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A polynomial interpolation algorithm for estimating a numerical function

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Abstract. We consider $(n+1)$ distinct numbers $x_0, x_1, ..., x_n$ and the values of the function f at these $(n+1)$ given points. We study the problem of estimating $f(x)$ at a point $x = x^*$, within a given accuracy ε, using polynomial interpolation. Based on an analysis of the linear interpolation and the error of interpolation, we present an algorithm which better exploits the results of the calculus, contributing to a decrease in the amount of work involved, respectively the computational cost necessary to approximate $f(x^*)$ with a given error. The material also presents numerical examples solved using this algorithm.

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1. Introduction

Regardless of wheather we are talking about experimental results or about tables, most of the functions are given only for a discrete set of points which make up a sample. Obviously the cases in which the value of the function is present in the table for the point that we need are seldom, and hence the necessity to call for the interpolation, which is a basic tool for the approximation of a given function.

Consider that we are given the values of the function f at $n+1$ distinct points $x_0, x_1, ..., x_n$. These are not necessarily equally spaced or even arranged in increasing order, the only main restriction is that they must be distinct. Occasionally the values of derivatives of f are also prescribed.

The polynomial interpolation, which is included in the principal class of the linear interpolation, is the basis of several types of numerical integration formulas and is used in the construction of extrapolation methods for integration and differential equations, (see [2], [3], [8]).

The polynomial interpolation problem consists of determining a polynomial P_n which takes the same values as f at the $n+1$ points $x_i,\ i = 0, 1, ..., n$, (see [5], [2]). We could write that for the $n+1$ given pairs of number $(x_k, f_k),\ k = 0, 1, ..., n$ with $x_i \neq x_j$ for $i \neq j$

$$P_n(x_k) = f_k, \ \ k = 0, 1, ..., n$$

2. Problem statement

We consider the problem of estimating $f(x)$ at a point $x = x^*$, within a given accuracy, using polynomial interpolation at distinct points x_i, $i = \overline{0, n}$, with a minimal computational cost.

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The main problem is that estimating the size of the error $f - P_n$ from a knowledge of the values of f at $x_0, x_1, ..., x_n$ alone, would be highly unlikely. Further information about f, such as bounds of his derivatives, are necessary, (see [7], [3]).
In consequence the degree of the polynomial needed for estimating $f(x^*)$ within a given accuracy ε is generally not known.
We will consider in the sequel the cases of the Lagrange, Newton and Hermite interpolating polynomial. Concerning the accuracy of polynomial interpolation, there is the following result, (see [2], [5]):

Proposition 2.1. *Let $[a, b]$ be any interval which contains all $n+1$ points $x_0, x_1, ..., x_n$. Let f, $f', ..., f^{(n)}$ exist and be continuous on $[a, b]$ and let $f^{(n+1)}$ exists for $x \in (a, b)$. Then, given any $x \in [a, b]$ there exist a number $\xi = \xi(x)$ in (a, b) so that*

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega(x) \tag{1}$$

where $\omega(x) = \prod_{i=0}^{n}(x - x_i)$, P_n denoting the interpolating polynomial at the points x_i, $i = \overline{0, n}$.

Corollary 2.1. *Under the hypotheses of Proposition 2.1, if we denote by $E(x)$ the error of interpolation, then*

$$\| E \|_\infty \leq \frac{1}{(n+1)!} \| f^{(n+1)} \|_\infty \| \omega \|_\infty \tag{2}$$

where $\| \phi \|_\infty = sup\{|\phi(t)|,\, t \in [a, b]\}$, $(\forall)\phi : [a, b] \to \mathbf{R}$ continue.

One can minimize the right side of (2) by choosing the interpolating points x_i, $i = \overline{0, n}$ as the zeros of the Cebyshev polynomials, (see [2],[5]),

$$T_{n+1} : (-1, 1) \to \mathbf{R}, \quad T_{n+1}(z) = \frac{1}{2^n} cos[(n + 1)arccos(z)]$$

Of course, this can be done if we have access to the values of f at these zeros.

