot{x}_n(t)) \leq \varepsilon_n,$$

hence

$$\lim_{n\to+\infty} d_{Gr(F)}(\theta_n(t), T(\theta_n(t))x_n, \dot{x}_n(t)) = 0,$$

from which we conclude that $d_{Gr(F)}(t, T(t)x, \dot{x}(t)) = 0$ and so, as F has a closed graph, we obtain $\dot{x}(t) \in F(t, T(t)x)$ for almost every $t \in [0,\tau]$. The proof is complete. □

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