Remark 2.1. *If the interpolating polynomial $P_n(x)$ is used to approximate $f(x)$ on the interval $[-1, 1]$ then one can chooses $x_i = cos\frac{\pi(2i + 1)}{2(n + 1)}$, $i = 0, 1, ..., n$.*
If $P_n(x)$ is used to approximate $f(x)$ on the interval $[a, b]$ then one can chooses the interpolating points given by $x_i = \frac{a + b}{2} + \frac{b - a}{2} cos\frac{\pi(2i + 1)}{2(n + 1)}$, $i = 0, 1, ..., n$.

Remark 2.2. *In practice the estimating problem of a function $f(x)$ at a point $x = x^*$ presents itself mostly in the form: what is desired is the value $f(x^*)$ approximated with a given accuracy ε. Based on the above results, this tolerance ε should not be selected smaller than $\frac{\| f^{(n+1)} \|_\infty}{(n + 1)!} \| \omega \|_\infty$ with respect to the zeros of the Cebyshev polynomials*

Let H_n be the Hermite interpolating polynomial which satisfies

$$\begin{cases} d^0(H_n) \leq m - 1 \\ H_n^{(k)} = f^{(k)}(x_i),\ 0 \leq i \leq n,\ 0 \leq k \leq \alpha_i \end{cases}$$

where

$$\begin{cases} x_i, \ i = \overline{0,n} \text{ are the interpolating points,} \\ \alpha_i, \ i = \overline{0,n} \text{ are positive integers,} \\ m = \sum_{i=0}^{n} (\alpha_i + 1) \end{cases}$$

H_n is the polynomial of lowest degree with the property that it agrees with the function f and with all its derivatives of order less than or equal to α_i at x_i, $i = \overline{0,n}$.

Remark 2.3. *If $\alpha_i = 0$, $i = \overline{0,n}$, $H_n(x)$ becomes the Lagrange polynomial which interpolates the function f at the points x_i, $i = \overline{0,n}$.*

Denoting by $E(x) = f(x) - H_n(x)$, $x \in [a,b]-$ an intervall which contains all points x_i, $i = \overline{0,n}$. Similar to Proposition 2.1, we have:

$$E(x) = \frac{f^{(m)}(\xi)}{m!}\omega_m(x), \ \ \xi = \xi(x), \ (\forall) x \in [a,b] \tag{3}$$

where $\omega_m(x) = \prod_{i=0}^{n} (x - x_i)^{\alpha_i + 1}$ and $f \in C^m[a,b]$.

Under some restrictive conditions one states the following result:

Proposition 2.2. *Consider that $f \in C^m[a,b]$ is an analytical one in the neighbourhood of $h = \frac{a+b}{2}$ with a convergence radius given by $r_c > \frac{3}{2}(b-a)$. Then the following statement holds:*

$$\| E \|_\infty \leq A \cdot \Big(\frac{b-a}{p}\Big)^m \cdot \Big(1 + (2^{m+1} - 1)\frac{b-a}{2p-b+a}\Big) \quad (\forall) p \in \Big(\frac{b-a}{2}, r_c\Big)$$

where $A = A(p)$ is a constant, and the sequence $(H_n)_{n \geq 1}$ of Hermite interpolating polynomials converges uniformly to the function f.

Proof. $f(x) = \sum_{i \geq 0} a_i (x-h)^i$, $\ (\forall) x \in (h - r_c, h + r_c)$

Choosing $p \in \Big(\frac{b-a}{2}, r_c\Big)$, $x = h + p$ then $\sum_{i \geq 0} a_i p^i$ converges.

In particular $\lim_{i \to \infty} a_i p^i = 0$. There is a constant $A = A(p)$ so that $|a_i| \leq \frac{A}{p^i}$, $i \geq 0$. Thus it follows that

$$|f^{(m)}(x)| \leq A \cdot \sum_{i \geq m} \frac{1}{p^i} \frac{i!}{(i-m)!} |x - h|^{i-m}$$

$(\forall) x \in [a,b] \subset (h - r_c, h + r_c)$ so that $|x - h| \in [0, \frac{b-a}{2}]$
Based on (3) one obtains

$$\| E \|_\infty \leq \frac{(b-a)^m}{m!} \| f^{(m)} \|_\infty \tag{4}$$

Taking into consideration that $(\forall) z \in \big[0, \frac{b-a}{2}\big] \subset [0, p)$

$$\sum_{i\geq m}\frac{1}{p^i}\frac{i!}{(i-m)!}z^{i-m}=\Big(\frac{1}{1-\frac{z}{p}}\Big)^{(m)}\leq\frac{m!\,p}{(p-\frac{b-a}{2})^{m+1}}=\frac{m!}{p^m}\Big(1+\frac{b-a}{2p-b+a}\Big)^{m+1}$$

and using (4) one gets

$$\| E \|_\infty\leq A\cdot\frac{(b-a)^m}{p^m}\cdot\Big(\frac{2p}{2p-b+a}\Big)^{m+1}=A\cdot\frac{p}{p-\frac{b-a}{2}}\cdot\Big(\frac{b-a}{p-\frac{b-a}{2}}\Big)^m$$

If $p\geq\frac{3}{2}(b-a)\Leftrightarrow r_c\geq\frac{3}{2}(b-a)$ then $\| f-H_n \|_\infty\to 0$ uniformly when $m\to\infty$.

Using the inequality $(1+\beta)^n\leq 1+(2^n-1)\beta,\ (\forall)n\geq 1,\ \beta\in[0,1]$ it follows that

$$\| E(x) \|_\infty\leq A\cdot\Big(\frac{b-a}{p}\Big)^m\cdot\Big(1+(2^{m+1}-1)\frac{b-a}{2p-b+a}\Big)$$

$\square$

3. Choice of interpolating points

Taking into consideration that the amount of work necessary to approximate $f(x)$ at a point $x=x^*$ within a given accuracy ε, by means of the polynomial interpolation is proportional to the degree of interpolating polynomial used, in order to have a minimal computational cost one proposes the following technique:

Denote by $P_k(x)$ the interpolating polynomial of degree$\leq k$ which interpolates the function f at distinct points $x_i,\ i=\overline{0,k}$. We calculate successively $P_i(x^*),\ i=0,1,...$, increasing the number of interpolation points and hence the degree of the interpolating polynomial, until the difference between two consecutive values becomes smaller than ε, and thus a satisfactory approximation $P_q(x^*)$ to $f(x^*)$ has been found.

The main problem is how to choose the interpolating points for the successive construction of the interpolating polynomials $P_i(x),\ i=0,1,....$

In order to minimize the computational cost we start with the points x_i,x_{i+1} so that $x^*\in(x_i,x_{i+1})$, considering that $z_0=x_i$, $z_1=x_{i+1}$ if x^* is close to x_i, or $z_0=x_{i+1}$, $z_1=x_i$ else.

The point z_2 will by the nearest point $x_j,\ j\in\{0,1,...,i-1,i+2,...,n\}$ to $x*$ and so on and so forth.

Thus we obtain a new set of points $z_0,z_1,...,z_n$ which have the same values as $x_0,x_1,...,x_n$ and which are ordered by means of distance to the point x^* :

$$|x^*-z_j|\geq|x^*-z_k|,\ \ k=0,1,...,j-1,\ \ j=0,1,...,n$$

In this way we can construct, successively, the interpolating polynomials $P_i(x),\ i=0,1,...,n$ associated to the interpolating points $z_0,z_1,...,z_i,\ i=1,2,...,n$.

4. Practical implementation

Based on the above analysis one can derive the following polynomial interpolation algorithm for estimating $f(x)$ at a point $x=x^*$, with a given accuracy ε, using the interpolation points $x_i,\ i=\overline{0,n}$:

1. determination of interpolating points x_p,x_{p+1} so that $x_p<x^*<x_{p+1}$;
2. if $|x^*-x_p|>|x^*-x_{p+1}|$ then $\begin{cases} z_0=x_{p+1}\\ z_1=x_p\end{cases}$, else $\begin{cases} z_0=x_p\\ z_1=x_{p+1}\end{cases}$
3. calculus of $P_1(x^*)$, $P_1(x)$ being the chosen interpolating polynomial with respect to the points z_0,z_1;

4. determination of the following interpolation point $z_2 \in \{x_0, x_1, ..., x_{p-1}, x_{p+2}, ..., x_n\}$ so that
$$|x^* - z_2| = min\{|x^* - x_i|,\ i \in \{0, 1, ..., p-1, p+2, ..., n\}\}$$
5. calculus of $P_2(x^*)$, $P_2(x)$ being the chosen interpolating polynomial with respect to the points z_0, z_1, z_2;
6. if $|P_1(x^*) - P_2(x^*)| \le \varepsilon$ then $f(x^*) \cong P_2(x^*)$;
   if not the case, we follow the steps 4-6 again, in order to add the following interpolation points $z_i,\ i = 3, 4, ...$ and to set up interpolating polynomials with respect to these new points, until the stopping criterion is accomplished.

Remark 4.1. *(i). If the successive differences $|P_i(x^*) - P_{i-1}(x^*)|$ begin to increase as i increases, one stops the algorithm concluding that ε was chosen unreasonably. In this situation one may consider that $f(x^*) \cong P_{i-1}(x^*)$ represents a better approximation;*
(ii). In order to obtain a minimal computational cost for the above algorithm we have to take into consideration that the evaluation of the interpolating polynomials $P_k(x^*),\ k = 1, 2, ...$ must be done in a manner that uses the previous calculations to as greater advantage as possible, (see [1]*);*
(iii). The algorithm can be used for evaluate $f(x)$ at a sequence of points x_i^*, $i = 0, 1, ..., r,\ r \in \mathrm{N}^*$.

5. Numerical examples

5.1 We consider the data

x_i	0	0.1	0.2	0.3	0.4	0.78	1.33
f_i	-1	-0.620500	-0.283987	0.006601	0.248424	0.677713	-0.230627

We wish to approximate $f(0.155)$, $f(0.947)$ within a given accuracy ε.

Using the presented algorithm we obtain:

x^*	accuracy ε	type of interpolating polynomial	number of used interpolating points	approximate value of $f(x^*)$
0.155	1e-2	Lagrange	3	-0.4297343
	1e-3	Lagrange	4	-0.4299099
	1e-5	Lagrange	5	-0.4299075
	2e-6	Lagrange	6	-0.4299082
0.947	3e-2	Newton	4	0.6035864
	1e-3	Newton	6	0.6004708
	1e-4	Newton	7	0.6005317

Remark 5.1. *The exact values are* $\begin{cases} f(0.155) = -0.4299082 \\ f(0.947) = 0.6005443 \end{cases}$

5.2 We consider the data

x_i	1	1.2	1.5	1.65	2.3	2.8	4.3
f_i	1	1.648721	2.718282	3.490343	7.389056	14.154040	20.085537

We wish to approximate $f(0.3),\ f(2.7)$ within a given accuracy ε.

Using the presented algorithm we obtain:

x^*	accuracy ε	type of interpolating polynomial	number of used interpolating points	approximate value of $f(x^*)$
0.3	1e-2	Newton	6	1.3485042
	3e-3	Newton	6	1.3505583
2.7	1e-2	Lagrange	7	14.8815651
	4e-4	Lagrangee	7	14.8798981

Remark 5.2. *The exact values are* $\begin{cases} f(0.3) = 1.3498595 \\ f(2.7) = 14.8797323 \end{cases}$

5.3 We consider the data

x_i	1	1.2	1.5	1.65	2.3	2.8	4.3
f_i	1.684370	2.199796	2.895113	3.223371	4.579691	5.592577	8.599632
f_i'	2.742245	2.443303	2.221171	2.159282	2.041032	2.014902	2.000737

We wish to approximate $f(1.8),\ f(3.1)$ within a given accuracy ε.

Using the presented algorithm we obtain:

x^*	accuracy ε	type of interpolating polynomial	number of used interpolating points	approximate value of $f(x^*)$
1.8	1e-3	Hermite	3	3.5438120
	3e-5	Hermite	4	3.5438051
	4e-7	Hermite	6	3.5438046
3.1	1e-3	Hermite	3	6.1959715
	1e-4	Hermite	4	6.1959610

Remark 5.3. *The exact values are* $\begin{cases} f(1.8) = 3.5438026 \\ f(3.1) = 6.1959327 \end{cases}$

6. Conclusions

In this paper we present an algorithm for estimating a numerical function $f(x)$ at a specific point $x = x^*$, based on the polynomial interpolation and the accuracy of interpolation.

Starting with an estimation of the size of the error we derive an algorithm which purpose is to evaluate $f(x^*)$ within a given accuracy and a minimal computational cost.

The numerical experiments performed by practical implementation of the above algorithm lead to the conclusion that this one offers a fast tool to evaluate $f(x)$ at a point x^* with a given reasonable accuracy ε, by means of an interpolating polynomial of suitable degree, and in consequence with a reduced amount of work.

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